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Opinion Dynamics and Political Persuasion

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Abstract

This paper proposes to adapt a simple disease spread model for political persuasion. More precisely, we observe how a policy presented by a leader prevails into a population divided in two groups: subscribers and resistants. At each date, agents from the two groups meet and influence each other due to the leader's *persuasion force*. If the leader's *persuasion force* dominates (is dominated), then some resistants (subscribers) become subscribers (resistants). Moreover, agents can also change their opinions simply because of the *attractive force* of each groups (intrinsic attraction). In the long run, it appears that a high *attractive force* can compensate a lack of *persuasion force* to ensure that more than half members subscribe to the policy presented by the leader. Such a situation is stable. Conversely, a high *persuasion force*, when the *attractive force* of the leader's group is relatively low, can generate the occurrence of a two-period cycle through a flip bifurcation such that the leader looses the majority from a period to another.

Keywords: Flip bifurcation, Opinion dynamics, Political persuasion, SIS models.

JEL: C61, D72

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1 Introduction

The literature on political persuasion is focused traditionally on strategic interactions[1]. For instance, following Austen-Smith (1990), a debate is seen as a costless process and then is modeled as a "cheap-talk" in the spirit of Crawford and Sobel (1982). In his paper, Austen-Smith (1990) questions, in particular, conditions for which debate matter, that is, conditions under which debate is able to modify the outcomes when two legislators interact. He observes that this occurs in two situations: (1) when legislators' preferences are not too dissimilar and (2) when the unique role of debate is the timing (that is the amount of information and the end of the decision process is exactly the same than the one at the very beginning). However, the way people form beliefs is not always due to strategic interactions. In particular, relying Grasnovetter (1973) and Cialdini (1984), Murphy and Shleifer (2004) argue that peoples' beliefs are influenced by social networks like friends and co-workers and then are driven by friendship, emotions or group identity. On this basis, they develop a simple model of equilibrium beliefs where agents influence each other if and only if they belong into the same network. Within their simple setup, Murphy and Shleifer (2004) discuss conditions for which a "grand coalition" arises (i.e. the whole population lies in the same network) and for which two equal sized networks represent the equilibrium.

Within the following paper, we propose to complete this literature on political persuasion by considering, as Murphy and Shleifer (2004), that peoples' opinions are influenced by social interactions. In their paper, Murphy and Shleifer (2004) focus on a static model where social interactions are modeled by an influence function which is increasing and concave in the network size. Inspired by the recent increasing use of disease spread models in economics[2], we rather propose a dynamic set up where an opinion is shared by a group of individuals. Social interactions are captured by the probability to met an agent with another opinion. The weight of the opinion in the society is growing with the size of the group. Hence, an opinion will prevail in the society when it is shared in the long run by a majority.

In epidemiology, there exists a wide range of disease spread models[3], accounting for lifelong immunity (SIR model), temporary immunity (SIRS model) as well as disease related fatality or not. While persuasion spreads an opinion as

[1] A general overview of this strand of litterature is given by Austen-Smith (1992).

[2] See, among others, Alvarez et al. (2021), Bosi et al. (2021) or Eichenbaum et al. (2021).

[3] The reader interested in a general presentation of the most used disease spread models in epidemiology is referred to Brauer and Castillo-Chavez (2012).

contamination spreads an infectious disease, there is no immunity, nor fatality, related to an opinion and then, the epidemiological model we focus on is the SIS model. In the simplest SIS model, the population is divided in two groups: the susceptibles and the infectives. At each date, a susceptible can meet an infective with a probability given by the share of infectives into the population. This meeting could results in effective infections captured by an infection rate. At the same time, some infectives recover from the disease and then, get back to the susceptible class. In its simplest form, the SIS model abstracts from both immunity and death. The purpose of the present paper is to adapt the simplest SIS model to account for the spread of an opinion and then, to build a new framework for discussing political persuasion.

The story we analyze is the following: a political leader makes a speech concerning a new policy in front of a set of individuals (for instance in front of deputies during a session at the French National Assembly or in front of the people through a TV show during an electoral campaign) that are concerned by this new policy. At the end of the talk, the set of individuals divides into two groups: the subscribers to the policy and the resistants. At each date, a subscriber can meet a resistant with a probability captured by the share of resistants into the population. From this encounter, a subscriber (resp. a resistant) can convince a resistant (resp. a subscriber) who becomes a subscriber (resp. a resistant). Indeed, while a change of group due to a meeting in the SIS model only goes from susceptibles to infectives, in an opinion spread context, the change of group due to a meeting goes in both ways: from subscribers to resistants and from resistants to subscribers. The direction of this change depends upon the *persuasion force* of the leader. If the leader's *persuasion force* dominates (resp. is dominated), then the change of group due to a meeting goes from resistants (resp. subscribers) to subscribers (resp. resistants). Moreover, in the SIS model, a spontaneous change of group (i.e. without any meeting) goes only from infectives to susceptibles because of disease recovery. In an opinion spread context, a spontaneous change of group (i.e. without any meeting) goes also in the two directions. Indeed, the two groups have an intrinsic *attractive force*, which stand for the symbolic attractive power of a group or for the attractiveness of the leader's personality.

Within this simple framework, we obtaine interesting results concerning both the long and the short run. In particular, we observe that, despite a lack of *persuasion force*, a high *attractive force* can ensure that more than half members subscribe to the leader's policy in the long run. Moreover, such a compensation appears to be sufficient to ensure the stability of this long run situation. Interestingly, when a lack of *attractive force* is compensated by a high *persuasion force*, the persuasion process can converge toward a two-period cycle (flip bifurcation) where the leader looses the majority from a period to another, leading to an unstable situation for the leader.

The paper is organized as follow. Section 2 presents the SIS epidemic model and section 3 proposes an adaptation of this epidemic model to account for political persuasion. The long run as well as the local dynamics are studied within sections 4 and 5 while section 6 proposes a simulation. Section 7 concludes the paper. All proofs are gathered in the Appendix.

2 The SIS epidemic model

In epidemiology, one of the most simple model to represent the spread of a communicable disease is the SIS model. A complete exposition of the discrete time version can be found in Allen (1994) or more recently in Goenka and Liu (2012). Within this section, we propose to present briefly this widely used epidemic framework in order to discuss modifications to adapt it to depict opinion dynamics. Let a population of size N divided in two groups. The first group of size S designates the *susceptibles* and corresponds to healthy individuals likely to contract the disease. The second group of size I designates the *infectives* and corresponds to individuals who are already ill. Of course, in the SIS model it is assumed that $N = S + I$. Susceptibles and infectives may interact over time and at each date t, susceptibles meet infectives with probability $\frac{I_t}{N_t}$ where I_t (resp. N_t) describes the size of the group of infectives (resp. population) at date t. In this model, the probability that a susceptible meets a infective is simply described by the ratio measuring the percentage of infected individuals in the population. When a susceptible meets an infective, there exists a positive probability that the infective contaminates the susceptible. This probability is approximated by $\varepsilon \in (0, 1)$ which denotes the *contamination* rate in the population. Consequently, the number of susceptibles who contracts the disease at date t is equal to $\varepsilon \frac{I_t}{N_t} S_t$. The SIS model allows that a share $\sigma \in (0, 1)$ of infectives recover from the disease and then, get back to the susceptible class. σ is called the *recovery* rate. To simplify the exposition, it is assumed here that the population remains constant over time and then, there is no death related to the infectious disease[4]. This assumption is clearly in line with an adaptation of the

[4] The reader interested in SIS models with disease fatalities is referred to Hethcote (1976) among others.

SIS model to opinion dynamics problem. The SIS model is dedicated to analyze how the interactions between the two groups affect their size, in particular, it allows to observe conditions under which the disease persists in the long run.

From the previous discussion, the number of susceptibles evolves as follow:

$$S_{t+1} = S_t \left(1 - \varepsilon \frac{I_t}{N_t}\right) + \sigma I_t \tag{1}$$

where $S_t \left(1 - \varepsilon \frac{I_t}{N_t}\right)$ is the number of healthy susceptibles at the date t and σI_t is the number of infectives cured of the disease at date t. Symmetrically, the number of infectives evolves according to the following equation :

$$I_{t+1} = I_t \left(1 - \sigma\right) + \varepsilon \frac{I_t}{N_t} S_t \tag{2}$$

where $I_t \left(1 - \sigma\right)$ is the number of infectives who did not recover at the date t and $\varepsilon \frac{I_t}{N_t} S_t$ is the number of susceptibles who contracts the disease at date t. Equations (1) and (2) form the so-called SIS model and allow to discuss conditions on ε and σ for which the infectious disease is eradicated in the long run or becomes endemic.

Similarly to a contamination, the spread of an opinion into a population arises because of interactions between agents. However, there are two main differences between opinion dynamics applied to political persuasion and disease dynamics. To point out those differences, let us assimilate *subscribers* of an opinion to the infectives and the *resistants* to susceptibles. Firstly, the spread of a disease goes only from sick individuals to healthy ones. Conversely, the spread of a political opinion can go both ways. Indeed, for example during a meeting, subscribers are able to convince resistants while resistants are also able to convince subscribers. Secondly, considering disease dynamics, a spontaneous change of state, that is without contacts, only goes from infectives to susceptibles (disease recovering). Conversely, regarding opinion dynamics applied to political persuasion, a subscriber can change her mind spontaneously as well as a resistant. The next section adapts the SIS model to take into account those two specificities.

3 Theoretical approach for political persuasion

Let us consider a committee with N members. The committee's leader is the bearer of a new policy that she initially presents to the committee members (that is, at $t = 0$). After the presentation of the policy, the N members of the committee divide into two groups. The group S (for subscriber) designates the group of individuals who subscribe to the leader's policy while the group R (for resistant) designates the group of individuals who are opposed to her policy. That is, at each date in time,

$$N_t = S_t + R_t \quad (3)$$

To keep things as simple as possible, the committee size is assumed to remain constant over time (i.e., $N_t = N_{t+1}$). From $t = 0$, and at each date in time, committee members interact on the policy initially presented by the leader. The probability for a resistant to meet a subscriber at date t is given by S_t/N_t and then, the number of resistant who meet a subscriber is given by[5] $(S_t/N_t)\, R_t$. From those interactions, a resistant is willing to change her political opinion with the probability $\alpha \in [0, 1]$. That is, at date t, $\alpha\,(S_t/N_t)\, R_t$ represents the number of resistants who change their political opinion and become subscribers. At the same time, the number of subscribers who changes their political opinion is given by $\beta\,(R_t/N_t)\, S_t$ and becomes resistants such that $\beta \in [0, 1]$ represents the probability for a subscriber to change her political opinion because of an encounter with a resistant. That is, the net flow of resistants who becomes subscribers because of interactions at date t is given by $(\beta - \alpha)\,(S_t/N_t)\, R_t$ while the net flow of subscribers who becomes resistants due to a meeting is given by $(\alpha - \beta)\,(R_t/N_t)\, S_t$. For further reference, let us introduce the leader's *persuasion force*, namely $\theta \equiv \alpha - \beta \in [-1, 1]$. If $\theta > 0$ (resp. $\theta < 0$), then $\alpha > \beta$ (resp. $\beta > \alpha$) which means that the leader's persuasion power dominates (is dominated).

Moreover, committee members can also change their political opinion spontaneously due to *i*) the symbolic power of attraction of a group to another or *ii*) the leader's ability to embody a character that committee members can identify with. For instance, in the French policy landscape, the symbolic attractive power of the General De Gaulle or of Jean Jaurès are commonly invoked respectively by "les Républicains" and "le parti socialiste". In the following, we assimilate those *attractive forces* by $\lambda \in [0, 1]$ and $\gamma \in [0, 1]$. More precisely, γ (resp. λ) represents the *attractive force* of subscribers (resp. resistants). That is, γR_t (resp. λS_t) depicts the number of resistants (resp. subscribers) who change spontaneously their political opinion at date t and then become subscribers (resp. resistants).

Considering jointly the *persuasion force* and the two *attractive forces*, the

[5] Symetrically, the number of subscriber who meet a resistant is given by $(R_t/N_t)\, S_t$ and trivially, $(R_t/N_t)\, S_t = (S_t/N_t)\, R_t$.

number of resistants and the number of subscribers evolve as follows:

$$R_{t+1} = R_t(1 - \theta s_t) - \gamma R_t + \lambda S_t \quad (4)$$

$$S_{t+1} = S_t(1 + \theta r_t) + \gamma R_t - \lambda S_t \quad (5)$$

Let us introduce $r_t \equiv R_t/N_t$ and $s_t \equiv S_t/N_t$, respectively the share of resistants and the share of subscribers into the committee. By considering equation (3), $r_t = 1 - s_t$ and since $N_t = N_{t+1}$, equation (5) becomes:

$$s_{t+1} = (\theta s_t + \gamma)(1 - s_t) + (1 - \lambda) s_t \equiv \varphi(s_t) \quad (6)$$

Interestingly, we remark that the evolution of s is independent of r. Thus, the study of equation (6) is sufficient to give informations about the capacity of the leader's opinion to invade the committee depending upon her *persuasion force* and the two *attractive forces.* The purpose of the rest of this paper is precisely to discuss this possibility. Due to its non-linear nature, equation (6) is not analytically solvable. To capture the behavior of s through time, we propose, as usual, to proceed in three steps. First of all, section 4 will explore situations where s stops its evolution, that is, situations where $s_t = s_{t+1}$ (i.e. the existence of one or more steady state). Secondly, the (local) stability of each possible steady state will be analyzed within section 5. Finally, numerical simulations will be proposed in section 6 to illustrate results obtained within sections 4 and 5.

4 Long run

Since equation (6) is not analytically solvable, we begin the exploration by studying the possibility that s stops to evolve, that is, we analyze if there exists one or more s such that $s_t = s_{t+1}$. A value of s such that $s = s_t = s_{t+1}$ is typically called a steady state. A steady state represents a situation we could expect in the long run. To determine if the political context described by equation (6) effectively converges to a steady state, it is also needed to study the dynamics of the problem (section 5).

By considering that $s_t = s_{t+1} = s$, equation (6) writes:

$$\phi(s) = \varphi(s) - s = -\theta s^2 + (\theta - \lambda - \gamma) s + \gamma = 0 \quad (7)$$

$\phi(s)$ is a second order equation. Obviously, ϕ is concave in s if $\theta > 0$, linear in s if $\theta = 0$ and convex in s if $\theta < 0$. Moreover,

$$\phi(0) = \gamma > 0 \quad (8)$$

$$\phi(1) = -\lambda < 0 \quad (9)$$

$\forall s \in (0,1)$, ϕ is continuous and then, by application of the intermediate value theorem, it follows from equations (8) and (9) that, whatever the value of θ, ϕ possesses always one root (see Fig.1) in the range $(0,1)$.

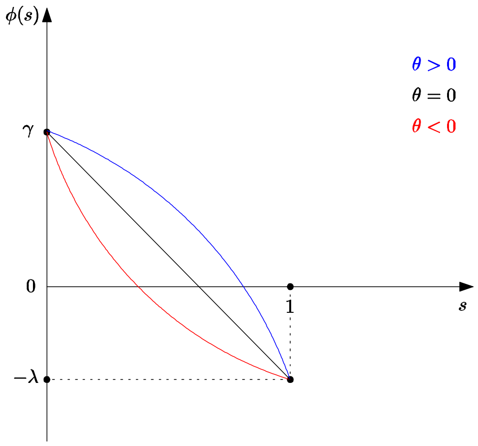

Fig. 1. $\phi(s)$ w.r.t θ.

The linearity of ϕ when $\theta = 0$ implies that this roots is unique. The concavity (resp. convexity) of ϕ when $\theta > 0$ (resp. $\theta < 0$) implies that there exists a second root which is negative (resp. higher than one). However, since, by definition, $s \in (0,1)$, it follows that equation (6) possesses a unique admissible steady state whatever the value of θ. More precisely, simple computation allows to obtain the two roots of equation (7) when $\theta \neq 0$, that is $\overline{s}$ and $\overline{\overline{s}}$ while, as discussed above, only one of those is an admissible steady state:

$$\overline{s} \equiv \frac{(\theta - \lambda - \gamma) + \sqrt{(\theta - \lambda - \gamma)^2 + 4\theta\gamma}}{2\theta}$$
$$\overline{\overline{s}} \equiv \frac{(\theta - \lambda - \gamma) - \sqrt{(\theta - \lambda - \gamma)^2 + 4\theta\gamma}}{2\theta}$$

Interestingly,

$$\overline{s} - \overline{\overline{s}} = \frac{\sqrt{(\theta - \lambda - \gamma)^2 + 4\theta\gamma}}{\theta} \lesseqgtr 0 \tag{10}$$

Considering considering (10), $\theta > 0$ ensures that $\overline{s} > \overline{\overline{s}}$. Recall the concavity of $\phi(s)$ in this case which indicates that $\overline{\overline{s}} < 0 < \overline{s} < 1$. That is, if $\theta > 0$, $\overline{s}$ is the only admissible steady state. If $\theta < 0$, $\overline{\overline{s}} > \overline{s}$, since $\phi(s)$ is convex in this case, it follows that $0 < \overline{s} < 1 < \overline{\overline{s}}$. That is, in this case too, $\overline{s}$ is the only admissible steady state. Finally, applying the L'Hôpital's rule we observe that both $\overline{s}$ and $\overline{\overline{s}}$ converge to the unique root of ϕ when $\theta = 0$:

$$\lim_{\theta \to 0} \overline{s} = \lim_{\theta \to 0} \overline{\overline{s}} = \frac{\gamma}{\lambda + \gamma}$$

This discussion leads to the following proposition.

Proposition 1 *Whatever* $\theta \in [-1, 1]$*, equation (6) possesses always a unique steady state given by* $\overline{s} \in [0, 1]$.

The steady state value $\overline{s}$ depends on *i*) the sign of the *persuasion force* θ that is the ability of the leader to convince the resistants to change their opinion and *ii*) the comparison between the *persuasion force* θ and the *attractive forces* of each group γ and λ. For a political leader, the existence of a unique steady state is meaningless if $\overline{s} < 1/2$. Indeed, to democratically impose her policy, it is necessary for the leader that more than half members are subscribers, that is $\overline{s} > 1/2$. The next proposition studies this possibility. For this purpose, let us introduce a threshold level for the *persuasion force*:

$$\hat{\theta} \equiv 2(\lambda - \gamma)$$

The threshold value of $\hat{\theta}$ depends on the comparison between the *attractive forces* of each group. More precisely, when the *attractive force* of resistants λ (resp. suscribers γ) dominates the *attractive force* of suscribers γ (resp. resistants λ), the threshold $\hat{\theta}$ becomes positive (resp. negative).

Proposition 2 *More than half members are subscribers in the long run, that is* $\overline{s} > 1/2$*:*

(1) when $\theta = 0$ *if and only if* $\hat{\theta} < 0$*,*

(2) when $\theta < 0$ *if and only if* $\hat{\theta} < \theta < 0$*,*

(3) when $\theta > 0$ *if and only if* $\hat{\theta} < \theta$*.*

Proposition 2 presents the different cases where the policy driven by the leader is successfully approved by the majority of the committee members in the long run. This success is closely related to the comparison between the *persuasion force* (i.e. θ) and the *relative attractive force* of resistants (captured by $\hat{\theta}$).

First of all, when the leader's persuasion power is neither dominant nor dominated, or equivalently, when the *persuasion force* is neutral (i.e. $\theta = 0$), more than half members subscribe to the leader's policy if and only if the *relative attractive force* of resistants is negative ($\hat{\theta} < 0$). More interestingly, when the *persuasion force* is positive ($\theta > 0$), the condition $\theta > \hat{\theta}$ is always verified when the *relative attractive force* of resistants is negative ($\hat{\theta} < 0$). This is not surprising because we have already observed that more than half members are subscribers when $\hat{\theta} < 0$ in the case where $\theta = 0$. This result is simply magnified when $\theta > 0$. Finally, when the *persuasion force* is negative ($\theta < 0$), it is also necessary (but not sufficient in this case) that the *relative attractive* *force* of resistants is negative ($\hat{\theta} < 0$) to ensure that more than half members are subscribers in the long run. Clearly, when $\theta < 0$, if $\hat{\theta} > 0$, then it is never possible to obtain $\overline{s} > 1/2$. In this case, a political leader is ensured to fail in imposing her policy.

This analysis clearly shows that the *relative attractive force* is a key determinant to impose her policy in all three cases of Proposition 2. Remember that the *attractive force* of each group stems for the symbolic power of attraction of a group to another or the leader's ability to embody a character that committee members can identify with. That is, a strategy to impose her specific policy could be to refer to an iconic figure, such that the reference to the General De Gaulle for most of right hand political parties during the 2022 French presidential election. Another strategy to improve the *attractive force* of the suscribers group could be also to clean the public picture of their political party. The leader has to adopt an attitude and a positioning in which the members of the resistants group could find themselves. For instance, Marine Le Pen is engaged since several years in a normalization strategy ("dédiabolisation") of her political party, notably by schifting away her father from the Front National and in renaming the "Front National" in "Rassemblement national".

In political competition, there is several examples where a candidate with a poor *persuasion force* (which corresponds to the case where $\theta \leq 0$) succeeded to win an election notably due to her *attractive force*. For instance, President François Hollande's victory in 2012 was a surprise for most political experts. One advanced reason lies in the failure of his principal opponent, the outgoing President Nicolas Sarkozy who was gradually discredited during his mandate through his hyperactivity and numerous scandals. But President François Hollande was substantially under estimated about his ability to win the election by the other candidates even whom from his own camp. He was strategically smarter and better advised on how to convince the most skeptical voters. To fit well with the majority of the electorate, his consultants in political communication advised him to appear as "Mister normal". The identification process even went as far as the media coverage of her spectacular weight loss. This strategy was successful since he succeeds to adapt his image to that of the majority electorate. In order to compare, during the second round of the electoral campaign of 2017, the victory of President Emmanuel Macron is the result of the combination of a strong *persuasion force* (notably during the second round debate) and a weak *attractive force* of his opponent Marine Le Pen who suffers from the history of her political party. A strong *persuasion force* coupled with a weak opponent *attractive force* has lead to a confortable victory where two thirds of the voters voted for him.

5 Local dynamics

The previous section has pointed out that there always exists a unique steady state in the opinion dynamics described by equation (6), namely $\overline{s}$, as well as conditions under which this steady state is such that more than half committee members are subscribers. To ensure that the dynamic process characterized by equation (6) allows to converge effectively toward this desirable situation, it is necessary to analyze the stability of $\overline{s}$. To capture the dynamics around the steady state, we focus on the slope of $\varphi(s)$ evaluated at $\overline{s}$ in the three cases considered within Proposition 2, that is, when the *persuasion force* is neutral ($\theta = 0$), negative ($\theta < 0$) and positive ($\theta > 0$).

$$\varphi'(\overline{s}) = 1 - (\lambda + \gamma) + \theta\,(1 - 2\overline{s}) \tag{11}$$

A steady state s is locally stable if and only if $|\varphi'(\overline{s})| < 1$ and locally unstable if and only if $|\varphi'(\overline{s})| > 1$. In one dimensional discrete time systems, a steady state looses its local stability through a local bifurcation[6] when $\varphi'(\overline{s}) = -1$ or when $\varphi'(\overline{s}) = 1$. If $\varphi'(\overline{s}) = -1$, the dynamic system does not converge toward a steady state but toward a periodic orbit, generating a two-period cycle[7]. If $\varphi'(\overline{s}) = 1$, the dynamic system undergoes a saddle-node (or fold) bifurcation[8]. When it occurs, two steady states collide and exchange their stability properties.

Proposition 3 *(Neutral persuasion force) If $\theta = 0$, the steady state is always locally stable.*

Proposition 3 implies that if the initial proportion of subscribers, s_0, among the committee members is close to the proportion at the steady state $\overline{s}$, then the dynamic process will converge toward $\overline{s}$, that is $\lim_{t\to+\infty} s_t = \overline{s}$. Considering jointly Propositions 2 and 3, it follows that if the *attractive force* of subscribers dominates the one of the resistants, then, despite the neutral leader's *persuasion force* (i.e. $\theta = 0$), the leader will effectively succeed to implement her policy, that is, $\lim_{t\to+\infty} s_t = \overline{s} > 1/2$.

[6] The reader interested in a general exposition of bifurcation theory is invited to refer to Kuznetsov (1998, in particular chapter 4) as well as to Grandmont (2008) among others.
[7] See Grandmont (2008, section 3.3) among others.
[8] See Grandmont (2008, section 3.2) among others.

Proposition 4 *(Negative persuasion force) If $\hat{\theta} < \theta < 0$, then the steady state is always locally stable.*

Focusing on case (2) of Proposition 2, $\hat{\theta} < \theta < 0$ implies that more than half committee members are subscribers at the steady state. Proposition 4 completes this result by pointing out that if s_0 is close to $\overline{s}$, then $\lim_{t \to +\infty} s_t = \overline{s} > 1/2$. That is, if the leader has a negative *persuasion force* compensated by a strong *relative attractive force* (see discussion after Proposition 2), it is sufficient to effectively succeed to impose her policy.

Now, let us introduce a new threshold for θ:

$$\theta^f \equiv \frac{\hat{\theta}}{2} + 2\sqrt{1 - \lambda\gamma}$$

Remark that $\theta^f > \hat{\theta}$ if and only if:

$$2\sqrt{1 - \lambda\gamma} > \lambda - \gamma \tag{12}$$

Proposition 5 *(Positive persuasion force) Let condition (12) holds and assume that $\theta > 0$ jointly with[9] $\theta > \hat{\theta}$:*

(1) If $\hat{\theta} < \theta < \theta^f$, then the steady state is always locally stable.

(2) If $\theta = \theta^f$, a two-period cycle emerges near the steady state through a flip bifurcation.

(3) If $\theta > \theta^f$, then the steady state is always locally unstable.

Simple computation allow to observe that $\theta^f < 1$ if and only if[10]:

$$\lambda > 1 - \gamma + 2\sqrt{1 - \gamma} \tag{13}$$

That is, periodic cycles and instability require a sufficiently high *attractive force* of resistants regarding the one of subscribers (see condition (13)) and a sufficiently high leader's *persuasion force* ($\theta \geq \theta^f$). It is interesting to remark that those two forces go in opposite direction: the high leader's *persuasion force* attracts committee members to subscribe while the high *attractive force* of resistants attracts committee members to resist. The magnitude and the simultaneity of those two opposite forces generates an instability. That is, despite the fact that $\overline{s} > 1/2$ (case (3) of Proposition 2), the leader is not able to reach a stable situation in the long run when $\theta > \theta^f$. Such a situation sounds like the yellow vest crisis after the first election of the French President Emmanuel

[9] Remember that $\hat{\theta} \lesseqgtr 0$.

[10] $\widehat{\theta} < \theta^f < 1$ if and only if condition (12) and (13) are simultanously met. We provide in the next section numerical simulations where this is true.

Macron: His *persuasion force* was high enough to win the presidential election ($\theta > \theta^f$) but at the same time his new born political party has not the power of a strong iconic figure like the ones of the opposition parties. From Proposition 5, it appears that the gap between President Macron's *persuasion force* and the weak relative *attractive force* of his political party could be seen as an explanation of the yellow vest crisis.

Proposition 3 and 4 hightlight that if a leader has a poor *persuasion force* ($\theta \leq 0$), she has to compensate it by a strong *attractive force* ($\hat{\theta} < \theta$). This combination is sufficient to guarantee the existence of a *winning* steady state that is $\lim_{t\to\infty} s = \overline{s} > 1/2$. Conversely, the case (2) and (3) of Proposition 5 shows that if the *attractive force* of the suscribers group is weak (condition (13)) then it is not always possible to compensate it with a strong *persuasion force* in order to guarantee the existence of a *winning* steady state.

6 Simulation

Within this section, we propose a numerical simulation using Python 3.9 in order to illustrate the loss of stability when $\theta > \theta^f$ (see Proposition 5). To do so, let $\gamma = 0.95$. Considering a value of $\lambda < 0.95$ ensure that $\theta^f > \hat{\theta}$ (see condition (12)). Moreover, $\forall\lambda > 0.49721$, $\theta^f < 1$ (see condition (13)). That is, for the rest of this numerical exploration, let us consider $\lambda = 0.85$. In this case,

$$\theta^f = 0.7775$$

Proposition 5 has pointed out the (local) convergence of s to $\overline{s}$ when $\theta < \theta^f$ as well the convergence to a two-period cycle when $\theta = \theta^f$ (i.e. flip bifurcation) and (local) instability of $\overline{s}$ when $\theta > \theta^f$. While the previous section was focused on a local analytical analysis of the dynamics, the numerical exploration allow us to observe what happens for values of s_0 far from $\overline{s}$. In other words, this numerical simulation allows us to approach the global dynamics described by equation (6) and to observe what happens when $\theta > \theta^f$. Indeed, the local instability could refer to a wide range of situations like an increase in the magnitude of the two-period cycle as well as to the birth of chaotic dynamics[11].

To illustrate Proposition 5, we propose to simulate the bifurcation diagram of equation (6) when $(\gamma, \lambda) = (0.95, 0.85)$. A bifurcation diagram shows the long run behavior of a dynamical system for different parameter values. Figure 2 depicts the bifurcation diagram for $\theta \in [0, 1]$ when $s_0 = 0.1$. As we can see, s converges to its steady state value for $\theta \in [0, 0.7775[$ while $\forall\theta \geq 0.7775$,

[11] See Grandmont (1985) among others.

s converges toward two distinct values (two-period cycle). Interestingly as θ increases, the magnitude of the two-period cycle increases too. In particular, we observe that the magnitude of the two-period cycle could be so high that the value of s alternates between two values: one is higher than 1/2 whereas the other is lower.

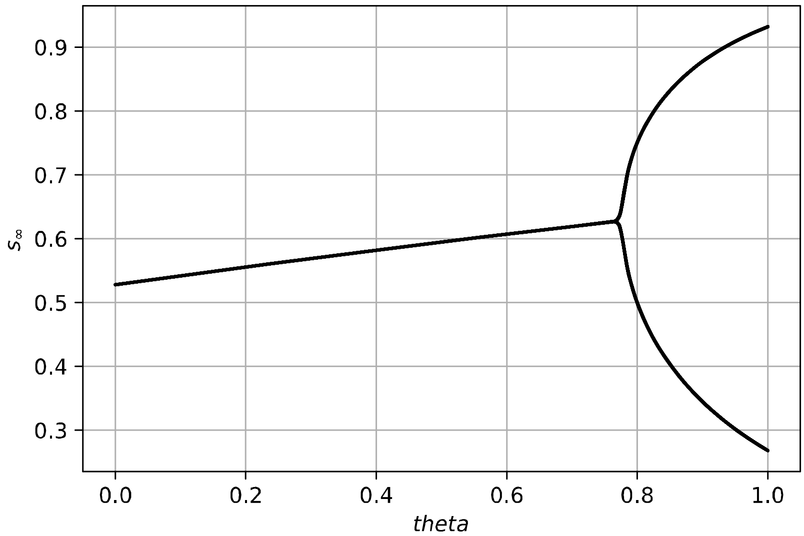

Fig. 2. Bifurcation diagram

To illustrate the two-period cycle arising when $\theta \geq 0.777\,5$, Figure 3 represents the time evolution of s when $\theta = 0.85$.

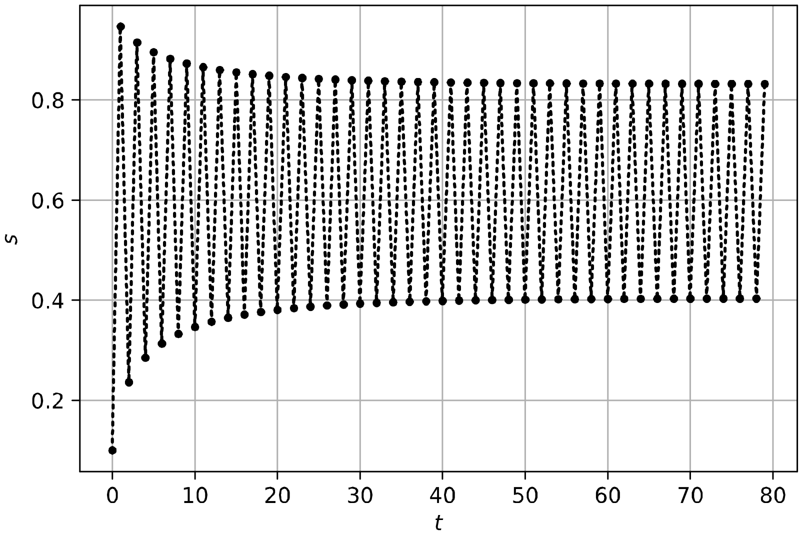

Figure 3. Trajectory of s when $\theta = 0.85$.

Within Figure 3, we observe that s fluctuates between the two values $s_{\min}$ and $s^{\max}$ such that $s_{\min} < 1/2 < s^{\max}$. This indicates that a leader with a high *persuasion force* ($\theta = 0.85$) looses the majority from a period to another.

7 Conclusion

Within this paper, we have adapted a simple disease spread model to account for the spread of a political opinion into a population. Two key forces has been introduced: the *persuasion force*, which denotes the ability of a leader to convince, and the *attractive force*, which captures the symbolic attractive power of a political party as well as the attractive power of a leader. Concerning the long run, we have observed in particular that a high *attractive force* can compensate a lack of *persuasion force* to ensure that more than half members subscribe to the policy presented by the leader. Interestingly, such a compensation is also sufficient to ensure the stability of such a long run situation. However, in case where the *persuasion force* is high, it appears that a lack of *attractive force* can imply the emergence of a two-period cycle, through a flip bifurcation, where the leader looses the majority from a period to another. The purpose of this paper was to develop a very simple model of opinion spread applied to political persuasion. This simplicity allows to propose a lot of modifications which are left for future researches.

Appendix

Proof of Proposition 2. Let us consider the different cases for the possible values of θ. Firstly, when $\theta = 0$, $\overline{s} > 1/2$ implies $\frac{\gamma}{\lambda+\gamma} > \frac{1}{2} \Leftrightarrow \gamma > \lambda \Leftrightarrow \hat{\theta} < 0$.

Now, focus on the case where $\theta < 0$:

$$\overline{s} > 1/2 \Leftrightarrow \sqrt{(\theta - \lambda - \gamma)^2 + 4\theta\gamma} < \lambda + \gamma$$

That is, $\overline{s} > 1/2$ if and only if $\theta > \hat{\theta}$.

Finally, focus on the case where $\theta > 0$:

$$\overline{s} > 1/2 \Leftrightarrow \sqrt{(\theta - \lambda - \gamma)^2 + 4\theta\gamma} > \lambda + \gamma$$

That is, $\overline{s} > 1/2$ if and only if $\theta < \hat{\theta}$.■

Proof of Proposition 3. Focus on equation (11) and assume that $\theta = 0$:

$$\varphi'(\overline{s}) = 1 - (\lambda + \gamma)$$

Since $\gamma < 1$ as well as $\lambda < 1$, then $-1 < \varphi'(\overline{s}) < 1$.■

Proof of Proposition 4. Focus on equation (11):

$$\varphi'(\overline{s}) = 1 - (\lambda + \gamma) + \theta\,(1 - 2\overline{s})$$

since $\hat{\theta} < \theta < 0$, that is $\overline{s} > 1/2$ (see Proposition 2), it follows that $\theta\,(1 - 2s) > 0$ and then, $\varphi'(s) > -1$. Moreover, focusing on the expression of $\overline{s}$,

$$\varphi'(\overline{s}) = 1 - \sqrt{(\theta - \lambda - \gamma)^2 + 4\theta\gamma} < 1$$

To sum up, $\hat{\theta} < \theta < 0$ ensures that $-1 < \varphi'(\overline{s}) < 1$.■

Proof of Proposition 5. Focus on the case where $\theta > 0$ jointly with $\theta > \hat{\theta}$ (i.e. $\overline{s} > 1/2$, see Proposition 2). It follows from equation (11) that $\varphi'(\overline{s}) < 1$. Now, let us discuss conditions under which $\varphi'(\overline{s}) > -1$:

$$\varphi'(\overline{s}) > -1 \Leftrightarrow \omega\,(\theta) < 0$$

with:

$$\omega\,(\theta) \equiv \theta^2 - 2\theta\,(\lambda - \gamma) + (\lambda + \gamma - 2)\,(\lambda + \gamma + 2)$$

It appears that $\omega\,(\theta)$ is a convex function such that $\omega\,(0) < 0$. That is, ω possesses two roots θ_0 and θ^f such that $\theta_0 < 0 < \theta^f$ and $\omega\,(\theta) < 0$ if and only if $0 < \theta < \theta^f$ (recall that $\theta > 0$). Simple computation allows to obtain the analytical expression of the two roots:

$$\theta_0 = \lambda - \gamma - 2\sqrt{1 - \lambda\gamma}$$
$$\theta^f = \lambda - \gamma + 2\sqrt{1 - \lambda\gamma}$$

That is, $\forall\theta < (>)\,\theta^f$, $\varphi'(\bar{s}) > (<) -1$ while $\varphi'(\bar{s}) = -1$ if and only if $\theta = \theta^f$.■

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