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Document de Travail n° 2024 – 23

Juin 2024

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Taste for nature and long-run cycles*

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June 3, 2024

Abstract

From a dynamic perspective, the existing literature on renewable resources in a Ramsey economy is puzzling. On the one hand, the central planner's solution leads to the occurrence of limit cycles around the lower steady state (Wirl, 2004); on the other hand, limit cycles arise in a market economy around the higher steady state (Bosi and Desmarchelier, 2018). To reconcile these findings, we study the competitive equilibrium of a discrete-time Ramsey-Cass-Koopmans model with a renewable resource, where preferences are represented by two different utility functions with Constant Static Elasticity of Substitution (CSES) and Constant Intertemporal Elasticity of Substitution (CIES). In the CSES case, we recover the dynamics highlighted by Wirl (2004), while, in the CIES case, the ones obtained by Bosi and Desmarchelier (2018). Moreover, this conclusion is robust under two alternative regeneration processes for the resource (power and logistic laws). In other words, the dynamics seems to depend more on the preference structure than on the market structure (central planner versus market economy).

Keywords: Ramsey model, reproduction law, pollution, two-period and limit cycles.

JEL codes: C61, E32, 044.

1 Introduction

Nature provides a wide range of services essential to life and human wellbeing. Following Sandifer et al. (2015), contact with nature brings both psychological and physiological health benefits, develop recreational, cultural and spiritual wellbeing, and promotes social interaction. Nature also supplies food, medicines and raw materials. From an economic perspective, most of these positive effects

*The authors acknowledge the financial support of the LABEX MME-DII (ANR-11-LBX-0023-01).

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are externalities even when economic agents are aware of the benefits of nature. Their economic decisions (consumption, production) affects nature which affects their wellbeing in turn. Larger economic activities stress more nature and reduce more the stock of renewable resources such as forests. Agents' wellbeing lowers in turn and affects their consumption demand. A lower demand entails a lower production and promotes the regeneration of ecosystems with a positive impact on welfare at the end. These cycles take place because of the role nature plays in human wellbeing: from a formal view, dynamics reproduce the predator-prey model introduced by Lotka (1907) and Volterra (1928).

Understanding the complex interactions between species, as in the predator-prey model, is essential for ecologists in suggesting conservation programs. Recognizing the cyclical properties of actions and feedbacks between nature (renewable resources) and economic activities is just as important to recommend economic policies serving environmental quality and wellbeing. Our paper aims to explore these complex dynamics based on the interaction between economic activities and nature in a standard Ramsey-Cass-Koopmans framework with a renewable resource.

Our study is not the first attempt to understand the role of natural resources in dynamic economies. Surprisingly, early work focused solely on the optimal depletion of a resource by a central planner.[1] They study how economic decisions (consumption and production) change the resource dynamics, how the latter affects the social welfare. For instance, Beltratti et al. (1994) propose a model where a renewable resource enters the utility function. Consumption impairs the resource dynamics driven by a bell-shaped regeneration process. The central planner maximizes what they call a "sustainable preference", that is a discounted utility increased by a term reflecting the concern for generations in the distant future. Beltratti et al. (1994) observe that the optimal solution leads to a saddle point with a positive resource level in the long run (resource preservation). Ayong Le Kama (2001) is also concerned with intergenerational equity, with three notable differences from Beltratti et al. (1994): (1) the resource enters not only the utility function as a good, but also the production function as an input; (2) instead of consumption, polluting production harms resource dynamics; (3) the central planner maximizes an undiscounted intertemporal utility with the Green Golden Rule à la Chichilnisky et al. (1995) (a generalization of the bliss point à la Ramsey (1928), representing the maximal utility in the long run, jointly given by consumption and resource). Ayong Le Kama (2001) recovers the main result of Beltratti et al. (1994): when the environmental impact of production is small, the economy converges to the Green Golden Rule (a unique saddle-point solution in the long run). Wirl (2004) introduces a resource in the utility function as in Beltratti et al. (1994) and maximizes a discounted social optimum as in Ayong Le Kama (2001): endogenous cycles can occur through a Hopf bifurcation. Interestingly, he proves the existence of two steady states (low and high resource level): the highest one is always saddle-path stable while the lowest loses its stability when a limit cycle arises around through a supercritical

[1] In their seminal contribution, Dasgupta and Heal (1974) consider an exhaustible resource.

Hopf bifurcation.

Wirl (2004) shows that endogenous cycles likely occur when a resource enters the utility function but he considers only the central planner's solution and, in fact, ignores the additional external effects of a natural resource in a market economy. These effects reinforce the mechanism behind cycles. More pertinently, Bosi and Desmarchelier (2018) study the welfare impact of a resource in a market economy and show how these externalities contribute to endogenous fluctuations. Not only they revisit Wirl (2004) from a market perspective, but also introduce non-separable preferences. Their utility, a composite CIES function, allows for both positive and negative effects of resource on the marginal utility of consumption. Even if they recover two steady states à la Wirl (2004), they prove that the lowest one is always unstable, while a Hopf bifurcation can only take place around the highest when consumption and nature are substitutes (negative effect of resource on marginal utility of consumption). Surprisingly, that is the converse of Wirl (2004): cycles arise only around the highest steady state instead of the lowest.

At first sight, the difference between Bosi and Desmarchelier (2018) and Wirl (2004) rests on their alternative approaches: a market economy instead of a central planner. However, a deeper insight reveals the role of separable preferences in Wirl (2004) whose utility function cannot be reduced to a particular case of the non-separable form in Bosi and Desmarchelier (2018). Then, an intriguing question arises: is it possible to find Wirl's main result (limit cycles around the lower steady state) in a market economy with a separable function as a special case of utility in Bosi and Desmarchelier (2018)? It is important to understand whether endogenous cycles are a robust feature of the Ramsey model with renewable resources.

Our aim is to solve this robustness puzzle. While both Wirl (2004) and Bosi and Desmarchelier (2018) are continuous-time models, we consider a discrete-time version of the Ramsey model where preferences are represented by two distinct utility functions: (1) Constant Static Elasticity of Substitution (CSES); (2) Constant Intertemporal Elasticity of Substitution (CIES). Preferences (1) allows for separability between consumption and resource à la Wirl (2004) as a special case, while preferences (2) are the same of Bosi and Desmarchelier (2018).

Interestingly, we recover the main result by Wirl (2004) (a Neimark-Sacker bifurcation[2] only around the lower steady state) in the more general CSES case. Therefore, what matters is the form of preferences, not regime (planners versus market). Also noteworthy and, in some respect, unsurprisingly, we recover Bosi and Desmarchelier (2018) under a CIES specification: limit cycles only arise around the higher steady state.

If the choice of the utility function matters for fluctuations, intriguing is also the role of the regeneration process. Beltratti et al. (1994) claims that this process should be bell-shaped. In this respect, they introduce a logistic law, a process widely used in ecology. This function is also considered by Ayong Le

[2]The discrete-time equivalent of the Hopf bifurcation.

Kama (2001), Wirl (2004) and by Bosi and Desmarchelier (2018). However, a power law is also bell-shaped and well-suited to represent a regeneration process of a renewable resource.

Bosi and Ha-Huy (2023) is a discrete-time Ramsey model with positive productive externalities from a renewable resource where two regeneration processes generate rich dynamics but richer under the power law. Indeed, while cycles of period two through a flip bifurcation are possible under both these laws, only the power law promotes the occurrence of limit cycles through a Neimark-Sacker bifurcation. Our paper focuses instead on externalities on preferences but compares the stability properties of both these regeneration processes with those obtained in Bosi and Ha-Huy (2023).

To sum up, we bridge and compare different segments of the existing literature on renewable resources from a unified perspective, combining two kinds of utility function with two types of regeneration process in a discrete-time Ramsey framework.

The paper is organized as follows. Section 2 introduces the fundamentals of the model. Section 3 presents the competitive equilibrium, while section 4 discusses the existence of an optimal solution. Section 5 develops the model when the regeneration process is a power law, while Section 6 when the regeneration process is a logistic law. Section 7 sums up and discusses the results. Section 8 concludes.

2 Fundamentals

In this section, we set up the model by specifying technology, preferences with a taste for nature, and alternative reproduction processes for nature.

2.1 Production

There is a large number of small price-taker producers, sharing the same technology: $F(K_{jt}, L_{jt}) = AK_{jt}^{\alpha}L_{jt}^{1-\alpha}$, where K_{jt} and L_{jt} represent, respectively, the capital and the labour demands of firm j in period t, with, as usual, $\alpha \in (0,1)$.

Let $r_t$and w_t be, respectively, the real interest rate and the wage rate at date t. The profit is given by $AK_{jt}^{\alpha}L_{jt}^{1-\alpha} - r_tK_{jt} - w_tL_{jt}$ and zero-profit conditions hold at equilibrium:

$$r_t = \alpha Ak_t^{\alpha-1} \text{ and } w_t = (1-\alpha)Ak_t^{\alpha}$$

where $k_t \equiv K_{jt}/L_{jt}$ denote the capital intensity, the same across the firms.

2.2 Preferences

As in Wirl (2004) and Bosi and Desmarchelier (2018), nature enters the utility function and consumer maximizes the intertemporal utility $\sum_{t=0}^{\infty}\beta^t u(c_t, N_t)$ under a sequence of budget constraints $c_t + \tilde{k}_{t+1} - (1-\delta)\tilde{k}_t \leq r_t\tilde{k}_t + w_tl_t$ where $\tilde{k}_t$ denotes the individual wealth. $\beta \in (0,1)$ and $\delta \in (0,1)$ are the discount factor and the capital depreciation rate respectively. We consider a market economy where consumers take the sequence $(N_t)_{t=1}^{\infty}$ as given. In other terms, nature is a pure externality.

Notice that, here, $\tilde{k}_t$ refers to individual consumer's capital supply, while $K_t \equiv \sum_j K_{jt}$ represents the firms' aggregate demand. To keep things as simple as possible, labour supply is inelastic, there is no population growth and the size of population is normalized to one: $L_t \equiv \sum_j L_{jt} = 1$. Thus, the individual capital coincides with the aggregate one: $\tilde{k}_t = K_t$, and

$$\tilde{k}_t = \frac{\sum_j K_{jt}}{\sum_j L_{jt}} = \frac{\sum_j k_t L_{jt}}{\sum_j L_{jt}} = k_t$$

The aggregate production is given by $Y_t = \sum_{j=1}^{n} AK_{jt}^{\alpha} L_{jt}^{1-\alpha} = Ak_t^{\alpha} \sum_{j=1}^{n} L_{jt} = Ak_t^{\alpha}$. The worker supplies one unit of labor: $l_t = 1$.

Assumption 1 *The utility function u is C^2, strictly increasing ($u_c(c_t, N_t) > 0$ and $u_N(c_t, N_t) > 0$) and strictly concave in c_t.*

Remark 1 *The utility function can be separable: $u(c_t, N_t) = v(c_t) + w(N_t)$; for instance, equal to $u(c_t, N_t) = \ln\left(c_t^{\alpha} N_t^{1-\alpha}\right) = \alpha \ln c_t + (1-\alpha) \ln N_{t+1}$, that is a function with zero cross derivatives: $u_{12} = 0$. However, the cases where nature affects the marginal utility of consumption are more interesting. Consumption and nature are substitutable or complementary goods if $u_{12} < 0$ and $u_{12} > 0$, respectively.*

Here, non-separability is formalized with two utility functions:

(1) CSES (Constant Static Elasticity of Substitution)

$$u(c, N) \equiv \left(\tau c^{\frac{\sigma-1}{\sigma}} + N^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}} \tag{1}$$

with elasticity σ,

(2) CIES (Constant Intertemporal Elasticity of Substitution)

$$v(g_t) \equiv \frac{g_t^{1-\frac{1}{\omega}}}{1-\frac{1}{\omega}} \tag{2}$$

where $\omega > 0$ is the constant elasticity of intertemporal substitution of the composite good $g_t = g(c_t, N_t)$.

Consider the CSES utility (1), where σ denotes the constant elasticity of substitution between consumption and nature, and $\tau > 0$ the taste for consumption. The relative taste for nature is given by $1 - \tilde{\tau}$ where $\tilde{\tau} \equiv \tau / (1+\tau) \in (0, 1)$ is the relative propensity to consumption. Utility (1) is equivalent to

$$\tilde{u}(c, N) \equiv \left[\tilde{\tau} c^{\frac{\sigma-1}{\sigma}} + (1-\tilde{\tau}) N^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}} \tag{3}$$

and, in the case of a unit elasticity ($\sigma \to 1$), to a Cobb-Douglas:[3]

$$\tilde{u}(c, N) \equiv c^{\tilde{\tau}} N^{1-\tilde{\tau}} \tag{4}$$

[3] Indeed, the Marginal Rate of Substitution of (1) and (3) becomes the MRS of (4) when $\sigma \to 1$.

We observe that, when $\sigma \to +\infty$, (3) becomes linear:

$$\tilde{u}(c, N) \equiv \tilde{\tau} c + (1 - \tilde{\tau}) N \tag{5}$$

In this respect, a separable utility between c and N can be viewed as a particular case of (1).

The homogeneity properties of (3) entails $\varepsilon_{cc} + \varepsilon_{cN} = 0$, that is

$$\varepsilon_{cN} = \frac{1}{\sigma} \frac{1}{1 + \tau \left(\frac{c}{N}\right)^{\frac{\sigma - 1}{\sigma}}} = -\varepsilon_{cc} > 0 \tag{6}$$

Thus, always $\varepsilon_{cc} < 0$ and $\varepsilon_{cN} > 0$ in the case of a CSES utility. In other terms, the CSES utility cannot capture negative cross effects: $u_{cN}(c, N) < 0$ (negative impact of nature on the marginal utility of consumption), but only weak substitutability ($\sigma \in (0, 1)$).

Consider now the CIES utility (2) and assume a Cobb-Douglas composite good $g_t = g(c_t, N_t) \equiv c_t^{1-\pi} N_t^{\pi}$ with $0 < \pi < 1$.

Let

$$\rho \equiv \frac{\pi}{1 - \pi} \in (0, \infty) \text{ and } \varphi \equiv \frac{1 + \rho\omega}{\omega + \rho\omega} \in \left(\frac{\rho}{1 + \rho}, \infty\right) \tag{7}$$

and define an equivalent utility function:

$$u(c_t, N_t) \equiv \frac{\left(c_t N_t^{\rho}\right)^{1-\varphi}}{1 - \varphi} \tag{8}$$

We observe that, $\omega > 0$ (positive elasticity of intertemporal substitution) is equivalent to restriction

$$\varphi > \frac{\rho}{1 + \rho} \equiv \pi \tag{9}$$

Lemma 2 *The utility function u is strictly increasing. Under the parameter restriction (9), it is also strictly concave. Nature has a positive effect on the marginal utility of consumption ($u_{cN} > 0$) if and only if $\varphi < 1$.*

According to Lemma 2, we consider two intervals: (1) If $\pi < \varphi < 1$, positive cross effects ($u_{cN} > 0$); (2) If $1 < \varphi < \infty$, negative cross effects ($u_{cN} < 0$).

As above, we introduce the partial elasticities:

$$\varepsilon_{cc} \equiv \frac{c u_{cc}(c, N)}{u_c(c, N)} = -\varphi < 0 \tag{10}$$

$$\varepsilon_{cN} \equiv \frac{N u_{cN}(c, N)}{u_c(c, N)} = \rho(1 - \varphi) > 0\,(< 0) \Leftrightarrow \varphi < 1\,(> 1) \tag{11}$$

The utility function (8) is the same considered by Bosi and Desmarchelier (2018). This function allows for both positive and negative effects of nature on marginal utility of consumption but, importantly, it is non-separable.

It is interesting to compare the dynamics arising in Wirl (2004) and Bosi and Desmarchelier (2018). Both the articles consider a continuous-time Ramsey-Cass-Koopmans framework with a renewable resource (nature) in the utility and show the existence of two steady states with low and high resource levels. However, there are two main differences: Wirl (2004) focuses on a planned economy with separable preferences, while Bosi and Desmarchelier (2018) on a market economy with non-separable preferences. Wirl (2004) shows that limit cycles can emerge only around the lower steady state (through a Hopf bifurcation), while Bosi and Desmarchelier (2018) prove that they can only around the higher steady state (through a Neimark-Sacker bifurcation, which is the discrete-time equivalent of the Hopf). Separability seems to play the key role in making this dynamic difference.

As we will see, CSES preferences generate cycles à la Wirl (2004), while CIES preferences, cycles à la Bosi and Desmarchelier (2018). According to (5), linearly separable preferences are a particular case of CSES preferences. In this respect, we conjecture that a CSES utility, compatible with separability, allows for dynamics à la Wirl (2004), while a CIES utility, incompatible with a separability, promotes cycles à la Bosi and Desmarchelier (2018).

The rest of the paper will address and deepen this issue to bridge the outcomes.

2.3 Regeneration process

Nature regenerate on its own through a general reproduction process. We plausibly conceive an accumulation process driven by two forces: a reproduction mechanism *stricto sensu*, say Φ, which depends on the state of nature, and a pollution effect, say Π_t, which always dampens natural accumulation:

$$N_{t+1} - N_t = \Phi\left(N_t\right) - \Pi_t \tag{12}$$

The pollution effect depends on human activities, for instance: (1) on production ($\Pi_t = bY_t = bAk_t^\alpha$) or, alternatively, (2) on consumption ($\Pi_t = bC_t = bc_t$).

In the following, as in Wirl (2004) and Bosi and Desmarchelier (2018), we consider laws of natural reproduction where pollution comes from production.[4] More explicitly, we study two alternative rules of nature accumulation: (1) power law; (2) generalized logistic.

(1) Power law:

$$N_{t+1} = aN_t^\varepsilon - bAk_t^\alpha \tag{13}$$

with $a, b > 0$ and $0 < \varepsilon < 1$, where a is the regeneration rate and b is the pollution rate.

In a world with no humans, $k_t = 0$ and $N_{t+1} = aN_t^\varepsilon$. The natural dynamics

$$N_t = a^{\frac{1-\varepsilon^t}{1-\varepsilon}} N_0^{\varepsilon^t}$$

[4] The reader interested in a Ramsey-Cass-Koopmans model where pollution comes from consumption rather than from production, is refereed to Heal (1982).

converge to the steady state in the long run:

$$\lim_{t\to\infty}\left(a^{\frac{1-\varepsilon^t}{1-\varepsilon}}N_0^{\varepsilon^t}\right)=a^{\frac{1}{1-\varepsilon}}$$

For instance, if $a=\varepsilon=1/2$ and $N_0=1/16$, we obtain $N_\infty=1/4$. The red curve in Figure 1 represents these dynamics.

(2) Generalized logistic law:

$$N_{t+1}-N_t=aN_t^{\varepsilon}\left(\bar{N}-N_t\right)-bAk_t^{\alpha} \tag{14}$$

with $0<a<1$, $0\leq\varepsilon\leq 1$ and $b,\bar{N}>0$.

In a world with no humans, $k_t=0$ and $N_{t+1}-N_t=aN_t^{\varepsilon}\left(\bar{N}-N_t\right)$.

When $\varepsilon=0$, we obtain a linear law $N_{t+1}-N_t=a\left(\bar{N}-N_t\right)$ and the natural dynamics $N_t=(1-a)^t N_0+\left[1-(1-a)^t\right]\bar{N}$ converge to the steady state $\lim_{t\to\infty}N_t=\bar{N}$ in the long run. For instance, if $a=1/2$ and $N_0=1/16$ and $\bar{N}=1/4$, we find $N_\infty=1/4$. The blue curve in Figure 1 represents the dynamics of a linear law. In the example we consider, power and linear laws generate close trajectories.

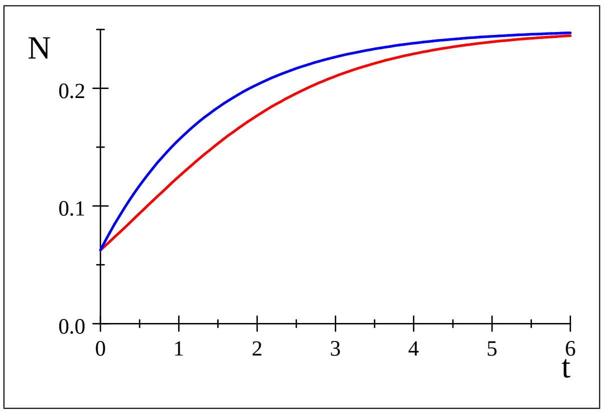

Fig. 1 Power and linear laws

When $\varepsilon=1$, we get the pure logistic law, often considered in biology to represent population dynamics. This law is also considered by Wirl (2004) and Bosi and Desmarchelier (2018).

It is interesting to compare processes (13) and (14). We observe that (13) is equivalent to $N_{t+1}-N_t=aN_t^{\varepsilon}-N_t-bAk_t^{\alpha}$. Thus, both processes write

$$N_{t+1}-N_t=\Phi_i\left(N_t\right)-bAk_t^{\alpha} \tag{15}$$

with $i=P,L$, where $\Phi_P\left(N_t\right)\equiv aN_t^{\varepsilon}-N_t$ and $\Phi_L\left(N_t\right)\equiv aN_t^{\varepsilon}\left(\bar{N}-N_t\right)$ are both strictly concave:

$$\begin{aligned}\Phi_P''\left(N_t\right) &= a\varepsilon\left(\varepsilon-1\right)N_t^{\varepsilon-2}<0\\ \Phi_L''\left(N_t\right) &= a\varepsilon\left(\varepsilon-1\right)N_t^{\varepsilon-2}\left(\bar{N}-N_t\right)-2a\varepsilon N_t^{\varepsilon-1}<0\end{aligned}$$

with $\Phi_P(0) = \Phi_L(0) = 0$, $\Phi'_P(0) = \Phi'_L(0) = +\infty$ and

$$\Phi_P\left(a^{\frac{1}{1-\varepsilon}}\right) = \Phi_L\left(\bar{N}\right) = 0$$

As we will see, both these laws of nature accumulation add a third dimension to the basic two-dimensional Ramsey-Cass-Koopmans (RCK) model. However, even if these processes look similar, non-linear dynamics might differ in some respect.

This issue, that is the role of natural law in promoting cycles, has been also tackled by Bosi and Ha-Huy (2024) in a discrete-time Ramsey-Cass-Koopmans model where nature generates positive productive externalities. Considering logistic and power laws to represent the natural regeneration, they find that both these processes promote the occurrence of two-period cycles through a flip bifurcation. However, they show also that limit cycles (through a Neimark-Sacker bifurcation) never take place under a logistic law but only under a power law. In the following, we will study how the choice of a regeneration law affects the local dynamics depending upon the preferences we consider (CSES versus CIES).

3 General equilibrium

Proposition 3 *The dynamic general equilibrium is driven by the following three-dimensional system with two predetermined state variables, k_t and N_t, and one non-predetermined choice variable, c_t:*

$$\frac{u_c(c_t, N_t)}{u_c(c_{t+1}, N_{t+1})} = \beta\left(1 - \delta + \alpha A k_{t+1}^{\alpha-1}\right) \quad (16)$$

$$c_t + k_{t+1} - (1-\delta) k_t = A k_t^{\alpha} \quad (17)$$

$$N_{t+1} - N_t = \Phi_i(N_t) - bAk_t^{\alpha} \quad (18)$$

jointly with the transversality condition[5] $\lim_{t\to\infty} \beta^t u_c(c_t, N_t) k_{t+1} = 0$.

This system is a two-dimensional RCK block augmented with a regeneration process of nature. The sequence of natural externalities $(N_t)_{t=0}^{\infty}$ directly affects the intertemporal smoothing by distorting the consumption-saving arbitrage (Euler equation). The magnitude of its impact depends on complementarity or substitutability between consumption and nature. Bosi et al. (2018) have considered a similar mechanism where the negative externalities of pollution P_t replace in the utility function the positive externalities of nature N_t. They have shown how the cross effects $u_{cN}(c_t, N_t) \gtrless 0$ (complementarity and substitutability) promote the occurrence of cycles.

[5] The transversality condition holds when the initial condition $\tilde{k}_0$ lies in a neighborhood of a stable steady state or a stable cycle, or inside an unstable cycle because the sequences $(c_t)_{t=0}^{\infty}$ and $\left(\tilde{k}_t\right)_{t=0}^{\infty}$ are uniformly bounded.

Remark 4 *Clearly, if the utility function is separable* $u(c_t, N_t) = v(c_t) + w(N_t)$*, the cross effect vanishes* $u_{cN}(c_t, N_t) = 0$ *and we recover the basic Ramsey model independent on the natural reproduction process:* $v'(c_t)/v'(c_{t+1}) = \beta\left(1 - \delta + \alpha A k_{t+1}^{\alpha-1}\right)$ *with* $c_t + k_{t+1} = (1-\delta)k_t + Ak_t^\alpha$*. Conversely, the reproduction process remains affected by human activities according to (18), that is by capital accumulation* $(k_t)_{t=0}^{\infty}$*.*

4 Market inefficiency

The planner internalizes the positive externalities of nature and maximizes the welfare, that is the representative agent's intertemporal utility function:

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t, N_t) \tag{19}$$

under a sequence of resource constraints instead of budget constraints:

$$c_t + k_{t+1} - (1-\delta)k_t = Ak_t^\alpha \tag{20}$$

As above, the population of consumers is normalized to one and $k_t = K_t$. The planner faces also a sequence of natural resource constraints:

$$N_{t+1} - N_t = \Phi_i(N_t) - bAk_t^\alpha \tag{21}$$

where the reproduction law depends on the model: $i = P, L$.

The existence of an optimal path is far from being trivial, because of the convex term $-bAk_t^\alpha$ in the law of natural reproduction appearing in the Lagrangian function.

Proposition 5 *There exists an optimal solution to program (19) under constraints (20) and (21).*

The rest of the paper is organized in two parts. First, we assume that the natural regeneration process is driven by a power law; second, by a logistic law. In each part, we will develop two subcases depending on the explicit utility function we consider: the CSES (i.e. (1)) and the CIES (i.e. (8)) utility functions.

5 Power law

The dynamic system (16)-(18) becomes:

$$\frac{u_c(c_t, N_t)}{u_c(c_{t+1}, N_{t+1})} = \beta\left(1 - \delta + \alpha A k_{t+1}^{\alpha-1}\right) \tag{22}$$

$$c_t + k_{t+1} = (1-\delta)k_t + Ak_t^\alpha \tag{23}$$

$$N_{t+1} = aN_t^\varepsilon - bAk_t^\alpha \tag{24}$$

5.1 Steady state

Let us introduce a positive upper bound for the TFP:

$$\bar{A} \equiv (\alpha\gamma)^{-\alpha} \left[(1-\varepsilon) (a\varepsilon)^{\frac{\varepsilon}{1-\varepsilon}} \frac{a}{b} \right]^{1-\alpha}$$

where, for notational simplicity, we set

$$\gamma \equiv \frac{\beta}{1-\beta(1-\delta)} \geq \beta \tag{25}$$

Notice that $\gamma = \beta$ when $\delta = 1$ (full capital depreciation).

Assumption 2 *$A \leq \bar{A}$.*

In the proof of the next proposition, we will see that Assumption 2 ensures the existence of at least one steady state.

Proposition 6 (multiple steady states) *(1) If $A < \bar{A}$, there are two steady states (k, c, N_1) and (k, c, N_2) with $0 < N_1 < N_2$, where*

$$k = (\alpha\gamma A)^{\frac{1}{1-\alpha}} > 0 \tag{26}$$

$$c = \left(\frac{1}{\alpha\gamma} - \delta \right) k > 0 \tag{27}$$

and the stationary levels of nature N_i are multiple solutions to equation

$$aN^{\varepsilon} = bAk^{\alpha} + N \tag{28}$$

(2) If $A = \bar{A}$, these two steady states coincide: $N_1 = N_2 > 0$.

(3) If $A > \bar{A}$, there is no steady state.

Remark 7 *We observe that (26)-(27) is precisely the Modified Golden Rule (MGR) of the standard Ramsey model: the regeneration process of nature has no impact on capital intensity and consumption level in the long run, which are unique and positive. However, the stationary levels of nature N_1 and N_2 in equation (28) depend on the MGR through the impact of pollution on the regeneration process: $\Pi = bAk^{\alpha}$.*

5.2 Local dynamics

We introduce the second-order partial elasticities:

$$\varepsilon_{cc} \equiv \frac{cu_{cc}(c,N)}{u_c(c,N)} \text{ and } \varepsilon_{cN} \equiv \frac{Nu_{cN}(c,N)}{u_c(c,N)} \tag{29}$$

As usual, $-1/\varepsilon_{cc}$ is the intertemporal elasticity of substitution in consumption while ε_{cN} captures the effect of nature on the marginal utility of consumption.

Let us also define some relevant blocks:

$$B \equiv \frac{b}{\gamma}\frac{k}{N} > 0,\ C \equiv \frac{1}{\alpha\gamma} - \delta > 0 \text{ and } E \equiv \varepsilon\left(1 + \frac{B}{\alpha}\right) > 0 \tag{30}$$

jointly with the most important expression involving the cross effects:

$$\eta \equiv \frac{1-\alpha}{\varepsilon_{cN}}\frac{\beta}{\gamma}C \tag{31}$$

Lemma 8 *System (22)-(24) is locally approximated around a steady state by the linear form:*

$$X_{t+1} = JX_t \tag{32}$$

where

$$X_t \equiv \left(\frac{dk_{t+1}}{k}, \frac{dc_{t+1}}{c}, \frac{dN_{t+1}}{N}\right)^T$$

and

$$J \equiv \begin{bmatrix} \frac{1}{\beta} & -C & 0 \\ \left(B + \frac{\eta}{\beta C}\right)\frac{\varepsilon_{cN}}{\varepsilon_{cc}} & 1 - \eta\frac{\varepsilon_{cN}}{\varepsilon_{cc}} & (1-E)\frac{\varepsilon_{cN}}{\varepsilon_{cc}} \\ -B & 0 & E \end{bmatrix} \tag{33}$$

denotes the Jacobian matrix.

Local dynamics depends on the location of the eigenvalues λ_1, λ_2 and λ_3 with respect to the unit circle in the Argand plane. The degree of stability depends on the number of eigenvalues inside the unit circle (with modulus less than one). The characteristic polynomial is given by $P(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) = \lambda^3 - T\lambda^2 + S\lambda - D$ where $T = \lambda_1 + \lambda_2 + \lambda_3$, $S = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3$ and $D = \lambda_1\lambda_2\lambda_3$ are the trace, the sum of principal minors of order two and the determinant of the Jacobian matrix. The sign of the characteristic polynomial when $\lambda = -1, 0, 1$ tells us about the location of the eigenvalues in the Argand plane. This method is well suited when the dynamic system is three-dimensional.

Lemma 9 *The characteristic polynomial at* -1, 0 *and* 1 *is given by*

$$P(-1) = -1 - T - S - D = (1+E)\left(\eta\frac{\varepsilon_{cN}}{\varepsilon_{cc}} - 2\frac{1+\beta}{\beta}\right) - 2BC\frac{\varepsilon_{cN}}{\varepsilon_{cc}} \tag{34}$$

$$P(0) = -D = -\frac{1}{\beta}E - BC\frac{\varepsilon_{cN}}{\varepsilon_{cc}} \tag{35}$$

$$P(1) = 1 - T + S - D = (1-E)\,\eta\frac{\varepsilon_{cN}}{\varepsilon_{cc}} \tag{36}$$

In order to know the sign of these values, we specifies the utility function by considering first a Constant Static Elasticity of Substitution (CSES) allowing only for positive cross effects and, then, a Constant Intertemporal Elasticity of Substitution (CIES) allowing for both negative and positive cross effects.

5.3 CSES preferences (positive cross effects)

Focus now on (1) and consider expressions (30) and (31) with, now, $\eta > 0$ since $\varepsilon_{cN} > 0$. We notice that the steady states (k, c, N_i) do not depend on preference parameters σ and τ.

These two parameters only enter η. Indeed, according to (6),

$$\eta = \sigma \left[1 + \tau \left(\frac{c}{N} \right)^{\frac{\sigma - 1}{\sigma}} \right] \frac{1 - \alpha}{\alpha \gamma} \left[1 - \beta + (1 - \alpha) \beta \delta \right]$$

Since c/N does not depend on (σ, τ), η varies from 0 to ∞ as σ and τ vary in their ranges from 0 to ∞.

In other terms, we can analyze the bifurcations with respect to η independently on the other parameters.

Lemma 10 *The characteristic polynomial $P(\lambda)$ takes the following values at $\lambda = -1, 0, 1$:*

$$P(-1) = 2BC - (1 + E) \left[\eta + 2 \left(1 + \frac{1}{\beta} \right) \right] \quad (37)$$

$$P(0) = BC - \frac{E}{\beta} \quad (38)$$

$$P(1) = \eta (E - 1) \quad (39)$$

Let us introduce the following critical values:

$$\varepsilon^* \equiv \frac{1 - \beta + (1 - \alpha) \beta \delta}{\beta + 1} \text{ and } b^* \equiv \alpha \gamma \frac{1 + \varepsilon}{\varepsilon^* - \varepsilon} \frac{(a \varepsilon)^{\frac{1}{1 - \varepsilon}}}{(\alpha \gamma A)^{\frac{1}{1 - \alpha}}}$$

jointly with the flip bifurcation value:

$$\eta_F \equiv 2 \frac{1 + \beta}{\beta} \left[\frac{\varepsilon^* B(N_1)}{\alpha (1 + \varepsilon) + \varepsilon B(N_1)} - 1 \right] \quad (40)$$

where $B(N_i) \equiv bk / (\gamma N_i)$ and N_1 is the lower steady state.

Proposition 11 (cycles of period two) *If $\varepsilon < \varepsilon^*$ and b is in a neighborhood of b^*, then there exists a critical value $\eta_F > 0$ such that cycles of period two generically arise around the lower steady state (c, k, N_1) through a flip bifurcation at $\eta = \eta_F$.*

If $\varepsilon > \varepsilon^*$, there is no room for cycles of period two through a flip bifurcation.

Focus now on the Neimark-Sacker bifurcation and its critical value:

$$\eta_N \equiv \frac{E}{\beta} - BC + \frac{\frac{E}{\beta} - BC - 1 + \beta - \beta E}{\beta BC - (1 - \beta) E} \quad (41)$$

Proposition 12 (limit cycles) *Limit cycles generically arise around the lower steady state N_1 through a Neimark-Sacker bifurcation at $\eta = \eta_N$ provided that*

$$(E-1)\frac{1-\beta}{\beta} - BC - 4 < \eta_N < (E-1)\frac{1-\beta}{\beta} - BC \tag{42}$$

There are no limit cycles around the higher steady state N_2.

Proposition 13 (locally determinacy) *When the system undergoes a Neimark-Sacker (NS) bifurcation, the equilibrium trajectory around the lower steady state (k, c, N_1) is locally determinate.*

Remark 14 *The unstable manifold associated to λ_1 is one-dimensional, while the center manifold associated to λ_2 and λ_3 is two-dimensional. The predetermined pair $\left(\bar{k}_t, \bar{N}_t\right)$ in a neighborhood of the steady state (k, c, N_1) defines a line $\left\{\left(\bar{k}_t, c_t, \bar{N}_t\right) : c_t \in \mathbb{R}\right\}$ in $\mathbb{R}^3$ which generically crosses the center manifold at $c_t = \bar{c}_t$. In other terms, the non-predetermined variable c_t takes a value $\bar{c}_t$ that neutralizes the unstable manifold.*

When the NS bifurcation is supercritical (subcritical), the limit cycle is attractive (repulsive). Then, there exists a unique equilibrium trajectory starting from $\left(\bar{k}_t, \bar{c}_t, \bar{N}_t\right)$ inside the limit cycle and converging to the cycle (to the steady state (k, c, N_1)) along a spiral lying on the two-dimensional center manifold.

5.4 CIES preferences (positive or negative cross effects)

Lemma 15 *The consumer's programs $\max \sum_{t=0}^{\infty} \beta^t v(g_t)$ and $\max \sum_{t=0}^{\infty} \beta^t u(c_t, N_t)$ are equivalent (they have the same solutions under the sequence of budget constraints).*

Therefore, in the following, we maximize the new intertemporal utility $\sum_{t=0}^{\infty} \beta^t u(c_t, N_t)$. We are especially interested in the sign of the cross effect $u_{cN}(c_t, N_t) \equiv \partial u_c(c_t, N_t)/\partial N_t$, that is the impact of nature on the marginal utility of consumption $u_c(c_t, N_t) \equiv \partial u(c_t, N_t)/\partial c_t$.

Let us define some relevant blocks:

$$A_1 \equiv \beta\left(\frac{1}{\alpha\gamma} - \delta\right)\frac{1-\alpha}{\gamma\varphi} > 0 \tag{43}$$

$$A_2 \equiv \left(\frac{1}{\alpha\gamma} - \delta\right)\frac{1-\varphi}{\varphi}\frac{b}{\gamma}\frac{k}{N} > 0 \Leftrightarrow \varphi < 1 \text{ (positive cross effects)} \tag{44}$$

$$B_5 \equiv \varepsilon + \frac{\varepsilon}{\alpha}\frac{b}{\gamma}\frac{k}{N} > 0$$

Lemma 16 *The characteristic polynomial $P(\lambda)$ takes the following values at $\lambda = -1, 0, 1$:*

$$P(-1) = 2\rho A_2 - (1 + B_5)\left(A_1 + 2\frac{1+\beta}{\beta}\right) \tag{45}$$

$$P(0) = \rho A_2 - \frac{B_5}{\beta} \tag{46}$$

$$P(1) = A_1(B_5 - 1) \tag{47}$$

We observe that ρ does not affect A_1 nor A_2 nor B_5. In the following, we will consider the solutions to $P(-1) = 0$ and $P(0) = 0$ associated to the steady states (k, c, N_1) and (k, c, N_2):

$$\rho_{Fi} \equiv \frac{1 + B_5(N_i)}{A_2} \left(\frac{A_1}{2} + \frac{1+\beta}{\beta} \right) \text{ and } \rho_{0i} \equiv \frac{1}{\beta} \frac{B_5(N_i)}{A_2}$$

Proposition 17 *If nature has a positive effect on the marginal utility of consumption ($\pi < \varphi < 1$) and $\rho_{02} < \rho < \rho_{F2}$, then the eigenvalues associated to (k, c, N_2) are real with $-1 < \lambda_1 < 0 < \lambda_2 < 1 < \lambda_3$ (equilibrium determinacy with saddle-path stability).*

Proposition 18 (saddle-node bifurcation) *When A crosses $\bar{A}$ from above, two steady states appear through a saddle-node bifurcation (generically).*

Proposition 19 (two-period cycles) *Under positive cross effects ($\pi < \varphi < 1$), a two-period cycle generically arises around N_i though a flip bifurcation at $\rho = \rho_{Fi}$ with $\rho_{F1} > \rho_{F2}$.*

Under negative cross effects ($\varphi > 1$), there is no room for flip bifurcations.

Remark 20 *We observe that, differently from Proposition 11, where two-period cycles arise only around the lower steady state N_1 (when $\varepsilon < \varepsilon^*$ and b is in a neighborhood of b^*), now both the steady states can experience a flip bifurcation. However, as ρ increases, two-period cycles first arise around the higher steady state, then around the lower one ($\rho_{F2} < \rho_{F1}$).*

Let us introduce the following critical value:

$$\rho_N^+ \equiv \frac{B_5 - D_N^+}{\beta A_2} \tag{48}$$

where

$$D_N^+ \equiv \frac{1}{2} \left[T - 1 + \sqrt{(T-3)^2 + 4A_1(1 - B_5)} \right] \tag{49}$$

Proposition 21 (limit cycles) *(1) There is no room for limit cycles around the lower steady state N_1 through a Neimark-Sacker (NS) bifurcation.*

(2) Limit cycles generically arise around the higher steady state N_2 through a NS bifurcation at $\rho = \rho_N^+$.

Proposition 22 (local determinacy) *When the system undergoes a Neimark-Sacker (NS) bifurcation (ρ crosses the critical value ρ_N^+), that is a limit cycle arises around the higher steady state (k, c, N_2), the equilibrium trajectory around this steady state is locally determinate.*

Interestingly and entirely, the dynamic explanation of local equilibrium uniqueness (Remark 14) still applies to this case.

6 Logistic law

The dynamic system (16)-(18) becomes:

$$\frac{u_c(c_t, N_t)}{u_c(c_{t+1}, N_{t+1})} = \beta\left(1 - \delta + \alpha A k_{t+1}^{\alpha-1}\right) \quad (50)$$

$$c_t + k_{t+1} = (1-\delta) k_t + A k_t^\alpha \quad (51)$$

$$N_{t+1} - N_t = a N_t^\varepsilon \left(\bar{N} - N_t\right) - bAk_t^\alpha \quad (52)$$

6.1 Steady state

Proposition 23 (multiple steady states) *Let $0 < \varepsilon \leq 1$ and, if $\varepsilon = 1$,*

$$a\varepsilon > \frac{bA(\alpha\gamma A)^{\frac{\alpha}{1-\alpha}}}{\bar{N}^2} \quad (53)$$

where γ is still given by (25).

Capital and consumption of steady state are still given by (26) and (27) (Modified Golden Rule).

Consider the function

$$\varphi(N) \equiv aN^\varepsilon - \frac{bAk^\alpha}{\bar{N} - N}$$

and let N^* be the unique solution to $\varphi'(N) = 0$ in the admissible range $\left[0, \bar{N}\right)$. A steady state is a positive solution N_i to $\varphi(N) = 0$.

(0) If $\varphi(N^*) < 0$, there are no steady states.

(1) If $\varphi(N^*) = 0$, there is a unique steady state ($N = N^*$).

(2) If $\varphi(N^*) > 0$, there are two solutions N_1 and N_2 with $N_1 < N^* < N_2$. Moreover,

$$N_1 < \frac{\varepsilon}{1+\varepsilon}\bar{N} < N_2 \quad (54)$$

6.2 Local dynamics

We define the second-order elasticities as in (29). In the case of a logistic law, expressions for B and C in (30) and for η in (31) remains the same. However, now, a more complicated expression

$$\tilde{E} \equiv 1 + \frac{B}{\alpha}\left(\varepsilon - \frac{N}{\bar{N} - N}\right) \quad (55)$$

replaces E in (30). (54) implies

$$\tilde{E}(N_2) < 1 < \tilde{E}(N_1) \quad (56)$$

Lemma 24 *System (50)-(52) is locally approximated around a steady state by the linear system (32), but now the Jacobian matrix is given by (33) with $\tilde{E}$ instead of E.*

Remark 25 *Surprisingly, the Jacobian matrix has the same form of the Jacobian matrix (33) of the power law, but expression (55), replaces E. We recover also the same characteristic polynomial with $\tilde{E}$ instead of E. In particular, expressions (34), (35) and (36) where, now, $\tilde{E}$ replaces E, and the dynamic analysis follows analogous lines.*

6.3 CSES preferences (positive cross effects)

Reconsider the CSES utility (1).

Lemma 26 *The characteristic polynomial evaluated at -1, 0 and 1 is still given by the values (37), (38) and (39) where, now, the expression $\tilde{E}$ replaces E.*

Remark 27 *We observe also that the steady states (k, c, N_i) do not depend on preference parameters σ and τ. These two parameters only enter η. Indeed, according to (6),*

$$\eta = \sigma \left[1 + \tau \left(\frac{c}{N}\right)^{\frac{\sigma-1}{\sigma}}\right] \frac{1-\alpha}{\alpha\gamma} \left[1 - \beta + (1-\alpha)\beta\delta\right]$$

Since c/N does not depend on (σ, τ), η varies from 0 to ∞ as σ and τ vary in their ranges from 0 to ∞.

We can compute the critical value η_F (that is σ_F or τ_F), such that a flip bifurcation takes place.

Proposition 28 (cycles of period two) *Cycles of period two generically arise around the steady state N_i when η crosses the flip bifurcation value*

$$\eta_{Fi} \equiv 2\frac{B(N_i)C}{1+\tilde{E}(N_i)} - 2\frac{1+\beta}{\beta}$$

provided that

$$\frac{B(N_i)C}{1+\tilde{E}(N_i)} > \frac{1+\beta}{\beta} \tag{57}$$

Since $B(N_i)C > 0$, inequality requires $1 + \tilde{E}(N_i) > 0$ as a necessary condition.

Remark 29 *Condition (57) depends on the steady state N_i we are considering and, therefore, changes. But it could hold in both steady states. Indeed, since $B(N_2) < B(N_1)$ and, according to (56), $\tilde{E}(N_2) < \tilde{E}(N_1)$, the effect of the steady state on the LHS of (57) is ambiguous.*

Let us introduce a new critical value

$$\eta_N \equiv \frac{\tilde{E}}{\beta} - BC + \frac{\frac{\tilde{E}}{\beta} - BC - 1 + \beta - \beta\tilde{E}}{\beta BC - (1-\beta)\tilde{E}} \tag{58}$$

According to Remark 27, we can fix σ and τ, that is η, as we wish without affecting the RHS of (58).

Proposition 30 (limit cycles) *Consider a parametric configuration such that*

$$N^* = \frac{\varepsilon}{1+\varepsilon}\bar{N} \tag{59}$$

with two steady states: $N_1 < N^* < N_2$ *(notice that (59) holds when* $N_1 = N_2$*). Limit cycles generically arise around the lower steady state* N_1 *through a Neimark-Sacker bifurcation at* $\eta = \eta_N$ *provided that*

$$\left(\tilde{E}-1\right)\frac{1-\beta}{\beta} - BC - 4 < \eta_N < \left(\tilde{E}-1\right)\frac{1-\beta}{\beta} - BC \tag{60}$$

There are no limit cycles around the higher steady state N_2*.*

Notice that, according to Remark 27, we can always move σ and τ, that is η, without affecting the steady states N^*, N_1 and N_2. Therefore, η can cross η_N (independent of σ and τ), while (59) and $N_1 < N^* < N_2$ remain true.

Proposition 31 (locally determinacy) *Under the assumptions of Proposition 30, when the system undergoes a Neimark-Sacker (NS) bifurcation, the equilibrium trajectory around the lower steady state* (k, c, N_1) *is locally determinate.*

6.4 CIES preferences (positive or negative cross effects)

Reconsider expressions A_1 and A_2 given by (43) and (44) with, now,

$$\tilde{B}_5 \equiv \tilde{E} \equiv 1 + \frac{1}{\alpha}\left(\varepsilon - \frac{N}{\bar{N}-N}\right)\frac{b}{\gamma}\frac{k}{N} \tag{61}$$

instead of B_5. As above, $A_1 > 0$, while $A_2 > 0$ if and only if $\varphi < 1$ (positive cross effects).

Lemma 32 *The values of the characteristic polynomial evaluated at* -1, 0 *and* 1 *become (45), (46) and (47) where, now,* $\tilde{B}_5 \equiv \tilde{E}$ *replaces* B_5*. Moreover,* $\tilde{B}_5(N_2) < 1 < \tilde{B}_5(N_1)$*.*

Proposition 33 (saddle-node bifurcation) *Let* $\varphi < 1$ *(positive cross effects) and*

$$\frac{1}{A_2}\frac{1}{\beta} < \rho < \frac{1}{A_2}\left(A_1 + 2\frac{1+\beta}{\beta}\right) \tag{62}$$

When the saddle-node bifurcation takes place giving rise to two steady states, in the neighborhood of the bifurcation point, N_1 *is less stable than* N_2*. More precisely, the eigenvalues associated to* N_1 *are all real and ranked as follows:* $-1 < \lambda_1 < 0 < 1 < \lambda_2 < \lambda_3$ *(two eigenvalues outside the unit circle), while the eigenvalues associated to* N_2 *are also real, but ranked as follows:* $-1 < \lambda_1 < 0 < \lambda_2 < 1 < \lambda_3$ *(two eigenvalues inside the unit circle).*

Notice that A_1 and A_2 don't depend on ρ, that is inequalities (62) are explicit. We observe that the economic system experiences dumping fluctuations in both the steady states because $-1 < \lambda_1 < 0$. Sufficient conditions for the occurrence of persistent cycles are provided in the next proposition.

Proposition 34 (two-period cycles) *If $\varphi < 1$ (positive cross effects), cycles of period two (through a flip bifurcation) generically arise around N_1 at $\rho = \rho_F(N_1) > 0$. Cycles of period two (through a flip bifurcation) can also arise generically around N_2, provided that $\rho_F(N_2) > 0$, that is*

$$2\frac{\gamma}{b}\frac{N_2}{k} + \frac{1}{\alpha}\left(\varepsilon - \frac{N_2}{\bar{N} - N_2}\right) > 0 \tag{63}$$

Notice that, if (63) holds, we don't know whether $\rho_F(N_2) < \rho_F(N_1)$.

Let

$$\rho_N^+ \equiv \frac{1}{A_2}\left(\frac{\tilde{B}_5}{\beta} - D_+\right)$$

where

$$D_+ \equiv \frac{1}{2}\left[T - 1 + \sqrt{(T-3)^2 + 4A_1\left(1 - \tilde{B}_5\right)}\right]$$

Proposition 35 (limit cycles) *Limit cycles generically arise through a Neimark-Sacker (NS) bifurcation around N_2 when ρ crosses the critical value ρ_N^+, provided that*

$$4(T+1) > A_1\left(1 - \tilde{B}_5\right) \tag{64}$$

There is no room for limit cycles around N_1.

In order to understand the role of the cross effects (the impact of nature on the marginal utility of consumption), let us consider the case when the capital share in total income is close to one (in a way, the model is close to a AK framework as in Bosi and Ha-Huy (2024)).

Corollary 36 *Let α be close to 1.*

Under negative cross effects ($\varphi > 1$), limit cycles generically arise around N_2 through a NS bifurcation at $\rho = \rho_N^+ > 0$.

Under positive cross effects ($\varphi < 1$), there are no limit cycles.

Proposition 37 (local determinacy) *We know that a limit cycle generically arises around N_2 when ρ crosses the critical value ρ_N^+. In this case, the equilibrium trajectory around the steady state is locally unique.*

As above, the dynamic interpretation of local equilibrium uniqueness (Remark 14) still applies to this case.

7 Interpretations

The following table sums up our results.

Regeneration process	Power law				Logistic law			
Bifurcation	Flip		NS		Flip		NS	
Steady state	N_1	N_2	N_1	N_2	N_1	N_2	N_1	N_2
CSES utility / $u_{cN} > 0$	$\boxed{Y}$	U	$\boxed{Y}$	N	$\boxed{Y}$	$\boxed{Y}$	$\boxed{Y}$	N
CIES utility / $u_{cN} > 0$	$\boxed{Y}$	$\boxed{Y}$	N	$\boxed{Y}$	$\boxed{Y}$	$\boxed{Y}$	N	$\boxed{Y}$
CIES utility / $u_{cN} < 0$	N	N	N	$\boxed{Y}$	U	U	N	$\boxed{Y}$

NS means Neimark-Sacker bifurcation. $\boxed{Y}$ denotes the possibility of the bifurcation we are focusing on for some parameter configuration. N means that any parameter configuration rules out the bifurcation. U stands for "uninformative": we are not able to establish the bifurcation occurrence from the characteristic polynomial.

From this table, the following conclusions can be roughly drawn.

(1) Under a CSES utility (with necessarily positive cross effects), two-period and limit cycles arise around the lower steady state whatever the regeneration process. Limit cycles are impossible around the higher steady state.

(2) Under a CIES utility with positive cross effects, two-period cycles arise around both the steady states whatever the regeneration process.

(3) Under a CIES utility with negative cross effects, there is no room for two-period cycles (at least under the power law). Surprisingly and contrarily to the CSES case, limit cycles arise only around the higher steady state whatever the regeneration process. To put it differently, we recover Wirl (2004) under a CSES utility and Bosi and Desmarchelier (2018) under a CIES utility. The kind of preferences seems to play the key role instead of the regime (central planner versus market economy). Our conclusions are robust regarding the regeneration process of nature (power versus logistic law).

8 Conclusion

Literature on the Ramsey model with renewable resource is puzzling. On the one hand, Wirl (2004) considers a central planner and separable preferences in consumption and resource, and he highlights the occurrence of limit cycles only around the lower steady state. On the other hand, Bosi and Desmarchelier (2018) study a market economy with non-separable (CIES) preferences and they observe the possibility of limit cycles only around the higher steady state.

In the spirit of Wirl (2004) and Bosi and Desmarchelier (2018), we have built a Ramsey-Cass-Koopmans model with a renewable resource in the utility, but, differently from Wirl (2004), we have used discrete time and focused on competitive equilibrium.

To compare and bridge their contributions, we have introduced two kind of preferences: a CSES utility turning out to be separable in the limit, as in Wirl (2004), and a non-separable CIES utility, as in Bosi and Desmarchelier (2018).

Under a CSES utility, we have recovered the limit cycles only around the lower steady state as in Wirl (2004); under a CIES utility, the limit cycles only around the higher steady state as in Bosi and Desmarchelier (2018).

We have shown the robustness of these conclusions under different natural regeneration processes (logistic and power laws).

To conclude, we observe that dynamics of a Ramsey-Cass-Koopmans model with a renewable resource in the utility depend more on the preference structure than on the market structure (centralized versus decentralized economy).

9 Appendix

Proof of Lemma 2

The utility function is strictly increasing:

$$\begin{aligned} u_c(c,N) &= \frac{(cN^\rho)^{1-\varphi}}{c} > 0 \\ u_N(c,N) &= \rho\frac{(cN^\rho)^{1-\varphi}}{N} > 0 \end{aligned}$$

Consider the Hessian matrix:

$$H \equiv \begin{bmatrix} u_{cc} & u_{cN} \\ u_{cN} & u_{NN} \end{bmatrix}$$

with

$$\begin{aligned} u_{cc}(c,N) &= -\varphi\frac{(cN^\rho)^{1-\varphi}}{c^2} < 0 \\ u_{cN}(c,N) &= \rho(1-\varphi)\frac{(cN^\rho)^{1-\varphi}}{cN} > 0 \Leftrightarrow \varphi < 1 \\ u_{NN}(c,N) &= \rho(\rho-1-\varphi\rho)\frac{(cN^\rho)^{1-\varphi}}{N^2} < 0 \Leftrightarrow \varphi > \frac{\rho-1}{\rho} \end{aligned}$$

Since $u_{cc} < 0$, the Hessian matrix H is negative definite (the utility is strictly concave) if and only if $\det H = u_{cc}u_{NN} - u_{cN}^2 > 0$ or, equivalently, (9) holds. Clearly, (9) implies $\varphi > (\rho-1)/\rho$, that is $u_{NN} < 0$. ■

Proof of Proposition 3

By maximizing the Lagrangian function

$$\sum_{t=0}^{\infty}\beta^t u(c_t,N_t) + \sum_{t=0}^{\infty}\lambda_t\left[r_t\tilde{k}_t + w_t l_t - c_t - \tilde{k}_{t+1} + (1-\delta)\tilde{k}_t\right]$$

we obtain the Euler equation $\lambda_t/\lambda_{t+1} = 1-\delta+r_{t+1}$ where $\lambda_t = \beta^t u_c(c_t,N_t)$, jointly with the budget constraint and the regeneration rule, now binding, and the transversality condition: $\lim_{t\to\infty}\lambda_t\tilde{k}_{t+1} = 0$.

The second-order conditions for utility maximization are also satisfied. Indeed, since the consumer takes the sequences $(N_t)_{t=0}^{\infty}$ and $(r_t)_{t=0}^{\infty}$ as given, the concavity of u with respect to the sequence $(c_t)_{t=0}^{\infty}$ imply the concavity of Lagrangian with respect to the sequence $(c_t)_{t=0}^{\infty}$. Hence, the first-order conditions turn out to be not only necessary but also sufficient for maximization.

Focus now on the general equilibrium. Since $l_t = 1$, $\tilde{k}_t = k_t$, $r_t = \alpha A k_t^{\alpha-1}$ and $w_t = (1-\alpha) A k_t^{\alpha}$, the budget constraint becomes a binding resource constraint: $c_t + k_{t+1} - (1-\delta) k_t = r_t k_t + w_t = A k_t^{\alpha}$.

Using $Y_t = A k_t^{\alpha}$, we obtain the system (16)-(18). ■

Proof of Proposition 5

We observe that, for $i = P, L$, there exists $\hat{N}$ such that $0 \leq N_t \leq \hat{N}$ for any t. Moreover, since $0 \leq k_{t+1} \leq A k_t^{\alpha}$, there exists $\hat{k}$ such that $0 \leq k_t \leq \hat{k}$, for every $t \geq 0$. Therefore, the sequences $(c_t)_{t=0}^{\infty}$, $(k_t)_{t=0}^{\infty}$ and $(N_t)_{t=0}^{\infty}$ are uniformly bounded and they belong to a compact set with respect to the product topology. This implies the existence of an optimal path $(c_t, k_t, N_t)_{t=0}^{\infty}$ (see also the proof of Proposition 1 in Bosi and Ha-Huy (2023)). ■

Proof of Proposition 6

At the steady state, system (22)-(24) simplifies:

$$1 = \beta \left(1 - \delta + \alpha A k^{\alpha-1}\right) \quad (65)$$

$$c = A k^{\alpha} - \delta k \quad (66)$$

$$N = a N^{\varepsilon} - b A k^{\alpha} \quad (67)$$

Solving (65)-(66), we recover the Modified Golden Rule (26)-(27). Equation (67) yields equation (28).

Let us write equation (28) as

$$h(N) = \Pi + N \quad (68)$$

with $h(N) \equiv a N^{\varepsilon}$. Since $\varepsilon \in (0,1)$, h is a strictly concave function with $h'(0^+) = +\infty$ and $h'(+\infty) = 0$. The LHS and the RHS of (68) have the same slope at $N = N^*$ solution to $h'(N) = 1$, that is at

$$N^* \equiv (a\varepsilon)^{\frac{1}{1-\varepsilon}}$$

Thus, $h(N)$ crosses the line $\Pi + N$ if and only if $h(N^*) \geq \Pi + N^*$ or, equivalently,

$$\Pi \leq a(1-\varepsilon)(a\varepsilon)^{\frac{\varepsilon}{1-\varepsilon}}$$

that is $A \leq \bar{A}$.

Clearly, if $A < \bar{A}$, then $h(N)$ crosses the line $\Pi + N$ twice at $N_1, N_2 > 0$; if $A = \bar{A}$, then the line $\Pi + N$ is tangent to $h(N)$ and the two steady states coalesce in a single point: $N_1 = N_2 > 0$. If $A > \bar{A}$, then $h(N)$ and the line $\Pi + N$ have no intersection. ■

Proof of Lemma 8

We linearize system (22)-(24) around an arbitrary steady state:

$$\begin{aligned}
-\frac{\eta}{C}\frac{dk_{t+1}}{k}+\frac{\varepsilon_{cc}}{\varepsilon_{cN}}\frac{dc_{t+1}}{c}+\frac{dN_{t+1}}{N} &= \frac{\varepsilon_{cc}}{\varepsilon_{cN}}\frac{dc_t}{c}+\frac{dN_t}{N}\\
\frac{dk_{t+1}}{k} &= \frac{1}{\beta}\frac{dk_t}{k}-C\frac{dc_t}{c}\\
\frac{dN_{t+1}}{N} &= -B\frac{dk_t}{k}+E\frac{dN_t}{N}
\end{aligned}$$

More compactly, we obtain (32). ■

Proof of Lemma 9

The trace, the sum of principal minors of order two and the determinant of the Jacobian matrix (33) are given by:

$$T = 1+\frac{1}{\beta}+E-\eta\frac{\varepsilon_{cN}}{\varepsilon_{cc}} \tag{69}$$

$$S = \frac{1}{\beta}+BC\frac{\varepsilon_{cN}}{\varepsilon_{cc}}+E\left(1+\frac{1}{\beta}-\eta\frac{\varepsilon_{cN}}{\varepsilon_{cc}}\right) \tag{70}$$

$$D = \frac{1}{\beta}E+BC\frac{\varepsilon_{cN}}{\varepsilon_{cc}} \tag{71}$$

Replacing in $P(\lambda)=\lambda^3-T\lambda^2+S\lambda-D$ and evaluating at -1, 0 and 1, we get expressions (34), (35) and (36). ■

Proof of Lemma 10

In the CSES case $\varepsilon_{cN}/\varepsilon_{cc}=-1$ and the trace, the sum of principal minors of order two and the determinant (69), (70) and (71) become

$$T = 1+\frac{1}{\beta}+E+\eta \tag{72}$$

$$S = \frac{1}{\beta}+E+\eta E+\frac{E}{\beta}-BC \tag{73}$$

$$D = \frac{E}{\beta}-BC \tag{74}$$

The values $P(-1)$, $P(0)$ and $P(1)$ are given by (34), (35) and (36), and become expressions (37), (38) and (39). ■

Proof of Proposition 11

We observe that

$$\begin{aligned}
P(-1) &= 2\left[BC-(1+E)\left(1+\frac{1}{\beta}\right)\right]-(1+E)\eta\\
&= 2\frac{1+\beta}{\beta}\left[\frac{B}{\alpha}(\varepsilon^*-\varepsilon)-(1+\varepsilon)\right]-(1+E)\eta
\end{aligned}$$

Thus, if $\varepsilon>\varepsilon^*$, then $P(-1)<0$ for any $\eta>0$ (no flip bifurcation).

Let $\varepsilon < \varepsilon^*$. Define

$$\begin{aligned} E(N) &\equiv \varepsilon + \frac{\varepsilon}{\alpha} B(N) \\ P_{-1}(N,\eta) &\equiv 2\frac{1+\beta}{\beta}\left[\frac{B(N)}{\alpha}(\varepsilon^* - \varepsilon) - (1+\varepsilon)\right] - [1 + E(N)]\eta \end{aligned}$$

and reconsider

$$N^* = (a\varepsilon)^{\frac{1}{1-\varepsilon}} \in (N_1, N_2)$$

because of Assumption 2. Clearly,

$$\begin{aligned} B(N_1) &> B(N^*) > B(N_2) \\ P_{-1}(N_1, 0) &> P_{-1}(N^*, 0) > P_{-1}(N_2, 0) \end{aligned}$$

When $P_{-1}(N^*, 0)$ is sufficiently close to 0, that is

$$B(N^*) \sim \alpha \frac{1+\varepsilon}{\varepsilon^* - \varepsilon}$$

or, equivalently, $b \sim b^*$, we have

$$P_{-1}(N_1, 0) > 0 > P_{-1}(N_2, 0)$$

Since N_1 does not depend on η and $P_{-1}(N_1, \eta)$ decreases continuously with η from $P_{-1}(N_1, 0)$ to $-\infty$, there exists a critical value η_F such that $P_{-1}(N_1, \eta_F) = 0$, that is (40). Since $B(N_1) > B(N^*)$, this value is positive. ■

Proof of Proposition 12

Without loss of generality, let λ_1 be a real eigenvalue and λ_2 and λ_3 be nonreal (conjugate) eigenvalues.

A Neimark-Sacker bifurcation generically occurs when the modulus of these nonreal eigenvalues crosses one, that is $\lambda_2\lambda_3 = 1$. Accordingly, we recompute (T, S, D):

$$\begin{aligned} D &= \lambda_1\lambda_2\lambda_3 = \lambda_1 && (75) \\ S &= \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = \lambda_1(\lambda_2 + \lambda_3) + 1 && (76) \\ &= \lambda_1(T - \lambda_1) + 1 = D(T - D) + 1 && (77) \end{aligned}$$

Therefore, a Neimark-Sacker bifurcation occurs at η_N solution to $S = D(T-D)+1$ or, equivalently, according to expressions (72)-(74):

$$\frac{1}{\beta} + E + \eta E + \frac{E}{\beta} - BC = \left(\frac{E}{\beta} - BC\right)\left(1 + \frac{1}{\beta} + E + \eta - \frac{E}{\beta} + BC\right) + 1$$

Solving for η, we find (41).

Moreover, the eigenvalues λ_2 and λ_3 have to be nonreal. Let us compute them. Noticing that

$$\begin{aligned} \lambda_2 + \lambda_3 &= T - D \\ \lambda_2 + \frac{1}{\lambda_2} &= T - D \\ \lambda_2^2 - (T - D)\lambda_2 + 1 &= 0 \end{aligned}$$

we get

$$\lambda_2 = \frac{T-D}{2} - \sqrt{\left(\frac{T-D}{2}\right)^2 - 1}$$
$$\lambda_3 = \frac{T-D}{2} + \sqrt{\left(\frac{T-D}{2}\right)^2 - 1}$$

These values are nonreal if and only if $|T-D| < 2$ or, equivalently, (42) holds.

Let us prove that the Neimark-Sacker bifurcation can arise only around N_1. At the steady state, $bAk^\alpha = aN^\varepsilon - N$ and

$$\frac{k}{N} = \frac{aN^{\varepsilon-1} - 1}{bAk^{\alpha-1}}$$

Since, according to the proof of Proposition 6, $N_1 < N^* < N_2$ and $\varepsilon < 1$, we have

$$\frac{k}{N_1} = \frac{aN_1^{\varepsilon-1} - 1}{bAk^{\alpha-1}} > \frac{aN^{*\varepsilon-1} - 1}{bAk^{\alpha-1}} > \frac{aN_2^{\varepsilon-1} - 1}{bAk^{\alpha-1}} = \frac{k}{N_2}$$

Replacing

$$N^* \equiv (a\varepsilon)^{\frac{1}{1-\varepsilon}}$$
$$k = (\alpha\gamma A)^{\frac{1}{1-\alpha}}$$

we find

$$\frac{k}{N_1} > \alpha\frac{\gamma}{b}\frac{1-\varepsilon}{\varepsilon} > \frac{k}{N_2}$$

and, according to (30),

$$E_1 \equiv \varepsilon + \frac{\varepsilon}{\alpha}\frac{b}{\gamma}\frac{k}{N_1} > \varepsilon + \frac{\varepsilon}{\alpha}\frac{b}{\gamma}\left(\alpha\frac{\gamma}{b}\frac{1-\varepsilon}{\varepsilon}\right) > \varepsilon + \frac{\varepsilon}{\alpha}\frac{b}{\gamma}\frac{k}{N_2} \equiv E_2$$

that is $E_1 > 1 > E_2$. Since $\eta_N > 0$ and

$$(E_2 - 1)\frac{1-\beta}{\beta} - BC < 0$$

the inequality on the RHS of (42) is violated, meaning that a Neimark-Sacker bifurcation never arises around N_2. ■

Proof of Proposition 13

The dynamic system has two predetermined variables, k_t and N_t, and one non-predetermined variable, c_t. When the NS bifurcation occurs two nonreal (conjugate) eigenvalues cross the unit circle, say, without loss of generality, λ_2 and λ_3. At the critical bifurcation point, we have $\lambda_2\lambda_3 = 1$ and, therefore, $D = \lambda_1\lambda_2\lambda_3 = \lambda_1$, a real eigenvalue.

Let us show that $|\lambda_1| > 1$. In this case, the equilibrium is locally determinate.

According to (42), since $\eta_N > 0$, a necessary condition for a NS bifurcation is

$$(E-1)\frac{1-\beta}{\beta} - BC > 0 \tag{78}$$

or, equivalently, according to (74),

$$D > \frac{1}{\beta} + E - 1 \tag{79}$$

We know from the proof of Proposition 12 that $E > 1$ at $N = N_1$. Therefore, (79) implies $D > 1/\beta > 1$, that is $\lambda_1 > 1$. ■

Proof of Lemma 15

Clearly, maximizing $\sum_{t=0}^{\infty} \beta^t v\left(g\left(c_t, N_t\right)\right)$ is equivalent to maximizing

$$p \sum_{t=0}^{\infty} \beta^t v\left(g\left(c_t, N_t\right)\right) = \sum_{t=0}^{\infty} \beta^t \left[pv\left(g\left(c_t, N_t\right)\right)\right]$$

where $p \equiv 1 + \rho \in (1, \infty)$ is a positive constant.

According to (7) and (9),

$$\omega = \frac{1-\pi}{\varphi - \pi}$$

We observe that

$$1 - \frac{1}{\omega} = (1+\rho)(1-\varphi) \tag{80}$$

and that

$$pv\left(g\left(c_t, N_t\right)\right) = (1+\rho)\frac{g\left(c_t, N_t\right)^{1-\frac{1}{\omega}}}{1-\frac{1}{\omega}} = (1+\rho)\frac{\left(c_t^{1-\pi} N_t^{\pi}\right)^{1-\frac{1}{\omega}}}{1-\frac{1}{\omega}} \tag{81}$$

Replacing (9) and (80) in (81), we obtain

$$pv\left(g\left(c_t, N_t\right)\right) = \frac{\left(c_t N_t^{\rho}\right)^{1-\varphi}}{1-\varphi} \equiv u\left(c_t, N_t\right)$$

■

Proof of Lemma 16

The Jacobian matrix (33) becomes

$$J \equiv \begin{bmatrix} \frac{1}{\beta} & -B_1 & 0 \\ -B_2 - B_3 B_4 & 1 + \beta B_1 B_2 & B_3\left(B_5 - 1\right) \\ -B_4 & 0 & B_5 \end{bmatrix}$$

where

$$B_1 \equiv \frac{1}{\alpha\gamma} - \delta,\ B_2 \equiv \frac{1-\alpha}{\gamma\varphi},\ B_3 \equiv \rho\frac{1-\varphi}{\varphi},\ B_4 \equiv \frac{b}{\gamma}\frac{k}{N} \text{ and } B_5 \equiv \varepsilon + \frac{\varepsilon}{\alpha}\frac{b}{\gamma}\frac{k}{N}$$

The trace, the sum of the principal minors of order two and the determinant are given by

$$\begin{aligned} T &= \frac{1+\beta}{\beta} + \beta B_1 B_2 + B_5 \\ S &= \left(\frac{1+\beta}{\beta} + \beta B_1 B_2\right) B_5 + \frac{1}{\beta} - B_1 B_3 B_4 \\ D &= \frac{1}{\beta} B_5 - B_1 B_3 B_4 \end{aligned}$$

Replacing $A_1 \equiv \beta B_1 B_2$ and $A_2 \equiv B_1 B_3 B_4 / \rho$, we obtain

$$T = \frac{1+\beta}{\beta} + A_1 + B_5 > 2 \quad (82)$$

$$S = \left(\frac{1+\beta}{\beta} + A_1\right) B_5 + \frac{1}{\beta} - \rho A_2 \quad (83)$$

$$D = \frac{1}{\beta} B_5 - \rho A_2 \quad (84)$$

We know that $P(\lambda) = \lambda^3 - T\lambda^2 + S\lambda - D$. Then, $P(-1) = -1 - T - S - D$, $P(0) = -D$, $P(1) = 1 - T + S - D$. Replacing (82), (83) and (84), we find (45), (46) and (47). ■

Proof of Proposition 17

Let

$$B_5(N_i) \equiv \varepsilon + \frac{\varepsilon}{\alpha} \frac{b}{\gamma} \frac{k}{N_i} \quad (85)$$

We rewrite (24) at the steady state (N_1 or N_2) as follows:

$$\frac{k}{N} = \frac{aN^{\varepsilon-1} - 1}{bAk^{\alpha-1}}$$

Since, according to the proof of Proposition 6, $N_1 < N^* < N_2$ and $\varepsilon < 1$, we have

$$\frac{k}{N_1} = \frac{aN_1^{\varepsilon-1} - 1}{bAk^{\alpha-1}} > \frac{aN^{*\varepsilon-1} - 1}{bAk^{\alpha-1}} > \frac{aN_2^{\varepsilon-1} - 1}{bAk^{\alpha-1}} = \frac{k}{N_2}$$

Replacing

$$\begin{aligned} N^* &\equiv (a\varepsilon)^{\frac{1}{1-\varepsilon}} \\ k &= (\alpha\gamma A)^{\frac{1}{1-\alpha}} \end{aligned}$$

we find

$$\frac{k}{N_1} > \alpha \frac{\gamma}{b} \frac{1-\varepsilon}{\varepsilon} > \frac{k}{N_2}$$

that is

$$B(N_1) = \frac{b}{\gamma} \frac{k}{N_1} > \alpha \frac{1-\varepsilon}{\varepsilon} > \frac{b}{\gamma} \frac{k}{N_2} = B(N_2)$$

Using (85), we have

$$B_5(N_1) = \varepsilon + \frac{\varepsilon}{\alpha}\frac{b}{\gamma}\frac{k}{N_1} > \varepsilon + \frac{\varepsilon}{\alpha}\left(\alpha\frac{1-\varepsilon}{\varepsilon}\right) > \varepsilon + \frac{\varepsilon}{\alpha}\frac{b}{\gamma}\frac{k}{N_2} = B_5(N_2)$$

that is $B_5(N_1) > 1 > B_5(N_2)$.

If $\varphi < 1$, then $A_2 > 0$ and $\rho_0 < \rho_F$. Then, if $\rho_0 < \rho < \rho_F$, we have $P(-1) < 0 < P(0)$. Moreover, $B_5 < 1$ implies $P(1) < 0$. Since $B_5(N_2) < 1$, the equilibrium is unique (locally determinate) because there are two eigenvalues inside the unit circle with two predetermined variables (k_t and N_t). ■

Proof of Proposition 18

According to Proposition 6, when $A < \bar{A}$, there are two steady states (k, c, N_1) and (k, c, N_2) with $N_1 < N_2$, while, when $A > \bar{A}$, there are no longer steady states. Therefore, when A crosses $\bar{A}$ from below, the two steady states collide ($N_1 = N_2 = N^*$) and disappears. At $A = \bar{A}$, we have $B_5 = 1$ or, equivalently, $P(1) = 0$, and the economy generically experiences a saddle-node bifurcation. ■

Proof of Proposition 19

The bifurcation values ρ_{Fi} are positive if and only if $A_2 > 0$, that is $\varphi < 1$ entailing $\varepsilon_{cN} > 0$. We observe that the flip bifurcation value associated to N_1 is higher than the one associated to N_2. Indeed,

$$\rho_{F1} \equiv \frac{1 + B_5(N_1)}{A_2}\left(\frac{A_1}{2} + \frac{1+\beta}{\beta}\right) > \frac{1 + B_5(N_2)}{A_2}\left(\frac{A_1}{2} + \frac{1+\beta}{\beta}\right) \equiv \rho_{F2}$$

because $B_5(N_1) > 1 > B_5(N_2)$. ■

Proof of Proposition 21

Without loss of generality, let λ_1 be a real eigenvalue and λ_2 and λ_3 be nonreal (conjugate) eigenvalues.

A Neimark-Sacker bifurcation generically arise when the modulus of these nonreal eigenvalues crosses one, that is $\lambda_2\lambda_3 = 1$. According to expressions (75), (76) and (77), a Neimark-Sacker bifurcation occurs at η_N solution to $S = D(T - D) + 1$ or, equivalently, according to expressions (82)-(84):

$$D^2 - (T - 1)D + \tilde{S} - 1 = 0 \tag{86}$$

where

$$\tilde{S} \equiv S - D = \frac{1}{\beta} + B_5(1 + A_1)$$

and T do not depend on ρ.

Solutions to (86) are

$$\begin{aligned} D_N^- &\equiv \frac{1}{2}\left[T - 1 - \sqrt{(T-3)^2 + 4A_1(1 - B_5)}\right] \\ D_N^+ &\equiv \frac{1}{2}\left[T - 1 + \sqrt{(T-3)^2 + 4A_1(1 - B_5)}\right] \end{aligned}$$

where the RHS does not depend on ρ.

Since $A_1 > 0$, these values are real if and only if

$$B_5 < 1 + \frac{1}{A_1}\left(\frac{T-3}{2}\right)^2 \tag{87}$$

Noticing that

$$D_N = \frac{1}{\beta}B_5 - \rho_N A_2$$

we define the critical values

$$\begin{aligned}\rho_N^- &\equiv \frac{1}{\beta}\frac{B_5}{A_2} - \frac{D_N^-}{A_2}\\ \rho_N^+ &\equiv \frac{1}{\beta}\frac{B_5}{A_2} - \frac{D_N^+}{A_2}\end{aligned}$$

Moreover, the eigenvalues λ_2 and λ_3 have to be nonreal. Let us compute them. Noticing that

$$\begin{aligned}\lambda_2 + \lambda_3 &= T - D\\ \lambda_2 + \frac{1}{\lambda_2} &= T - D\\ \lambda_2^2 - (T-D)\,\lambda_2 + 1 &= 0\end{aligned}$$

we get

$$\begin{aligned}\lambda_2 &= \frac{T-D}{2} - \sqrt{\left(\frac{T-D}{2}\right)^2 - 1}\\ \lambda_3 &= \frac{T-D}{2} + \sqrt{\left(\frac{T-D}{2}\right)^2 - 1}\end{aligned}$$

These values are nonreal if and only if $|T-D| < 2$. Thus, we require

$$-2 < T - D_N < 2 \tag{88}$$

(1) Let us show that D_N^- violates (88) in both the steady states N_1 and N_2, that is there is no room for limit cycles at $\rho = \rho_N^-$ through a NS bifurcation whatever the steady state.

Under (87), since $T > 2$, the LHS inequality of (88) is satisfied:

$$T - D_N^- = \frac{1}{2}\left[1 + T + \sqrt{(T-3)^2 + 4A_1(1-B_5)}\right] > 0 > -2$$

Focus on the RHS. $T - D_N^- < 2$ if and only if

$$\sqrt{(T-3)^2 + 4A_1(1-B_5)} < 3 - T$$

that is, since $A_1 > 0$, if and only if

$$T < 3 \text{ and } B_5 > 1 \tag{89}$$

(1.1) N_1 violates $T < 3$. Indeed, since $A_1 > 0$ and $B_5(N_1) > 1$, according to (82),

$$T = \frac{1+\beta}{\beta} + A_1 + B_5(N_1) > \frac{1+\beta}{\beta} + B_5(N_1) > 2 + 1 = 3 \tag{90}$$

(1.2) N_2 violates $B_5 > 1$. Indeed, $B_5(N_2) < 1$.

Therefore, N_1 and N_2, both violate (89), that is (88).

(2) Focus now on D_N^+ and consider restrictions (87) and (88).

We observe that $T - D_N^+ > -2$ if and only if

$$\sqrt{(T-3)^2 + 4A_1(1-B_5)} < T + 5$$

that is, since $T > 0$, if and only if $(T-3)^2 + 4A_1(1-B_5) < (T+5)^2$, that is $4 + 4T - A_1 + A_1B_5 > 0$ or, equivalently, according to (82), $4 + \beta(8 + 3A_1 + 4B_5 + A_1B_5) > 0$, which is always true since $A_1, B_5 > 0$.

We observe that $T - D_N^+ < 2$ if and only if

$$T - 3 < \sqrt{(T-3)^2 + 4A_1(1-B_5)}$$

that is, under (87), if and only if

$$T < 3 \text{ or } (T > 3 \text{ and } B_5 < 1) \tag{91}$$

(2.1) Consider N_1. According to (90), $T > 3$ and $B_5(N_1) > 1$. Then, N_1 violates (91), that is restrictions (88): there is no room for a NS bifurcation around N_1 at $\rho = \rho_N^+$.

(2.2) Consider N_2. We have $B_5(N_2) < 1$. Thus, both restrictions (87) and (91) are satisfied. Then, generically, a limit cycle arise around N_2 at $\rho = \rho_N^+$. ■

Proof of Proposition 22

As seen in the proof of Proposition 13, the dynamic system has two predetermined variables, k_t and N_t, and one non-predetermined variable, c_t. When the NS bifurcation occurs two nonreal (conjugate) eigenvalues cross the unit circle, say, without loss of generality, λ_2 and λ_3. At the critical bifurcation point, we have $\lambda_2\lambda_3 = 1$ and, therefore, $D = \lambda_1\lambda_2\lambda_3 = \lambda_1$, a real eigenvalue.

Let us show that $|\lambda_1| > 1$. In this case, the equilibrium is locally determinate.

We observe that $B_5 < 1$ when $N = N_2$ (see the proof of Proposition 21). Then

$$T - 3 + \sqrt{(T-3)^2 + 4A_1(1-B_5)} > 0$$

and, according to (49), $\lambda_1 = D_N^+ > 1$. ■

Proof of Proposition 23

At the steady state, respectively, equations (50)-(51) give:

$$\begin{aligned} Ak^{\alpha-1} &= \frac{1}{\alpha\gamma} && (92) \\ c &= \left(Ak^{\alpha-1}-\delta\right)k && (93) \end{aligned}$$

Solving (92) for k and replacing the LHS of (92) in the RHS of (93), we obtain (26) and (27).

(52) at the steady state is equivalent to $\varphi(N)=0$.

We observe that

$$\begin{aligned} \varphi(0) &= -\frac{bAk^{\alpha}}{\bar{N}}<0 \\ \varphi\left(\bar{N}^{-}\right) &= -\infty \\ \varphi'(N) &= a\varepsilon N^{\varepsilon-1}-\frac{bAk^{\alpha}}{\left(\bar{N}-N\right)^2} \\ \varphi''(N) &= a\varepsilon(\varepsilon-1)N^{\varepsilon-2}-\frac{2bAk^{\alpha}}{\left(\bar{N}-N\right)^3}<0 \end{aligned}$$

because $0<\varepsilon\leq 1$.

Let N^* be the unique solution to $\varphi'(N)=0$, that is to

$$a\varepsilon N^{\varepsilon-1}=\frac{bAk^{\alpha}}{\left(\bar{N}-N\right)^2}$$

provided that $\varphi'(0^+)>0$. This is always true if $\varepsilon<1$. If $\varepsilon=1$, we require

$$a\varepsilon>\frac{bAk^{\alpha}}{\bar{N}^2}$$

that is (53).

Thus, $\varphi(N)=0$ has no solution if $\varphi(N^*)<0$, one solution N^* if $\varphi(N^*)=0$, two solutions $N_1<N^*<N_2$ if $\varphi(N^*)>0$.

We know that $\varphi''(N)<0$. In the case of two steady states, we have

$$\varphi'(N_1)>\varphi'(N^*)=0>\varphi'(N_2)$$

Focus on N_1. Then,

$$\varphi'(N_1)=aN_1^{\varepsilon}\frac{\varepsilon}{N_1}-\frac{bAk^{\alpha}}{\left(\bar{N}-N_1\right)^2}>0 \tag{94}$$

Since $\varphi(N_1)=0$, we obtain

$$aN_1^{\varepsilon}=\frac{bAk^{\alpha}}{\bar{N}-N_1}$$

and, replacing in (94),

$$N_1<\frac{\varepsilon}{1+\varepsilon}\bar{N}$$

Similarly, we get

$$N_2 > \frac{\varepsilon}{1+\varepsilon}\bar{N}$$

■

Proof of Proposition 24

We linearize system (50)-(52) around an arbitrary steady state:

$$\begin{aligned}
-\frac{\eta}{C}\frac{dk_{t+1}}{k}+\frac{\varepsilon_{cc}}{\varepsilon_{cN}}\frac{dc_{t+1}}{c}+\frac{dN_{t+1}}{N} &= \frac{\varepsilon_{cc}}{\varepsilon_{cN}}\frac{dc_t}{c}+\frac{dN_t}{N} \\
\frac{dk_{t+1}}{k} &= \frac{1}{\beta}\frac{dk_t}{k}-C\frac{dc_t}{c} \\
\frac{dN_{t+1}}{N} &= -B\frac{dk_t}{k}+\tilde{E}\frac{dN_t}{N}
\end{aligned}$$

More compactly, we obtain (33) with, now, $\tilde{E}$ instead of E. ■

Proof of Lemma 26

According to Lemma 24, the Jacobian matrix is given by (33) with $\tilde{E}$ instead of E. Moreover, as in the proof of Lemma 10, $\varepsilon_{cN}/\varepsilon_{cc}=-1$ implies that the values of $P(\lambda)$ at -1, 0 and 1 are precisely given by (37), (38) and (39) with, now, $\tilde{E}$ instead of E. ■

Proof of Proposition 28

The flip bifurcation value η_{Fi} is solution to:

$$P(-1)=2B(N_i)C-\left[1+\tilde{E}(N_i)\right]\left(\eta+2\frac{1+\beta}{\beta}\right)=0$$

(57) implies $\eta_{Fi}>0$. ■

Proof of Proposition 30

Without loss of generality, let λ_1 be a real eigenvalue and λ_2 and λ_3 be nonreal (conjugate) eigenvalues.

A Neimark-Sacker bifurcation generically arise when the modulus of these nonreal eigenvalues crosses one, that is $\lambda_2\lambda_3=1$. According to expressions (75), (76) and (77), a Neimark-Sacker bifurcation occurs at η_N solution to $S=D(T-D)+1$ or, equivalently, according to expressions (72)-(74) with $\tilde{E}$ instead of E:

$$\frac{1}{\beta}+\tilde{E}+\eta\tilde{E}+\frac{\tilde{E}}{\beta}-BC=\left(\frac{\tilde{E}}{\beta}-BC\right)\left(1+\frac{1}{\beta}+\tilde{E}+\eta-\frac{\tilde{E}}{\beta}+BC\right)+1$$

Solving for η, we find (58).

$$\eta_N\equiv\frac{\tilde{E}}{\beta}-BC+\frac{\frac{\tilde{E}}{\beta}-BC-1+\beta-\beta\tilde{E}}{\beta BC-(1-\beta)\tilde{E}}$$

Moreover, the eigenvalues λ_2 and λ_3 have to be nonreal. Let us compute them. Noticing that

$$\begin{aligned}\lambda_2 + \lambda_3 &= T - D \\ \lambda_2 + \frac{1}{\lambda_2} &= T - D \\ \lambda_2^2 - (T - D)\lambda_2 + 1 &= 0\end{aligned}$$

we get

$$\begin{aligned}\lambda_2 &= \frac{T-D}{2} - \sqrt{\left(\frac{T-D}{2}\right)^2 - 1} \\ \lambda_3 &= \frac{T-D}{2} + \sqrt{\left(\frac{T-D}{2}\right)^2 - 1}\end{aligned}$$

These values are nonreal if and only if $|T - D| < 2$ or, equivalently, (60) holds.

Let us prove that the Neimark-Sacker bifurcation can arise only around N_1.

Consider a parametric configuration such that

$$\frac{N^*}{\bar{N} - N^*} = \varepsilon$$

that is (59) holds.

Notice that, when $N_1 = N^* = N_2$, it is the case, because

$$\frac{N_1}{\bar{N} - N_1} = \frac{aN_1^{1+\varepsilon}}{bAk^\alpha} < \frac{aN^{*1+\varepsilon}}{bAk^\alpha} = \frac{1}{\varepsilon}\left(\frac{N^*}{\bar{N} - N^*}\right)^2 < \frac{N_2}{\bar{N} - N_2} = \frac{aN_2^{1+\varepsilon}}{bAk^\alpha}$$

since $\varphi(N_i) = 0$ and $\varphi(N^*) = 0$ with $0 \leq N_1 \leq N^* \leq N_2$.

In this case,

$$\frac{N_1}{\bar{N} - N_1} < \frac{N^*}{\bar{N} - N^*} = \varepsilon < \frac{N_2}{\bar{N} - N_2}$$

Let

$$\tilde{E}_i - 1 = \frac{B}{\alpha}\left(\varepsilon - \frac{N_i}{\bar{N} - N_i}\right)$$

Then, $\tilde{E}_1 > 1 > \tilde{E}_2$. Since $\eta_N > 0$ and

$$\left(\tilde{E}_2 - 1\right)\frac{1-\beta}{\beta} - BC < 0$$

the inequality on the RHS of (60) is violated, meaning that a Neimark-Sacker bifurcation never arises around N_2. ■

Proof of Proposition 31

The dynamic system has two predetermined variables, k_t and N_t, and one non-predetermined variable, c_t. When the NS bifurcation occurs two nonreal (conjugate) eigenvalues cross the unit circle, say, without loss of generality, λ_2

and λ_3. At the critical bifurcation point, we have $\lambda_2\lambda_3 = 1$ and, therefore, $D = \lambda_1\lambda_2\lambda_3 = \lambda_1$, a real eigenvalue.

Let us show that $|\lambda_1| > 1$. In this case, the equilibrium is locally determinate.

According to (60), since $\eta_N > 0$, a necessary condition for a NS bifurcation is

$$\left(\tilde{E} - 1\right) \frac{1-\beta}{\beta} - BC > 0$$

or, equivalently, according to (74) with $\tilde{E}$ instead of E,

$$D > \frac{1}{\beta} + \tilde{E} - 1 \tag{95}$$

We know that, under the Assumptions of Proposition 30 that $\tilde{E} > 1$ at $N = N_1$. Therefore, (95) implies $D > 1/\beta > 1$, that is $\lambda_1 > 1$. ■

Proof of Lemma 32

Reconsider the utility function (8) with $\rho > 0$ and $\varphi > \rho/(1+\rho)$. According to (10) and (11), we obtain $\varepsilon_{cN}/\varepsilon_{cc} = \rho(1 - 1/\varphi)$. Replacing in (69), (70) and (71) with $\tilde{E}$ instead of E, we find

$$\begin{aligned} T &= 1 + \frac{1}{\beta} + \tilde{E} + \rho\eta\frac{1-\varphi}{\varphi} \\ S &= \frac{1}{\beta} - \rho BC\frac{1-\varphi}{\varphi} + \tilde{E}\left(1 + \frac{1}{\beta} + \rho\eta\frac{1-\varphi}{\varphi}\right) \\ D &= \frac{1}{\beta}\tilde{E} - \rho BC\frac{1-\varphi}{\varphi} \end{aligned}$$

Using expressions (30) for B and C, and expression (31) for η, we get expressions (82), (83) and (84) for the trace, the sum of principal minors of order two and the determinant where expressions A_1 and A_2 are precisely given by (43) and (44) but, now, (61) replace B_5.

Focus on the sign of

$$\tilde{B}_5 - 1 \equiv \frac{1}{\alpha}\left(\varepsilon - \frac{N}{\bar{N} - N}\right)\frac{b}{\gamma}\frac{k}{N} \tag{96}$$

that is of

$$\varepsilon - \frac{N}{\bar{N} - N}$$

at N_1 and N_2.

Consider the regeneration process of nature (52). At the steady state, $\omega(N) \equiv aN^{\varepsilon}\left(\bar{N} - N\right) = bAk^{\alpha}$. ω is a bell-shaped function. We have shown that $N_1 < N^* < N_2$, where N^* is solution to $\varphi'(N) = 0$. Since k does not depend on N, N_1 is located on the upward-sloping branch and N_2 on the downward-sloping branch of ω.

We observe that

$$\omega'(N) \equiv a\varepsilon N^{\varepsilon-1}\left(\bar{N} - N\right) - aN^{\varepsilon}$$

and $\omega'(N) > 0$ if and only if

$$\varepsilon > \frac{N}{\bar{N} - N}$$

Put is differently,

$$\frac{N_1}{\bar{N} - N_1} < \varepsilon < \frac{N_2}{\bar{N} - N_2} \tag{97}$$

Therefore, according to (96), $\tilde{B}_5(N_1) - 1 > 0$ and $\tilde{B}_5(N_2) - 1 < 0$. ■

Proof of Proposition 33

When a saddle-node bifurcation takes place $N_1 = N_2 = N^*$. Then, according to Lemma 32, $\tilde{B}_5(N_1) = \tilde{B}_5(N_2) = 1$ and

$$\begin{aligned} P(-1) &= 2\left(\rho A_2 - A_1 - 2\frac{1+\beta}{\beta}\right) \\ P(0) &= \rho A_2 - \frac{1}{\beta} \\ P(1) &= 0 \end{aligned}$$

We notice also that $A_1, A_2 > 0$ and they don't depend on ρ. Inequalities (62) are equivalent to $P(-1) < 0$ and $P(0) > 0$. Moreover, when N_1 and N_2 are distinct but in a neighborhood of N^*,

$$P(1)|_{N=N_2} = A_1\left[\tilde{B}_5(N_2) - 1\right] < 0 < A_1\left[\tilde{B}_5(N_1) - 1\right] = P(1)|_{N=N_1}$$

By continuity, when N_1 and N_2 are close enough, the corresponding characteristic polynomials are also close, they cross the vertical line $\lambda = -1$ below zero and the vertical line $\lambda = 0$ above zero. Moreover, $P(\lambda)|_{N=N_2}$ crosses the vertical line $\lambda = 1$ below zero, while $P(\lambda)|_{N=N_1}$, above. Since $\lim_{\lambda\to-\infty} P(\lambda) = -\infty$ and $\lim_{\lambda\to\infty} P(\lambda) = \infty$, we have that the curve $P(\lambda)|_{N=N_2}$ crosses once each interval $(-1, 0)$, $(0, 1)$ and $(1, \infty)$, while the curve $P(\lambda)|_{N=N_1}$ crosses once the interval $(-1, 0)$ and generically twice the interval $(1, \infty)$. The proposition follows. ■

Proof of Proposition 34

We introduce the following critical value:

$$\rho_F \equiv \frac{1 + \tilde{B}_5}{2A_2}\left(A_1 + 2\frac{1+\beta}{\beta}\right)$$

solution to $P(-1) = 0$. If $\varphi < 1$, then $A_2 > 0$ and we have $P(-1) > 0$ if and only if $\rho > \rho_F$.

We can specify ρ_F in the two steady states:

$$\rho_F(N_i) \equiv \left[2\frac{\gamma}{b}\frac{N_i}{k} + \frac{1}{\alpha}\left(\varepsilon - \frac{N_i}{\bar{N} - N_i}\right)\right]\frac{A_1 + 2\frac{1+\beta}{\beta}}{2\left(\frac{1}{\alpha\gamma} - \delta\right)\frac{1-\varphi}{\varphi}}$$

with $i = 1, 2$.

The ratio on the right is positive and does not depend on N.

According to (97), we have $\rho_F(N_1) > 0$, but $\rho_F(N_2)$ can be negative. ■

Proof of Proposition 35

According to the proof of Proposition 12, a NS bifurcation generically occurs if and only if

$$S = D(T - D) + 1 \tag{98}$$

$$|T - D| < 2 \tag{99}$$

We observe that $S = D + T + \left(\tilde{B}_5 - 1\right) A_1 - 1$.

(98) becomes:

$$D(T - D) + 1 = D + T + \left(\tilde{B}_5 - 1\right) A_1 - 1 \tag{100}$$

As before, let us choose ρ as bifurcation parameter and remark that only D depends on ρ. Solving (100) for D gives:

$$\begin{aligned} D_- &\equiv \frac{1}{2}\left[T - 1 - \sqrt{(T-3)^2 + 4A_1\left(1 - \tilde{B}_5\right)}\right] \\ D_+ &\equiv \frac{1}{2}\left[T - 1 + \sqrt{(T-3)^2 + 4A_1\left(1 - \tilde{B}_5\right)}\right] \end{aligned}$$

The corresponding critical values for ρ are:

$$\begin{aligned} \rho_N^- &= \frac{1}{A_2}\left(\frac{\tilde{B}_5}{\beta} - D_-\right) \\ \rho_N^+ &= \frac{1}{A_2}\left(\frac{\tilde{B}_5}{\beta} - D_+\right) \end{aligned}$$

Notice that at $N = N_2$, we have $\tilde{B}_5 < 1$ and

$$(T-3)^2 + 4A_1\left(1 - \tilde{B}_5\right) > 0 \tag{101}$$

then, the two critical values ρ_N^- and ρ_N^+ are real. When $N = N_1$, $\tilde{B}_5 > 1$ and we require the additional restriction (101) holds.

Focus on condition (99), which is equivalent to $-2 < T - D < 2$.

$T - D_- < 2$ is equivalent to

$$T - 3 + \sqrt{(T-3)^2 + 4A_1\left(1 - \tilde{B}_5\right)} < 0 \tag{102}$$

A necessary condition is $4A_1\left(1 - \tilde{B}_5\right) < 0$, that is $\tilde{B}_5 > 1$. According to Lemma 32, this holds only when $N = N_1$. Thus, no NS bifurcation for $N = N_2$ at $\rho = \rho_N^-$.

Moreover, since $\tilde{B}_5(N_1) > 1$,

$$T(N_1) = \frac{1+\beta}{\beta} + A_1 + \tilde{B}_5(N_1) > 3$$

and, therefore,

$$T(N_1) - 3 + \sqrt{[T(N_1) - 3]^2 + 4A_1\left[1 - \tilde{B}_5(N_1)\right]} > 0$$

which violates (102) (notice that $\rho_N^- \in \mathbb{R}$ always requires $(T-3)^2 + 4A_1\left(1 - \tilde{B}_5\right) > 0$).

Thus, no NS bifurcation for $N = N_1$ at $\rho = \rho_N^-$.

Summing up, $\rho = \rho_N^-$ is not a NS bifurcation point.

Focus now on ρ_N^+. Condition (99) is equivalent to $-2 < T - D < 2$.

$T - D_+ < 2$ is equivalent to

$$(T-3) - \sqrt{(T-3)^2 + 4A_1\left(1 - \tilde{B}_5\right)} < 0 \tag{103}$$

If $\tilde{B}_5 > 1$, then

$$T = \frac{1+\beta}{\beta} + A_1 + \tilde{B}_5 > 3$$

and

$$(T-3) - \sqrt{(T-3)^2 + 4A_1\left(1 - \tilde{B}_5\right)} > 0$$

According to Lemma 32, ρ_N^+ cannot a NS bifurcation point for N_1.

If $\tilde{B}_5 < 1$, (103) always holds. Therefore, when $N = N_2$, $T - D_+ < 2$ holds.

$-2 < T - D_+$ is equivalent to

$$T + 5 > \sqrt{(T-3)^2 + 4A_1\left(1 - \tilde{B}_5\right)}$$

that is to

$$T + 5 > 0 \text{ and } 4(T+1) > A_1\left(1 - \tilde{B}_5\right)$$

or, equivalently, to (64) since $\tilde{B}_5 < 1$ and, hence, $T + 5 > T + 1 > 0$.

Therefore, if (64) holds, the system undergoes a NS bifurcation at $\rho = \rho_N^+$ and a limit cycle arises around the steady state N_2. ■

Proof of Corollary 36

We know from Proposition 35 that there are no limit cycles around N_1.

Consider N_2. When α is close to one, $A_1 \approx 0$, $T \approx 1 + 1/\beta + \tilde{B}_5$ and

$$D_+ \approx \frac{1}{2}(T - 1 + |T-3|)$$

Focus on the term $|T-3|$.

If $T-3<0$, then $D_+ \approx 1$ and

$$\rho_N^+ \approx \frac{1}{A_2}\left(\frac{\tilde{B}_5}{\beta}-1\right)$$

If $T-3>0$, then $D_+ \approx T-2$ and

$$\rho_N^+ \approx \frac{1-\beta}{\beta}\frac{\tilde{B}_5-1}{A_2}$$

We observe also that

$$T-3 \approx \frac{1+\beta}{\beta}+\tilde{B}_5-3<0 \Leftrightarrow \tilde{B}_5<2-\frac{1}{\beta}$$

Notice also that $2-1/\beta<\beta$.

(1) If $\tilde{B}_5(N_2)<2-1/\beta$, then $T-3<0$ and $\tilde{B}_5<\beta$, that is $\tilde{B}_5/\beta-1<0$ and

$$\rho_N^+(N_2) \approx \frac{1}{A_2}\left[\frac{\tilde{B}_5(N_2)}{\beta}-1\right]>0 \Leftrightarrow A_2<0 \Leftrightarrow \varphi>1$$

Thus, generically, limit cycles around N_2 through a NS bifurcation if $\varphi>1$ (negative cross effects). Conversely, no limit cycles around N_2 when $\varphi<1$ (positive cross effects).

(2) If $2-1/\beta<\tilde{B}_5(N_2)$, then $T-3>0$ and

$$\rho_N^+(N_2) \approx \frac{1-\beta}{\beta}\frac{\tilde{B}_5(N_2)-1}{A_2}>0 \Leftrightarrow A_2<0 \Leftrightarrow \varphi>1$$

since $\tilde{B}_5(N_2)<1$ (Lemma 32).

As above, generically, limit cycles around N_2 if $\varphi>1$ (negative cross effects). Conversely, no limit cycles when $\varphi<1$ (positive cross effects).

Summing up, we have the corollary. ■

Proof of Proposition 37

As seen in the proof of Proposition 22, the dynamic system has two predetermined variables, k_t and N_t, and one non-predetermined variable, c_t. When the NS bifurcation occurs two nonreal (conjugate) eigenvalues cross the unit circle, say, without loss of generality, λ_2 and λ_3. At the critical bifurcation point, we have $\lambda_2\lambda_3=1$ and, therefore, $D=\lambda_1\lambda_2\lambda_3=\lambda_1$, a real eigenvalue.

Therefore, the eigenvalue λ_1 associated to N_2 is equal to D_+ when $\rho=\rho_N^+$. Let us show that $|D|>1$ in this case, that is the equilibrium is locally determinate around N_2. Indeed, $D_+>1$ is equivalent to

$$T-3+\sqrt{(T-3)^2+4A_1\left(1-\tilde{B}_5\right)}>0$$

which always holds since $\tilde{B}_5<1$ when $N=N_2$. ■

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