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Allocating the common costs of a public service operator: an axiomatic approach

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Abstract

Accurate cost allocation is a challenge for both public service operators and regulatory bodies, given the dual objectives of ensuring essential public service provision and maintaining fair competition. Operators have the obligation to provide essential public services for all individuals, which may incur additional costs. To compensate this, the operators receive state aids, which are determined by an assessment of the net cost associated with these obligations. However, these aids introduce the risk of distorting competition, as operators may employ them to subsidize competitive activities. To avoid this risk, a precise cost allocation method that adequately assess the net cost of these obligations becomes necessary. Such a method must satisfy specific properties that effectively prevent cross-subsidization. In this paper, we propose a method grounded in cooperative game theory that offers a solution for allocating common costs between activities and obligations in public service provision. We adopt a normative approach by introducing a set of desirable axioms that prevent cross-subsidization. We provide two characterizations of our proposed solution on the basis of these axioms. Furthermore, we present an illustration of our method to the allocation of common costs for a public service operator.

Keywords: Cooperative game theory; Cost allocation; Public service; Cross-subsidization

JEL codes: C71; L51

1. Introduction

1.1. Context and problematic

Accurate cost allocation poses a significant challenge for public service operators tasked with fulfilling **services of general economic interest** (SGEI) as well as for regulatory bodies (see Choné et al., 2000, 2002). Despite the important of SGEI, the European Commission has not given a clear definition of the concept[1]. Ølykke and Møllgaard (2016) define the SGEI

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[1] SGEI are regulated by Article 14 Treaty of European Union and Protocol no. 26 on Service of General Interest in the Charter of fundamental rights, OJ 2010C 83/389.

concept on the basis of economic theory as "the strenghteing of a component of a network that under provides services to a significant share of the population." In other words, SGEI is the **obligation** to provide access to essential public services for all individuals within a country at affordable prices. Such services are provided through major network infrastructures, such as telecommunication, energy, mail services or rail transport. Public service operators share the common characteristic of having multiple **activities** to carry out their services. The SGEI imposes a constraint on operators, requiring them to provide a greater range of activities compared to what would be expected under competitive conditions, i.e., without SGEI. For example, Article 3(1) Regulation (EU) 2015/2120 lays down measures concerning open internet access.[2] It imposes that individuals "shall have the right to access and distribute information and content, use and provide applications and services, and use terminal equipment of their choice, irrespective of the end-user's or provider's location or the location, origin or destination of the information, content, application or service, via their internet access service." Consequently, the telecom network must be able to provide internet access to the most remote villages, even if it implies more costs than benefits. The constraint of SGEI leads to additional costs to be supported by public service operators. In order to remain competitive, an operator constrained by SGEI can receive some aid from the state. The amount of aid is determined through an assessment of the net cost of providing the SGEI. It is accepted by European law but, according to the article 107(1) of the Treaty on the functioning of the European union (TFEU),"aid granted by a member state shall not distort competition by favoring activities in competitive market." Indeed, there is the risk that an operator declares an inflated assessment of the net cost of providing the SGEI. This would allow him to perceive more state aid than necessary, which may not be employed to ensure the SGEI but to make its activities more competitive. In other words, this is a risk of **cross-subsidization** from obligations to activities.[3] Preventing this risk is a challenge for the regulators and appropriate measure should be undertaken.

To obtain a precise assessment of the net cost of providing the SGEI and to avoiding cross-subsidization, an appropriate cost allocation method should be employed. However, in some relevant cases, the existing methods commonly employed in practice are not fully satisfactory. We present two of them. The top-down **Activity-Based Costing** (see Bruns and Kaplan, 1987) method is generally used to allocate the **common costs** of the activities. The latter are difficult to allocate because, by definition, they are not directly attributable to a specific activity. The objective of this method is to ascertain the cost drivers of activities, which entails establishing a cause-and-effect relationship between common costs and activities. The allocation of common costs is done proportionally to the cost drivers of the respective activities. Although this method is commonly used, it has some drawbacks. If they are no cost drivers, the common costs are allocated in proportion to the costs directly attributable to the activities, which lacks accuracy. Moreover, this method focuses solely on activities, it is not suitable for an operator in charge of obligations. Alternatively, Cremer et al. (2000) and Panzar (2000) recommended a

[2]Regulation (EU) 2015/2120 of the European parliament and the council of 25 November 2015 laying down measures concerning open internet access and amending Directive 2002/22/EC on universal service and users' rights relating to electronic communications networks and services and Regulation (EU) No 531/2012 on roaming on public mobile communications networks within the Union.

[3]The opposite phenomenon exists; that is, it is possible to have cross-subsidization from obligations to activities. In other words, a portion of the net cost of the obligations is borne by the activities. In practice, according to regulatory bodies, operators must have an incentive to reduce the cost of their obligations. Consequently, this kind of cross-subsidization is acceptable and should not be discouraged.

profitability cost approach. The burden that the SGEI imposes on an operator is equal to the difference in the operator's supported costs under two comparative scenarios: one with SGEI and one without SGEI. To identify the services or products that would not be offered by the operator in the absence of the SGEI, a counterfactual scenario is constructed. This scenario presents a hypothetical situation wherein the operator solely considers its business strategy, without the constraint of SGEI. It portrays the operator's competitive positioning in the market. When the operator provides two or more SGEI, the profitability cost approach successfully gives the net cost of the set of SGEI but need to be adapted to calibrate the cost of each SGEI. One possibility is to use a two-step profitability approach. This method, however, does not allow to take into account all counterfactual scenarios on which a precise cost allocation can rely.

1.2. Methodology and results

This article proposes an alternative approach for allocating the common costs of a multiple activity operator responsible for potentially multiple SGEI. The suggested method is grounded in **cooperative game theory** and offers a comprehensive solution to the allocation problem, considering activities, obligations and more counterfactual scenarios then the methods currently used. A cooperative game with transferable utility (a TU-game for short) describes a situation where players can form coalitions to generate some worth. One fundamental issue in TU-game theory is how to redistribute the worth generated by the grand coalition, in which all players cooperate. The **Shapley value** stands as a widely recognized solution to this problem. It is computed by averaging the **marginal contributions** of each player to coalitions across all possible orderings of the players. A player's marginal contribution to a coalition denotes the difference in worth generated when that player joins the coalition.

In the context of this paper, we view the activities of an operator and the multiple SGEI constraining him as separate players. In the following, we employ the generic term obligation instead of SGEI. The set of all players is denoted by N. A coalition $E \subseteq N$ contains some activities subject to some obligations. It is assumed that the common costs $v(E)$ associated with operating each coalition $E \subseteq N$ is known. In practice, major companies have developed mathematical methods to assess these common costs. The couple (N, v) is called the **public service game**. A solution for public service games can be viewed as an allocation of the common costs among the activities and the obligations. The marginal contribution of a player to a coalition can be understood as the **incremental cost** incurred when incorporating an activity (or an obligation) into the coalition. Adhering to the profitability cost approach, a solution for public service games based on incremental costs can offer valuable insights in assessing the expenses associated with the activities and the obligations. While the Shapley value could be a potential candidate solution, it does not fulfill a crucial property necessary for preventing cross-subsidization, which we discuss in the next paragraph.

To better see this, we adopt a **normative approach** by introducing several desirable properties, called axioms, to determine an appropriate solution for public service games. Some of these axioms are designed to prevent cross-subsidization, while others are adaptations of classical axioms from TU-games to our context. The first axiom, **Independence of activities from obligations** (IAO), requires that a change in the set of obligations has no effect on the share of the common costs allocated to the activities. Consequently, it prevents cross subsidization from obligation to activities. It appears that the Shapley value fails (IAO). In addition, we propose the **Restricted equal treatment of equals** (RET), inspired by the axiom of Equal treatment of equal (see Shapley, 1953). This axiom implies that if two players, both in the set of activities or in the set of obligations, have the same incremental costs, then they receive the same allocation. Then, the classical axiom **Efficiency** (E), requires that the common costs of the grand coalition should be entirely allocated among the players. Next, **Monotonicity** (M), proposed by Young (1985), imposes that a player whose incremental costs weakly increase does not end up with a lower allocation. The interpretation is that if an operator employs a cost reduction technology to operate an activity, then the cost allocated to that activity should decrease.

We demonstrate that there is a unique solution that satisfies these four axioms. This solution is called the ***PS* value** and is formally defined in Section 4.1. It incorporates most of the advantages of the Shapley value while preventing cross-subsidization. The PS value is also the unique solution that satisfies (E), (IAO) and the axiom of **Restricted balanced contributions** (RBC). This last axiom, inspired by the Balanced contributions axiom proposed by Myerson (1980), states that for any two players, both in the set of activities or in the set of obligations, the amount that each player would gain or lose by the other player's withdrawal from the game should be equal.

1.3. Related literature

The issue of cross-subsidization within a competitive market has long piqued the interest of researchers. We present a (non-exhaustive) list of **previous studies** (both theoretical and empirical) addressing this issue. Faulhaber (1975) defines and analyzes cross-subsidization in companies with economies of joint production. To conduct his study, he adopts a cooperative game approach. Contrary to our article, he studies cross-subsidization through the core of the game, i.e., the set of allocations that are efficient and desirable for the players. Puelz and Snow (1994) builds upon the work of Akerlof (1978) and conduct an empirical study to investigate cross-subsidization in the market for automobile collision insurance. Parsons (1998) provides a survey on cross-subsidization in the context of telecommunication industry. Moulin and Sprumont (2005) propose two cost-sharing methods for idiosyncratic commodity (e.g., mail delivery). In their framework, consumers demand comparable commodities and are responsible for their own demand, the cross-subsidization is possible between consumers. Moulin and Sprumont (2006) propose two cost-sharing theories in which consumers demand comparable commodities. They conduct an axiomatic study and discuss concepts of responsibility and cross-subsidization. Dormady et al. (2019) conduct an empirical study and show evidence that retail restructuring leads to two types of cross-subsidization. Finally, Chen and Rey (2019) investigate cross-subsidization that occurs when companies possessing distinct comparative advantages engage in competition for consumers who exhibit preferences for heterogeneous products. They model a situation in which firms subsidize poorly performing products with the profits made on well-selling products. Contrary to these studies, our agents are not companies or consumers, but the activities and the obligations within a public service operator.

1.4. Plan

The rest of the paper is organized as follows. We provide preliminaries on TU-games in Section 2. Section 3 presents public service games and introduces axioms for solutions to these games. Section 4 introduces the PS value and provides two axiomatic characterizations. Moreover, this section contains an **illustration** of the PS value to a public service operator. We allocate the common costs of the network of a public service operator between its activities and its obligations. As an **additional content**, public service games are compared with the model of TU-games with a priority structure (see Béal et al., 2022) in Section 5. A priority structure reflects some exogenous rights, needs or hierarchical constraints between the cooperating players. Béal et al. (2022) introduced the Priority value for TU-games with a priority structure. This value takes both the marginal contributions of the players and the priority structure into account. We show that, to each public service games, it is possible to associate a TU-game with a priority structure. Moreover, we show that the PS value coincides with the Priority value of this associated game. Finally, Section 6 concludes.

2. Preliminaries

This section contains basics on transferable utility games. It presents the Shapley value along with two of the value's most well-known axiomatic characterizations proposed by Myerson (1980) and Young (1985). Moreover, we present a result proposed by Pintér (2015), which states that the characterization of the Shapley value due to Young (1985) holds on certain sub-classes of transferable utility games.

2.1. Transferable utility games

Let $N \subset \mathbb{N}$ be a non-empty and finite set of **players**. Each subset $E \in 2^N$ is referred to as a **coalition** of cooperating players. The grand coalition N represents a situation in which all players cooperate. The coalition $\emptyset$ represents a situation in which no player cooperates, it is called the empty coalition. For each $E \in 2^N$, the integer $|E| \in \mathbb{N}$ denotes the cardinality of coalition E.

A transferable utility game, or simply a **TU-game**, is a couple (N, v) consisting of a finite player set $N \subset \mathbb{N}$ and a characteristic function $v : 2^N \to \mathbb{R}$, with the convention that $v(\emptyset) = 0$. The real number $v(E)$ can be interpreted as the worth the players in E generate when they cooperate. A TU-game is **sub-additive** if the union of any two disjoint coalitions has a worth lower than the sum of their individual worth. Formally, $(N, v) \in \mathbb{G}$ is sub-additive if

$$\forall E, F \in 2^N, E \cap F = \emptyset, \quad v(E \cup F) \leq v(E) + v(F).$$

In some TU-games, a larger coalition always implies a higher worth. Such TU-games are called **monotonic** TU-games. Formally, $(N, v) \in \mathbb{G}$ is monotonic if

$$\forall E, F \in 2^N, \quad E \subseteq F \implies v(E) \leq v(F).$$

The class of TU-games with the finite set of players $N \subset \mathbb{N}$ is denoted by $\mathbb{G}^N$. The class of all TU-games with a finite set of players is denoted by $\mathbb{G} = \cup_{N \subset \mathbb{N}} \mathbb{G}^N$. The sub-class of monotonic TU-games with a finite set of players is denoted by $\mathbb{G}^m$.

Pick any $N \subset \mathbb{N}$ and any $(N, v) \in \mathbb{G}^N$. Since 2^N is a finite set, a characteristic function $v : 2^N \to \mathbb{R}$ can be described by the vector $(v(E))_{E \in 2^N} \in \mathbb{R}^{2^N}$. Therefore, the class $\mathbb{G}^N$ is a linear sub-space of $\mathbb{R}^{2^N}$. It follows that (N, v) can be expressed as a linear combination of $2^N - 1$ linearly independent games, since $v(\emptyset) = 0$. In his original paper, Shapley (see Shapley, 1953) identifies a salient basis on which v can be decomposed as

$$v = \sum_{E \in 2^N \setminus \{\emptyset\}} \Delta_v(E) u_E, \quad \text{where} \quad u_E(F) = \begin{cases} 1 \text{ if } E \subseteq F, \\ 0 \text{ otherwise,} \end{cases} \tag{1}$$

where each (N, u_E) is called the **unanimity game** with ruling coalition $E \in 2^N$. The coordinate $\Delta_v(E)$ is the **Harsanyi dividend** (see Harsanyi, 1959) of E. This dividend reflects the net surplus generated by E, and is formally defined as

$$\Delta_v(E) = v(E) - \sum_{F \subset E} \Delta_v(F).$$

Pick any $(N, v) \in \mathbb{G}$. In some situations, we may want to consider sub-games of (N, v) on a smaller set of players, but with the same characteristic function. Pick any $E \in 2^N$. The **sub-game** $(E, v_E) \in \mathbb{G}$ is defined as

$$\forall F \in 2^E, \quad v_E(F) = v(F).$$

If no confusion arises, we simply denote (E, v_E) by (E, v).

The **marginal contribution** of a player $i \in N$ to a coalition $E \in 2^{N\setminus\{i\}}$ in (N, v) is given by $v(E \cup \{i\}) - v(E)$. Two players $i, j \in N$ are **equal** in (N, v) if they have the same marginal contributions to coalitions, i.e., $v(E \cup \{i\}) = v(E \cup \{j\})$ for each $E \in 2^{N\setminus\{i,j\}}$. In a unanimity game (N, u_E), all players in E are equal.

2.2. The Shapley value

One of the basic issues in the theory of cooperative TU-games is as follows: "If the grand coalition forms, how to divide its worth among the players". This issue may be addressed by values for TU-games. In a TU-game $(N, v) \in \mathbb{G}$, each player $i \in N$ may receive a payoff. A payoff vector $x \in \mathbb{R}^{|N|}$ is a $|N|$-dimensional vector that assigns a payoff $x_i \in \mathbb{R}$ to each player $i \in N$. A single-valued solution, or a value, is a map f that assigns a unique payoff vector $f(N, v)$ to each $(N, v) \in \mathbb{G}$. The Shapley value (see Shapley, 1953) is probably the most prominent value for TU-games. It can be defined as follows.

Assume that the grand coalition forms step by step, starting from the empty coalition. At each step, a player enters the coalition. The players enter according to a linear order $\theta \in \Theta_N$ over N, where Θ_N is the set of all possible linear orders over N. The step $\theta(i)$ is the step at which $i \in N$ enters. When a player enters, it contributes positively or negatively to the worth of the coalition. The Shapley value attributes, to each player, its average contribution to the coalitions assuming that the linear orders in Θ_N occur with equal probability. The Shapley value can also be defined as the map that divides the Harsanyi dividend of each coalition equally among the members of the coalition.

Definition 1 (Shapley value). *The Shapley value is the value Sh on $\mathbb{G}$ defined, for each $(N, v) \in \mathbb{G}$, as*

$$\forall i \in N, \quad Sh_i(N, v) = \frac{1}{|\Theta_N|} \sum_{\theta \in \Theta_N} \Big[v(E^{\theta,i} \cup \{i\}) - v(E^{\theta,i}) \Big].$$

where $E^{\theta,i} = \{j \in N : \theta(j) < \theta(i)\}$ is the set of predecessors of $i \in N$ with respect to θ. Equivalently, the Shapley value can be defined as

$$\forall i \in N, \quad Sh_i(N, v) = \sum_{\substack{E \in 2^N \\ E \ni i}} \frac{\Delta_v(E)}{|E|}.$$

2.3. Axiomatic characterizations

Next, we present the axiomatic characterizations of the Shapley value proposed by Myerson (1980) and Young (1985). Let us introduce axioms that define properties for any value f on $\mathbb{G}$.

(E) Efficiency. For each $(N, v) \in \mathbb{G}$, $\sum_{i \in N} f_i(N, v) = v(N)$.

(BC) Balanced contributions. For each $(N, v) \in \mathbb{G}$ and each $i, j \in N$,

$$f_i(N, v) - f_i(N \setminus \{j\}, v) = f_j(N, v) - f_j(N \setminus \{i\}, v).$$

The first axiom (E) transcribes the simple idea that the total worth of the grand coalition should be entirely allocated among the players. The second axiom, (BC), addresses the fairness between any pair of players. It states that for any two players, the amount that each player would gain or lose by the other's withdrawal from the game should be equal. Combining (E) and (BC) leads to a characterization of the Shapley value.

Theorem 1 (Myerson (1980)). *A solution f on $\mathbb{G}$ satisfies (E) and (BC) if and only if $f = Sh$.*

(ET) Equal treatment of equals. For each $(N, v) \in \mathbb{G}$, if two distinct players $i, j \in N$ are equals, then $f_i(N, v) = f_j(N, v)$.

Let us introduce a few definitions to properly present the next axiom. Denote by Θ_N the set of all linear orders on N. Pick any $\theta \in \Theta_N$. The game $(N, \theta v)$ is defined as

$$\forall E \in 2^N, \quad \theta v(\cup_{i \in E}\{\theta(i)\}) = v(E).$$

In this game, each player in E has a new label but keeps the same marginal contributions to coalitions.

(A) Anonymity. For each $(N, v) \in \mathbb{G}$ and each linear order θ on N,

$$f_i(N, v) = f_{\theta(i)}(N, \theta v).$$

(M) Monotonicity. For each $(N, v), (N, w) \in \mathbb{G}$ and each $i \in N$ such that

$$\forall E \in 2^{N \setminus \{i\}}, \quad v(E \cup \{i\}) - v(E) \geq w(E \cup \{i\}) - w(E), \text{ it holds that } f_i(N, v) \geq f_i(N, w).$$

Axiom (ET), originally introduced by Shubik (1962), advocates that two players with the same marginal contribution to coalitions should receive the same payoff. Second, (A) ensures that the payoff of a player does not depends on its label. Observe that (A) implies (ET). The converse is not true. Finally, (M) compares the payoffs of a single player in two different situations. This axiom guarantees that a player whose marginal contributions weakly increases does not end up with a lower payoff. Combining (A) and (M) with (E) results in another characterization of the Shapley value.

Theorem 2 (Young (1985)). *A solution f on $\mathbb{G}$ satisfies (E), (M) and (A) if and only if $f = Sh$.*

It appears that relaxing (A) into (ET) does not affect the result of Theorem 2. In fact, Pintér (2015) shows that (E), (M) and (ET) characterize the Shapley value on certain sub-classes of TU-games (including the full class of TU-games). In particular, their result holds on monotonic TU-games.

Theorem 3 (Pintér (2015)). *A solution f on $\mathbb{G}^m$ satisfies (E), (M) and (ET) if and only if $f = Sh$.*

3. Public service games and axioms for values

In this section, we define public service games as monotonic TU-games whose player set is partitioned into two sub-sets of players: the set of activities and the set of obligations. The characteristic function of a public service game specifies the common costs of operating any subset of activities under the constrains of any subset of obligations. Consequently, this game is not sub-additive since adding an obligation to a coalition of activities systematically increases the common costs, as it can be seen in the example in Section 4.2. A value for public service games can be viewed as a mean to allocate the common costs among the activities and the obligations. To determine a relevant value for public service games, we adopt an axiomatic approach. We introduce several axioms that can be viewed as desirable properties for a value in the context of public service games. Some of them are natural adaptations of the axioms introduced in the preliminaries (see sub-section 2.3) to the context of public service games, while others describe new properties. We provide two separate results that both claim the uniqueness of a value satisfying certain sets of axioms.

3.1. Game formulation and notation

Fix $\mathbb{A} \subset \mathbb{N}$ the non-empty and finite set of **activities**. In addition, fix $\mathbb{O} \subset \mathbb{N}$ the non-empty and finite set of universal service **obligations**, or obligations for short, where $\mathbb{A} \cap \mathbb{O} = \emptyset$.

Definition 2 (Public service game). *A public service game is a TU-game $(N, v, \{A, O\})$ where the set of players is partitioned into two sub-sets $N = A \cup O$, $O \subseteq \mathbb{O}$ and $A \subseteq \mathbb{A}$. The characteristic function $v : 2^N \to \mathbb{R}$ is a monotonic map. If no confusion arises, we simply denote $(N, v, \{A, O\})$ by (N, v). The class of all public service games is denoted by $\mathcal{P}$.*

We reinterpret several concepts related to TU-games introduced in the Preliminaries within the framework of public service games. Pick any public service game $(N, v) \in \mathcal{P}$. The characteristic function v measures the **common costs** $v(E)$, $E \in 2^N$, of operating the subset of activities $E \cap A$ under the constrains of the subset of obligations $E \cap O$. The real $v(N)$ corresponds to the **common costs** of operating the whole set of activities under the constrains of the whole set of obligations. In practice, the operator bears these common costs.

In the context of public service games, a marginal contribution $v(E \cup \{i\}) - v(E)$, where $i \in N$ and $E \in 2^{N \setminus \{i\}}$, corresponds to the **incremental cost** generated by adding the activity (or the obligation) i to the set E. A public service game (N, v) is monotonic since adding an activity or an obligation systematically leads to higher common costs. Therefore, all incremental costs are non-negative.

Consider the sub-game (E, v), $E \in 2^N$, where the set of activities is reduced to $E \cap A$ and the set of obligations is reduced to $E \cap O$. Obviously, $(E, v) \in \mathcal{P}$. Such sub-game represents a

counterfactual in which the operator had to downsize its activities and/or is subject to less obligation constrains.

A particular case of sub-game is (A, v), where all obligations have been removed, i.e., the operator conducts its activities without the constrain of any obligation. In this counterfactual, the operator is a **competitive** company and is completely free to decide the number and nature of the activities it offers. We call (A, v) the **activity game**. Observe that public service games without any obligations can be viewed as standard (monotonic) TU-games. Indeed, if $O = \emptyset$, then the partition of the player set becomes meaningless and $(N, v, \{A, \emptyset\})$ can be viewed as a standard TU-game.

The sub-game (O, v) is a counterfactual in which the set of activities is empty. It corresponds to a situation where the operator cannot offer any activity, despite being under contract of universal service obligation.

A payoff vector can be viewed as an **allocation** of the common costs among activities and obligations. A value for public service games specifies an allocation to each game in $\mathcal{P}$.

3.2. A uniqueness result based on a monotonicity axiom

In this sub-section, we introduce two axioms that can be viewed as desirable properties for a value f on $\mathcal{P}$. The first axiom states that the common costs allocated to activities should not depend on the number of obligations. This allows to avoid situations in which an operator manipulates its number of obligations in order to reduce the common costs allocated to its activities, i.e., the cross-subsidization from obligations to activities.

(IAO) Independence of activities from obligations. For each $(N, v) \in \mathcal{P}$,

$$\forall i \in A, \forall j \in O, \quad f_i(N, v) = f_i(N \setminus \{j\}, v).$$

It should be noted that (IAO) loses its significance when considering public service games whose set of obligations is empty. The second axiom can be viewed as a weak version of (ET). It guarantees that equal activities and equal obligations receive equal allocations. This serves to prevent any form of cross-subsidization between activities or obligations.

(RET) Restricted equal treatment of equals. Pick any $(N, v) \in \mathcal{P}$ and any two equal players $i, j \in N$. If $i, j \in A$ or if $i, j \in O$, then

$$f_i(N, v) = f_j(N, v).$$

Observe that (RET) reduces to (ET) when considering public service games whose set of obligations is empty. Furthermore, observe that an activity and an obligation may not be allocated the same amount, even if they are equal players in terms of contributions. This discrepancy arises from the fact that they are inherently different types of entities.

While (IAO) prevents cross-subsidization between the sets of obligations and activities, (RET) prevents cross-subsidization within these sets. Together, these two axioms guarantee the total absence of cross-subsidization. It appears that there is at most one value that satisfies these two axioms alongside (E) and (M).

Proposition 1. *There is at most one value on $\mathcal{P}$ that satisfies (E), (IAO), (M) and (RET).*

Proof. Pick any value f on $\mathcal{P}$ that satisfies (E), (IAO), (M) and (RET). Let us show that f is uniquely determined. Pick any $(N, v) \in \mathcal{P}$. By successive application of (IAO), one obtains

$$\forall i \in A, \quad f_i(N, v) = f_i(A, v).$$

The sub-game (A, v) can be viewed as a monotonic TU-game. Moreover, (RET) reduces to (ET) since $O = \emptyset$ in (A, v). Since f satisfies (E), (RET) and (M), by Theorem 3, $f(A, v)$ is uniquely determined. Therefore, $f_i(N, v)$ is uniquely determined for each $i \in A$.

It remains to show that $f_i(N, v)$ is uniquely determined for each $i \in O$. To that end, we must introduce several notations. The characteristic function v can be decomposed as a linear combination of unanimity games (1). Pick any coalition size $1 \leq e \leq |N|$. Define the highest Harsanyi dividend of a coalition of size e as

$$\Delta_e^+ = \max_{\substack{E \subseteq N \\ |E| = e}} \Delta_v(E).$$

For each coalition $E \in 2^N$, denote the difference between $\Delta_{|E|}^+$ and E's Harsanyi dividend by

$$\overline{\Delta}(E) = \Delta_{|E|}^+ - \Delta_v(E). \tag{2}$$

Observe that $\overline{\Delta}(E) \geq 0$. Define the game (N, u^+) in which the Harsanyi dividend of any coalition is entirely determined by its size by

$$\forall F \in 2^N, \quad u^+(F) = \sum_{E \in 2^N} \Delta_{|E|}^+ u_E(F).$$

Observe that (N, u^+) is a symmetric game, i.e., all players are equal in this game. Moreover, (N, u^+) is a monotonic game. To see this, observe that

$$\begin{aligned} u^+ &= \sum_{E \in 2^N} (\Delta_v(E) + \overline{\Delta}(E)) u_E \\ &= v + \sum_{E \in 2^N} \overline{\Delta}(E) u_E. \end{aligned} \tag{3}$$

Since $\overline{\Delta}(E) \geq 0$ for each $E \in 2^N$, $(N, \sum_{E \in 2^N} \overline{\Delta}(E) u_E)$ is a monotonic game. Since (N, v) is a monotonic game and the sum of two monotonic games is a monotonic game, it follows that (N, u^+) is a monotonic game. Therefore, $(N, u^+) \in \mathcal{P}$. In the rest of the proof, we will use (3) under the form

$$v = u^+ - \sum_{E \in 2^N} \overline{\Delta}(E) u_E, \tag{4}$$

to emphasize that v can be rewritten by using a symmetric game. Finally, define the set of coalitions whose difference (2) is non-null in (N, v) as

$$\mathcal{T}(N, v) = \{E \in 2^N : \overline{\Delta}(E) \neq 0\}.$$

We show the uniqueness of each $f_i(N,v)$, $i \in O$, by induction on the cardinality of $\mathcal{T}(N,v)$.

Initialization: If $|\mathcal{T}(N,v)| = 0$, then by (4), it holds that $v = u^+$ and (N,v) is a symmetric game. Let us show that $f(N,v)$ is uniquely determined. In (N,v), all players in N are equal. Observe that, by (E), $\sum_{i\in A} f_i(A,v) = v(A)$. Therefore, by (E) and (IAO),

$$\begin{aligned}\sum_{i\in O} f_i(N,v) &= v(N) - \sum_{i\in A} f_i(N,v)\\ &= v(N) - \sum_{i\in A} f_i(A,v)\\ &= v(N) - v(A).\end{aligned}$$

Then by (RET),

$$\forall i \in O, \quad f_i(N,v) = \frac{1}{|O|}(v(N) - v(A)).$$

We have shown that if $|\mathcal{T}(N,v)| = 0$, then $f(N,v)$ is uniquely determined.

Next, if $|\mathcal{T}(N,v)| = 1$, then $v = u^+ - \overline{\Delta}(E)u_E$, for some $E \in 2^N$. Let us show that $f(N,v)$ is uniquely determined. Pick any $i \notin E$. Player i has the same marginal contribution in (N,v) as in (N,u^+). By (M), $f_i(N,v) = f_i(N,u^+)$. Applying a similar reasoning than for the case $|\mathcal{T}(N,v)| = 0$, we obtain that $f(N,u^+)$ is uniquely determined. Therefore, each $f_i(N,v)$, $i \notin E$, is uniquely determined. It remains to show that each $f_i(N,v)$, $i \in E$, is uniquely determined.

If $|E| = 1$, then uniqueness of $f(N,v)$ is trivial. Assume that $|E| > 1$. Observe that all players in E are equal in (N,v). If $E \cap O = \emptyset$, then E only contains elements of A and the proof is done. Assume that $E \cap O \neq \emptyset$. By (E),

$$\sum_{i\in E\cap O} f_i(N,v) = v(N) - \sum_{i\in E\cap A} f_i(N,v) - \sum_{i\notin E} f_i(N,v). \tag{5}$$

At the beginning of the proof, we showed that $f_i(N,v)$ is uniquely determined for each $i \in A$. Moreover, we know that $f_i(N,v)$ is uniquely determined for each $i \notin E$. Consequently, the right hand side of (5) is uniquely determined. Then, by (RET),

$$\forall i \in E\cap O, \quad f_i(N,v) = \frac{1}{|E\cap O|}(v(N) - \sum_{i\in E\cap A} f_i(N,v)) - \sum_{i\notin E} f_i(N,v).$$

This shows that $f_i(N,v)$ is uniquely determined for each $i \in E \cap O$. We have shown that if $|\mathcal{T}(N,v)| = 1$, then $f(N,v)$ is uniquely determined.

Hypothesis: Fix $0 \leq K < 2^N - |N|$ and assume that for each $(N,v) \in \mathcal{P}$, $|\mathcal{T}(N,v)| = K$, $f(N,v)$ is uniquely determined.

Induction: Pick any $(N,v) \in \mathcal{P}$ such that $|\mathcal{T}(N,v)| = K+1$. Let us show that $f(N,v)$ is uniquely determined. Consider the coalition

$$I = \bigcap_{E\in\mathcal{T}(N,v)} E.$$

Observe that all the players in I are equal in (N, v). Pick any $i \notin I$. There exists a coalition $E \in \mathcal{T}(N, v)$ such that $i \notin E$. Therefore, player i has the same marginal contributions in (N, v) and in $(N, v + \overline{\Delta}(E)u_E)$. Observe that $(N, v + \overline{\Delta}(E)u_E) \in \mathcal{P}$. By (M), it holds that

$$f_i(N, v) = f_i(N, v + \overline{\Delta}(E)u_E).$$

However, observe that $|\mathcal{T}(N, v + \overline{\Delta}(E)u_E) = K|$. By the Hypothesis, $f(N, v + \overline{\Delta}(E)u_E)$ is uniquely determined. Therefore $f_i(N, v)$ is uniquely determined for any $i \notin I$. Next, pick any $i \in I$.

If $I \cap O = \emptyset$, then I only contains elements of A and the proof is completed by (RET). Assume that $I \cap O \neq \emptyset$. By (E),

$$\sum_{i \in I \cap O} f_i(N, v) = v(N) - \sum_{i \notin I} f_i(N, v) - \sum_{i \in I \cap A} f_i(N, v). \tag{6}$$

Observe that the right hand side of (6) is uniquely determined by similar arguments than in the initialization. Then by (RET),

$$\forall i \in I \cap O, \quad f_i(N, v) = \frac{1}{|I \cap O|}\Big(v(N) - \sum_{i \notin I} f_i(N, v) - \sum_{i \in I \cap A} f_i(N, v)\Big).$$

This shows that $f_i(N, v)$ is uniquely determined for each $i \in I \cap O$. We have shown that each $f(N, v)$ is uniquely determined when $|\mathcal{T}(N, v)| = K + 1$. This concludes the induction. We have shown that $f(N, v)$ is uniquely determined, which concludes the proof. □

3.3. A uniqueness result based on a balanced contributions axiom

In this sub-section, we introduce an additional axiom: Restricted balanced contributions (RBC). Similarly to (RET), (RBC) restricts the prescription of (BC). The axiom applies the fairness requirement of (BC) to activities (or to obligations) only.

(RBC) Restricted balanced contributions. Pick any $(N, v) \in \mathcal{P}$ and any $i, j \in N$. If $i, j \in A$ or if $i, j \in O$, then

$$f_i(N, v) - f_i(N \setminus \{j\}, v) = f_j(N, v) - f_j(N \setminus \{i\}, v).$$

Observe that (RBC) reduces to (BC) for public service games whose set of obligations is empty. We show that there is at most on value that satisfies this axiom alongside (E) and (IAO).

Proposition 2. *There is at most one value on $\mathcal{P}$ that satisfies (E), (IAO) and (RBC).*

Proof. Assume that a value f on $\mathcal{P}$ satisfies (E), (IAO) and (RBC). Let us show that f is uniquely determined. By successive application of (IAO), one obtains

$$\forall i \in A, \quad f_i(N, v) = f_i(A, v).$$

Observe that $(A, v) \in \mathcal{P}$ Since f satisfies (RBC) and (E), by Theorem 1, $f(A, v)$ is uniquely determined. Therefore $f_i(N, v)$ is uniquely determined for each $i \in A$. By (E), it holds that

$$\sum_{i \in O} f_i(N, v) = v(N) - \sum_{i \in A} f_i(N, v). \tag{7}$$

Observe that the right hand side of (7) is uniquely determined. Let us show that $f_i(N, v)$ is uniquely determined for each $i \in O$. We proceed by induction on the number of obligations in O. The actual number of obligations in N is denoted by L.

Initialization: the case where $|O| = 0$ is trivial. If $|O| = 1$, then by (7), $f_i(N, v)$, $i \in O$, is uniquely determined.

Hypothesis: assume that each $f_i(N, v)$, $i \in O$, is uniquely determined when $|O| = K$, for some $K < L$.

Induction: let us show that each $f_i(N, v)$, $i \in O$, is uniquely determined when $|O| = K + 1$. Without loss of generality, relabel the obligations in O by $O_1, O_2, \ldots, O_{K+1}$. Applying (RBC), we can obtain the following system of K equations

$$\begin{cases} f_{O_1}(N, v) - f_{O_1}(N \setminus \{O_2\}, v) = f_{O_2}(N, v) - f_{O_2}(N \setminus \{O_1\}, v) \\ f_{O_2}(N, v) - f_{O_2}(N \setminus \{O_3\}, v) = f_{O_3}(N, v) - f_{O_3}(N \setminus \{O_2\}, v) \\ \vdots \\ f_{O_K}(N, v) - f_{O_K}(N \setminus \{O_{K+1}\}, v) = f_{O_{K+1}}(N, v) - f_{O_{K+1}}(N \setminus \{O_K\}, v). \end{cases}$$

Observe that each negative term in the system is uniquely determined by the Hypothesis. Combining this system with (7), we obtain a system of $K + 1$ linearly independent equations with $K + 1$ unknowns. It follows that $f_i(N, v)$ is uniquely determined for each $i \in O$. This concludes the induction. Therefore $f_i(N, v)$ is uniquely determined for each $i \in N$. This concludes the proof of Proposition 2. □

Remark 1. Observe that the Shapley value fails to satisfy (IAO), but it satisfies (RET) and (RBC).

4. Characterization and illustration of a value for public service games

This section presents a value for public service games. This value is computed by applying the Shapley value to appropriate sub-games. It also satisfies all the axioms invoked in Proposition 1 and Proposition 2. Consequently, this leads to two axiomatic characterizations of the value. We point out that each characterization is tight by showing logical independence between the invoked axioms. Finally, we provide an operational illustration of the value.

4.1. The value and its characterizations

Pick any $(N, v) \in \mathcal{P}$. The associated **activity game** is the sub-game (A, v). In the activity game (A, v), we measure the common costs of operating any subset of activities, assuming there is no obligation. As mentioned in sub-section 3.1, the sub-game (A, v) is a counterfactual in which the operator is competitive. In addition, the associated **obligation game** is the game (O, v^O) defined as

$$\forall E \in 2^O, \quad v^O(E) = v(A \cup E) - v(A).$$

In the obligation game (O, v^O), we measure the impact of any subset of obligation on the common costs of operating the whole set of activities. Using the associated activity game and the obligation game, we define the PS value of any public service game. This value allocates, to each activity, its Shapley value in the activity game, and to each obligation, its Shapley value in the obligation game.

Definition 3. *The PS value is defined, for each* $(N, v) \in \mathcal{P}$*, by*

$$\forall i \in N, \quad PS_i(N, v) = \begin{cases} Sh_i(A, v) & \text{if } i \in A \\ Sh_i(O, v^O) & \text{if } i \in O. \end{cases}$$

The next theorem constitutes the main result of this paper. It asserts that the PS value stands as the unique solution for public service games, satisfying two distinct sets of axioms: (E), (IAO), (M), and (RET); as well as (E), (IAO), and (RBC).

Theorem 4. *The PS value is the only value on* $\mathcal{P}$ *that satisfies the following combinations of axioms:*

(i) (E), (IAO), (M) and (RET).

(ii) (E), (IAO) and (RBC).

Proof. First, we show that the PS value satisfies (E), (M), (RET), (RBC) and (IAO) on $\mathcal{P}$. By Theorem 1, the Shapley value satisfies (E), (M), (ET) and (BC). Pick any $(N, v) \in \mathcal{P}$. By considering the definitions of the associated games (A, v) and (O, v^O), and observing that the PS value is derived from the application of the Shapley value to these associated games, it becomes evident that PS inherently satisfies (E), (M) and (RBC). Moreover, observe that if any two players $i, i' \in A$ (or $i, i' \in O$) are equal in (N, v), then they are equal in (A, v) (or in (O, v^O)). Since the Shapley value satisfies (ET), it follows that PS satisfies (RET). Since $PS_i(N, v) = Sh_i(A, v)$ for each $i \in A$, it holds that $PS_i(N \setminus \{j\}, v) = Sh_i(A, v)$ for any $j \in O$. Thus, PS satisfies (IAO).

We have shown that the PS value satisfies (E), (M), (RET), (RBC) and (IAO) on $\mathcal{P}$. By Proposition 1, we know that there is a unique solution on $\mathcal{P}$ satisfying (E), (IAO), (M) and (RET); and by Proposition 2 we know that there is a unique solution on $\mathcal{P}$ satisfying (E), (IAO) and (RBC). Consequently we can conclude that the PS value is the only solution satisfying (E), (IAO), (M) and (RET); as well as (E), (IAO), and (RBC). □

The axioms invoked in Theorem 4 are logically independent, as shown by the following alternative solutions:

- The null value f defined, for each $(N, v) \in \mathcal{P}$ and each $i \in N$, as $f_i(N, v) = 0$, satisfies all the axioms except (E).

- The Shapley value satisfies all the axioms except (IAO).

- The value f defined, for each $(N, v) \in \mathcal{P}$, as

$$\forall i \in A, \quad f_i(N, v) = \frac{v(A)}{N} \quad \text{and} \quad \forall i \in O, \quad f_i(N, v) = \frac{v(N) - v(A)}{N},$$

satisfies (E), (IAO) and (RET), but fails (M).

- Let $w = (w_i)_{i\in\mathbb{N}}$ be a (countable and infinite) family of strictly positive weights. Consider the Weighted Shapley value (see Kalai and Samet, 1987) Sh^w defined, for each $(N, v) \in \mathbb{G}$, as

$$\forall i \in N, \quad Sh_i^w(N, v) = \sum_{\substack{E\in 2^N \\ i\in E}} \frac{w_i}{\sum_{j\in E} w_j} \Delta_v(E).$$

  The value f defined, for each $(N, v) \in \mathcal{P}$, as

$$\forall i \in A, \quad f_i(N, v) = Sh_i^w(A, v) \quad \text{and} \quad \forall i \in O, \quad f_i(N, v) = Sh_i^w(O, v^O),$$

  satisfies (E), (IAO) and (M), but fails (RET).

- The value f defined, for each $(N, v) \in \mathcal{P}$, as

$$\forall i \in A, \quad f_i(N, v) = \frac{v(A)}{N} \quad \text{and} \quad \forall i \in O, \quad f_i(N, v) = \frac{v(N) - v(A)}{N},$$

  satisfies (E) and (IAO), but fails (RBC).

Remark 2. In this axiomatic study, we consider two axioms related to cross-subsidization: (IAO) and (RET). The former ensures no cross-subsidization from obligations to activities, and the latter ensures no cross-subsidization between activities or between obligations. An alternative axiom to consider is Independence of obligations from activities (IOA). Such an axiom ensures no cross-subsidization from activities to obligations. Formally, the axiom prescribes, for each $(N, v) \in \mathcal{P}$,

$$\forall i \in O, \forall j \in A, \quad f_i(N, v) = f_i(N, \backslash\{j\}, v).$$

By combining (E), (IOA) and (RET), it is possible to characterize an alternative value $\tilde{PS}$, which is defined as

$$\forall i \in N, \quad \tilde{PS}_i(N, v) = \begin{cases} Sh_i(O, v) & \text{if } i \in O \\ Sh_i(A, v^A) & \text{if } i \in A, \end{cases}$$

where for each $E \in 2^A$, $v^A(E) = v(O \cup E) - v(O)$. This value simply permutes the roles of A and O compared to the PS value. It appears that there is no value on $\mathcal{P}$ that satisfies (E), (IAO) and (IOA). To see this, consider a value satisfying (IAO) and (IOA). Observe that (IAO) and (IOA) respectively imply

$$\sum_{i\in A} f_i(N, v) = v(A) \quad \text{and} \quad \sum_{i\in O} f_i(N, v) = v(O).$$

Since there is no guarantee that $v(N) = v(A) + v(O)$, it follows that f may fail (E). This means that it is not possible guarantee no cross-subsidization from obligations to activities and from activities to obligations at the same time.

4.2. An illustration to a public service operator

In this section the public service game and the PS value are applied to allocate the common costs of the network of a public service operator.

The operator provides three main activities, they may correspond to essential activities such as telecommunication services, or to non-essential activities such as advertising. Formally, The set of activities is given by $A = \{a, b, c\}$. Generally, obligations can be of two main types: an obligation to provide basic products and an obligation of territory development. For example, in Belgium the Postal service act requires postal operator to fulfill these obligations:

- Article 15(1) of the Belgium Postal Services Act provides a list of basic postal products that must be offered, "the universal postal service includes[...] the collection, sorting, transport and delivery of postal items weighing up to 2 kg; [...] of unit-rate parcels up to 10 kg; [...] of unit-rate parcels from other member states weighing up to 20 kg; (and) registered and insured mail services."
- Article 16(1) states that the Belgium postal operator must have a network sufficiently developed in order to "all communes in the Kingdom, [...], are provided with at least one access point for the deposit of the postal items."

The set of obligations is given by $O = \{p, t\}$ with $\{p\}$ the obligation of basic products and $\{t\}$ the obligation of territory development. Let $N = \{\{a, b, c\} \cup \{p, t\}\}$ be the set of players and $v : 2^N \to \mathbb{R}$ the characteristic function. It measures the common costs supported by the operator when it offers any subset of activities under the constrains of any subset of obligations. Then, we obtain the public service game $(N, v) \in \mathcal{P}$.

Coalition	Worth	Coalition	Worth
$\{a\}$	800	$\{a, b, p\}$	2400
$\{b\}$	200	$\{a, b, t\}$	3000
$\{c\}$	400	$\{a, c, p\}$	2300
$\{p\}$	0	$\{a, c, t\}$	2800
$\{t\}$	0	$\{a, p, t\}$	3100
$\{a, b\}$	900	$\{b, c, p\}$	1400
$\{a, c\}$	930	$\{b, c, t\}$	2200
$\{a, p\}$	2200	$\{b, p, t\}$	2100
$\{a, t\}$	2900	$\{c, p, t\}$	200
$\{b, c\}$	500	$\{a, b, c, p\}$	2700
$\{b, p\}$	1240	$\{a, b, c, t\}$	3100
$\{b, t\}$	2000	$\{a, b, p, t\}$	3150
$\{c, p\}$	200	$\{a, c, p, t\}$	3200
$\{c, t\}$	200	$\{b, c, p, t\}$	2200
$\{p, t\}$	0	$\{a, b, c, p, t\}$	3300
$\{a, b, c\}$	1100		

Table 1: Public service game

We can make several remarks on table 1. The common costs of operating coalitions $\{p\}$, $\{t\}$ and $\{p, t\}$ are equal to 0. Without any activity, the obligation of basic products $\{p\}$ cannot be met and the simple logo of the operator is not enough to establish territory development $\{t\}$. In other words, activities are required to give substance to these obligations. Observe that (N, v) is a monotonic game but not sub-additive. For instance, The common costs generated by the union of $\{a, b\}$ with $\{p\}$ is $v(\{a, b, p\}) = 2400$, which greatly surpasses the sum of their individual common costs, i.e,. obligations do not benefit from economies of scale.

From table 1, we can construct the activity game (A, v) and the obligation game (O, v^O). The activity game represents the network's common cost under competitive conditions, i.e., without obligations (see table 2).

Coalition	Worth
$\{a\}$	800
$\{b\}$	200
$\{c\}$	400
$\{a, b\}$	900
$\{a, c\}$	930
$\{b, c\}$	500
$\{a, b, c\}$	1100

Table 2: Activity game

The obligation game (O, v^O) models the incremental impact of the obligations on the common cost. It is described in table 3. For instance, the worth of the coalition $\{p\}$ is given by $v^O(\{p\}) = v(\{a, b, c, p\}) - v(\{a, b, c\})$ (see Definition 3).

Coalition	Worth
$\{p\}$	1600
$\{t\}$	2000
$\{p, t\}$	2200

Table 3: Obligation game

In Table 4, we apply the PS value on (N, v). We obtain an allocation of the common costs among the activities and the obligations. For clarity, we also express these allocations in terms of percentage.

$i \in N$	a	b	c	p	t
$PS_i(N, v)$	672	157	272	900	1300
%	20.4%	4.7%	8.2%	27.3%	39.4%

Table 4: Allocation of the common costs – PS value

We also apply the Shapley value to the public service game (N, v). This leads to an alternative allocation of the common costs among the activities and the obligations, which is given in Table 5.

$i \in N$	a	b	c	p	t
$Sh_i(N,v)$	1124	562	444	395	775
%	34%	17%	13%	12%	24%

Table 5: Allocation of the common costs – Shapley value

Observe that the Shapley value assigns fewer common costs to obligations. This is due to the fact that, contrary to the PS value, the Shapley value does not satisfy (IAO). Consequently, the presence of obligations has an impact on the cost allocated to activities. To see this, assume that the operator is longer subjected to the obligation of territory development, i.e., the obligation $\{t\}$ is removed from the public service game. We obtain a new public service game $(N \setminus \{t\}, v)$, in which only one obligation remains: $\{p\}$. Observe that the withdrawal of $\{t\}$ has no impact on the allocations prescribed by the PS value to the activities. However, the allocations prescribed by the Shapley value to the activities greatly changes, as shown by Table 6. This demonstrates that the Shapley value fails (IAO), and thus, may not be viewed as a desirable allocation rule in the context of public service games.

$i \in N$	a	b	c	p
$PS_i(N,v)$	672	157	272	1600
$Sh_i(N,v)$	1133	367	699	501

Table 6: Allocation of the common costs - PS value and Shapley value

As an additional remark, observe that, the PS value allocates $v^O(\{p\})$ to the remaining obligation $\{p\}$. This amount coincides with the profitability cost approach defined in the Introduction (see sub-section 1.1).

5. Additional content

This section focuses on cooperative games with a priority structure as introduced and studied by Béal et al. (2022) and Lowing and Techer (2022). In particular, Béal et al. (2022) proposed the Priority value as a single-valued solution for TU-games with a priority structure. In this section, we show that a TU-game with a priority structure can be constructed from any public service game. In addition, we show that the PS value of a public service game can be viewed as the Priority value of the associated TU-game with a priority structure.

5.1. Games with a priority structure

Let $N \subset \mathbb{N}$ be a non-empty and finite set of players. A priority structure on N reflects some exogenous rights, different needs, merit or hierarchical constraints between the players. A **partially ordered set**, or simply a poset, on N is a reflexive, antisymmetric and transitive binary relation on N. A priority structure on N can be represented by a poset $(N, \succeq)$ on the player set N. The relation $i \succeq j$ means that i has priority over j. The poset $(N, \succeq^0)$ containing no priority relations among pairs of distinct players is called the trivial poset. The class of all priority structures is denoted by $\mathbb{S}$. A poset $(N, \succeq)$ gives rise to the asymmetric binary relation $(N, \succ)$: $i \succ j$ if $i \succeq j$ and $i \neq j$. Two distinct players i and j are **incomparable** in $(N, \succeq)$ if neither $i \succeq j$ nor $j \succeq i$. For each player $i \in N$, define the **priority group** of i, denoted by $\uparrow_{\succ} i$, as the set of players having priority over i in $(N, \succeq)$

$$\uparrow_{\succ} i = \{j \in N : j \succ i\}.$$

For each nonempty $E \in 2^N$, the **sub-poset** $(E, \succeq^E)$ of $(N, \succeq)$ induced by E is defined as follows: for each $i \in E$ and $j \in E$, $i \succeq^E j$ if $i \succeq j$. We will also use the notation $(E, \succeq)$ instead of $(E, \succeq^E)$. A player $i \in N$ is a priority player in $(E, \succeq)$ if, for $j \in E$, the relation $j \succeq i$ implies $i = j$. The non-empty subset of **priority players** in $(E, \succeq)$ is defined as

$$Z(E, \succeq) = \{i \in E : \forall j \in E \text{ such that if } j \succeq i, \text{ then } j = i\}.$$

A TU-game with a priority structure is a triplet $(N, v, \succeq)$ such that $(N, v) \in \mathbb{G}$ and $(N, \succeq) \in \mathbb{S}$. The class of TU-games with a priority structure is denoted by $\mathbb{GS}$. Béal et al. (2022) introduce the Priority value on $\mathbb{GS}$. This value divides the Harsanyi dividend of each coalition among the priority players of this coalition.

Definition 4 (Priority value, Béal et al. (2022)). *The Priority value is the value Pr on $\mathbb{GS}$ defined, for each $(N, v, \succeq) \in \mathbb{GS}$, as*

$$\forall i \in N, \quad Pr_i(N, v, \succeq) = \sum_{\substack{E \in 2^N \\ Z(E,\succeq) \ni i}} \frac{\Delta_v(E)}{|Z(E, \succeq)|}. \tag{8}$$

The Priority value extends the Shapley value (see Shapley, 1953). Indeed the two values coincide when $(N, \succeq)$ is the trivial poset.

5.2. Relationships with public service games

Consider situations in which the set of players is partitioned into several priority classes $(N_1, , \ldots, N_q)$. Each priority class contains incomparable players that have priority over each player in the next class. Formally,

$$\forall i, i' \in N, \quad [i \succ i'] \iff [i \in N_p, i' \in N_{p'} \implies p < p'].$$

The set of players over which player $i \in N_p$ has priority is $N_{>p} = \bigcup_{p'>p} N_{p'}$. Denote the class of priority structures organized by classes by $\mathbb{C}$, and the class of TU-games with a priority structure in $\mathbb{C}$ by $\mathbb{GC}$. The sub-class of monotonic TU-games with a priority structure in $\mathbb{C}$ is denoted by $\mathbb{G}^m\mathbb{C}$. Béal et al. (2022) showed that the Priority value can be expressed by using the Shapley value on $\mathbb{GC}$.

Proposition 3 (Béal et al. (2022)). *Pick any $(N, v, \succeq) \in \mathbb{GC}$. For each $p \leq q$,*

$$\forall i \in N_p, \quad Pr_i(N, v, \succeq) = Sh_i(N_p, v^p),$$

where $v^p(E) = v(E \cup N_{>p}) - v(N_{>p})$ for each $E \subseteq N_p$.

Pick any public service game $(N, v) \in \mathcal{P}$. By definition, N is partitioned into two distinct sets A and O, and v is a monotonic map. The two sets A and O can be viewed as priority classes. Assume that players in O always have priority over players in A. Formally, assume that there exists a partial order $(N, \succeq)$ such that for any $i \in A$ and any $j \in O$, $j \succ i$. Consequently, we can construct a monotonic TU-game with a priority structure organized in two classes $(N, v, \succeq) \in \mathbb{G}^m\mathbb{C}$ from the public service game $(N, v) \in \mathcal{P}$. By applying the Priority value to the resulting game $(N, v, \succeq)$, and by using Proposition 3, we obtain, for each $i \in A$, $Pr_i(N, v, \succeq) = Sh_i(A, v^A) = Sh_i(A, v_A) = Sh_i(A, v) = PS_i(N, v)$, where $v_A(E) = v(E)$ for each $E \subseteq A$. Similarly, we obtain, for each $i \in O$, $Pr_i(N, v, \succeq) = Sh_i(O, v^O) = PS_i(N, v)$, where $v^O(E) = v(E \cup A) - v(A)$ for each $E \subseteq O$. This shows that the PS value of any public service game $(N, v) \in \mathcal{P}$ can be viewed as the Priority value of the corresponding game with a priority structure $(N, v, \succeq) \in \mathbb{G}^m\mathbb{C}$. Consequently, the following alternative expression of the PS value holds.

Definition 5. *The PS value is defined, for each $(N, v) \in \mathcal{P}$, as*

$$\forall i \in N, \quad PS_i(N, v) = \sum_{\substack{E \in 2^N \\ T(E) \ni i}} \frac{\Delta_v(E)}{|T(E)|},$$

where $T : 2^N \to 2^N$ is defined as

$$\forall E \in 2^N, \quad T(E) = \begin{cases} O \cap E & \text{if } O \cap E \neq \emptyset \\ E & \text{otherwise.} \end{cases}$$

The map T plays the same role as the map $Z(., \succeq) : 2^N \to 2^N$. To see this, pick any $(N, v) \in \mathcal{P}$ and consider its associated game with a priority structure $(N, v, \succeq) \in \mathbb{G}^m\mathbb{C}$. Pick any $E \in 2^N$. The set $Z(E, \succeq)$ contains the priority players in $(E, \succeq)$. Since obligations have priority over activities in $(N, v, \succeq)$, it follows that $Z(E, \succeq)$ consists of all obligations in E, i.e., $O \cap E$, assuming that $O \cap E$ is non-empty. In case, $O \cap E$ contains no players, the priority players in $(E, \succeq)$ is just E since it contains only activities, i.e., $E \cap A = E$.

Recall that the Shapley value (see Definition 1) distributes the Harsanyi dividend of each coalition equally among the players in the coalition. In contrast, the PS value distributes the Harsanyi dividend of each coalition equally among the obligations in the coalition, or among the activities in the coalition if there is no obligation in it. The interpretation is that any dividend produced by coalitions comprising obligations ought to be attributed solely those obligations. Consequently, the activities remain unaffected by these dividends, thus supporting the idea of (IAO) and upholding the principle of non cross-subsidization from obligations to activities.

6. Conclusion

This paper proposes a new methodology grounded in cooperative game theory to allocate the common costs of a public service operator among its activities and obligations. We introduce public service games and provide a solution to these games: the PS value. This solution verifies desirable axioms that prevent different forms of cross-subsidization. We apply our methodology to a case study from the network of a public service operator. Finally, we compare our approach with TU-games with a priority structure to highlight technical similarities.

For future research, it would be interesting to apply our methodology to alternative case studies stemming from public service sectors, including telecommunications, the energy industry, railway companies, and others. Furthermore, our methodology employs the well-known Shapley value. It would be interesting to explore alternative solution concepts from cooperative game theory and investigate their implications in our framework and methodology.

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