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Minimax regret in the 11-20 money request game

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Abstract

Arad and Rubinstein's 11-20 money request game nicely triggers level-*k* reasoning. Yet we show that mixed-strategy minimax regret, in a general class of money-request games, mimics a level-*k* reasoning, at least if the number of players is supposed to decrease in the depth of reasoning. We also show that, in this class of games, the minimax regret probability distribution is the exact reverse of the mixed-strategy Nash equilibrium distribution, and that minimax regret leads to a larger expected payoff than the Nash equilibrium payoff.

Keywords: minimax regret, level-k reasoning, money-request game, Nash equilibrium

JEL Classification: C72

1.Introduction

Arad and Rubinstein (2012) presented the 11-20 money request game as the game that naturally triggers level-*k* reasoning. In this 2-player game, each player requests an amount of money. This amount is an integer between 11 and 20. Each player receives the amount he asks for. And a player gets an additional amount of 20 if he requests exactly one unit less than the other player.

Arad and Rubinstein (2012) are partly right when claiming that this game proves the existence of level-k reasoning, in that the level-k behavior cannot be easily obtained by other common ways of reasoning. As a matter of facts, there is no dominated strategy, so iterative dominance cannot lead to the same behavior as level-k reasoning (by contrast to the guessing game, see Nagel 1995), and there is also no pure strategy Nash equilibrium (by contrast to the guessing game). Moreover, the Pareto dominant states are strategically unstable, and conflict between the two players is circumscribed in that each player gets the amount he asks for, regardless of what is played by the opponent. What is more, by contrast to many games, the level-0 behavior in this game seems to find consensus: by playing 20, a player is sure to get 20, which is a very large (unusual) maxmin payoff. Hence a player who does not want to do an iterative reasoning, surely requests 20. Finally, the game is very easy and needs few (almost no) cognitive skills to do a level-k reasoning. By contrast to guessing games where each additional step needs to calculate a new mean, here each additional step just consists in decreasing the requested amount by 1; this fact namely explains that a slightly simplified version of Arad and Rubinstein's game has been used in experiments with very young children, from five years old (see Fe et al, 2020). For all these reasons, the 11-20 money request game seems to be a good game to test level-*k* reasoning.

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Yet, as observed by Li and Rong (2018), by contrast to other level-*k* testing games like the guessing game, in which the players always get the same amount (1 for the winner and 0 for the losers), the payoff obtained by a player in the money request game strongly depends on the chosen number. The maximal and minimal amounts a player can win with a given integer are increasing in the played number (except for 20): a player gets at least 19 and at most 39 by playing 19 but only at least 11 and at most 31 by playing 11. This fact may trigger behavioral traits that are different from level-*k* reasoning. Li and Rong (2018) mentioned risk aversion, that indeed leads players to playing large numbers more frequently.

Minimax regret (Renou and Schlag, 2010, Halpern and Pass, 2012) is another way to approach the game, in that the 11-20 money request game exhibits a strong strategic uncertainty: according to Pearce's concept (1984), each amount is rationalizable (because each request x from 11 to 19 is the best response to the request x+1, and 20 is the best response to the request 11). So it is difficult to anticipate the other's behavior and each decision may generate a regret. Players may take regret into account. In fact, mixed-strategy minimax regret mimics level-*k* reasoning up to a certain depth of reasoning, providing one assumes that the number of players is decreasing in the depth of reasoning. So, in some way, level-*k* behavior can also be obtained by regret minimization despite it conveys a completely different philosophy. Garcia-Pola (2020) already observed a link between level-1 reasoning and iterative pure-strategy minimax regret. But pure-strategy minimax regret only focuses on maximal regrets, which is very restrictive. Mixed-strategy minimax regret better exploits all the regrets in the game and leads to a much more interesting link between minimax regret and level-*k* reasoning.

The aim of this short paper is to show the link between mixed-strategy minimax regret and level- *k* reasoning as well as the link between mixed-strategy minimax regret and the mixed -strategy Nash equilibrium in a general version of the 11-20 money request game. We also show that the players' expected payoff is larger with minimax regret than with the Nash equilibrium. Finally we share some behavioral comments out of a classroom experiment.

In section 2, we start by showing how 410 students played the 11-20 money request game in a classroom experiment at the Strasbourg University. In section 3 we establish the mixed-strategy minimax regret behavior in the 11-20 money request game, before looking for this behavior in a general money request game. We show that minimax regret mimics a level-*k* reasoning in which the percentage of players is decreasing in the depth of reasoning. In section 4 we show that the mixed-strategy minimax regret distribution is the exact reverse of the mixed-strategy Nash equilibrium distribution, and we discuss the larger expected payoff it gives rise to. Section 5 gives some behavioral comments out of the Strasbourg's classroom experiment: Arad and Rubinstein's 11-20 money request game reveals to be richer than expected by its authors.

2. 11-20 money request game, a classroom experiment

The classroom experiment was run during a third-year class in game theory at the faculty of economics and management of the university of Strasbourg in the academic year 2021-2022. The students played the game before knowing the concepts of dominance and Nash equilibrium, but they knew the notion of a normal form matrix game. So, when taking their decision, they had in front of them the normal form matrix of the 11-20 money request game (matrix 1). They were also invited to explain their decision in a few sentences. In total 410 students participated to the experiment. The question asked to the student was: “Imagine you are player 1 and you are confronted to another student, player 2, randomly selected in the amphitheater. Which amount do you ask for?” To facilitate the reading of the matrix, player 1’s payoffs in the matrix were written in red.

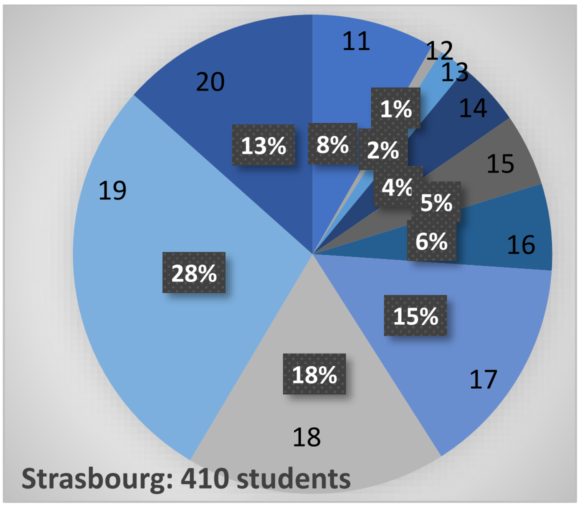
		Pl2									
		11	12	13	14	15	16	17	18	19	20
	11	(11,11)	(31,12)	(11,13)	(11,14)	(11,15)	(11,16)	(11,17)	(11,18)	(11,19)	(11,20)
	12	(12,31)	(12,12)	(32,13)	(12,14)	(12,15)	(12,16)	(12,17)	(12,18)	(12,19)	(12,20)
	13	(13,11)	(13,32)	(13,13)	(33,14)	(13,15)	(13,16)	(13,17)	(13,18)	(13,19)	(13,20)
	14	(14,11)	(14,12)	(14,33)	(14,14)	(34,15)	(14,16)	(14,17)	(14,18)	(14,19)	(14,20)
Pl1	15	(15,11)	(15,12)	(15,13)	(15,34)	(15,15)	(35,16)	(15,17)	(15,18)	(15,19)	(15,20)
	16	(16,11)	(16,12)	(16,13)	(16,14)	(16,35)	(16,16)	(36,17)	(16,18)	(16,19)	(16,20)
	17	(17,11)	(17,12)	(17,13)	(17,14)	(17,15)	(17,36)	(17,17)	(37,18)	(17,19)	(17,20)
	18	(18,11)	(18,12)	(18,13)	(18,14)	(18,15)	(18,16)	(18,37)	(18,18)	(38,19)	(18,20)
	19	(19,11)	(19,12)	(19,13)	(19,14)	(19,15)	(19,16)	(19,17)	(19,38)	(19,19)	(39,20)
	20	(20,11)	(20,12)	(20,13)	(20,14)	(20,15)	(20,16)	(20,17)	(20,18)	(20,39)	(20,20)

Matrix1: 11-20 money request game

The results are given in table 1 and represented in figure 1.

Requested amount	*11*	*12*	*13*	*14*	*15*	*16*	*17*	*18*	*19*	*20*
Number of students	34	3	8	18	20	24	61	72	115	55

Table 1: Strasbourg university’s classroom experiment (2021/2022)



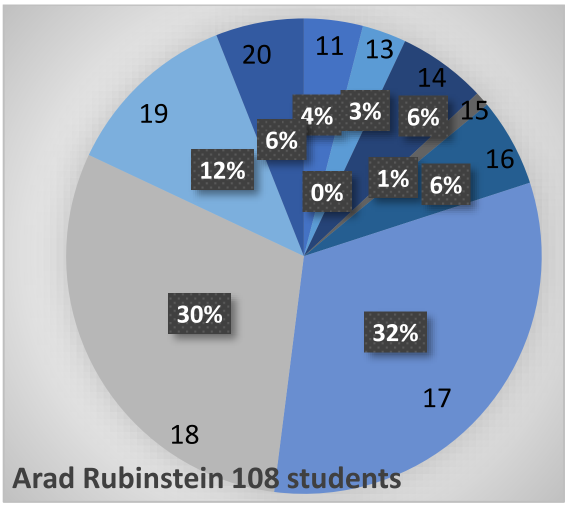


Figure 1: Strasbourg University’s Classroom experiment Figure 2: Arad and Rubinstein’s experiment

These results are rather different from Arad and Rubinstein’s results (see figure 2); the students at the university of Strasbourg played more often 19 and 11 and less often 18 and 17.

It follows from the classroom experiment that the increasing link between payoffs and requested amounts plays an important role. Many students view 19 as a sure way to get a large payoff with an opportunity to get 39 if the other player is cautious and wants to get 20 for sure. In fact, as will be commented longer in the conclusion, level-0 and level-1 reasoning play a role but are not the main motivations behind the requests 20 and 19. Level-*k* reasoning is more present in the students' explanations when they choose 18, 17 and even 16.

Even if the students' comments do not always fit with level-*k* reasoning, figure 1 is in line with a usual observation (namely in guessing games) according to which the number of players doing a level-*k* reasoning is decreasing in the depth of reasoning (k >0). So 28% of the students choose19 (level-1 reasoning), 17.6% of them play18 (level-2 reasoning), 14.9% of them play 17 (level-3 reasoning) and so on, down to 12 (see table 1). Only the fact that 8.3% of the students play 11 is not in accordance with this fact.

In the same way, if we omit the 8.3% of students playing 11, or better if we could redirect them to the amount 20, we observe that the students' behavior also rather well fits with the mixed-strategy minimax regret behavior given in figure 3. We now turn to this concept.

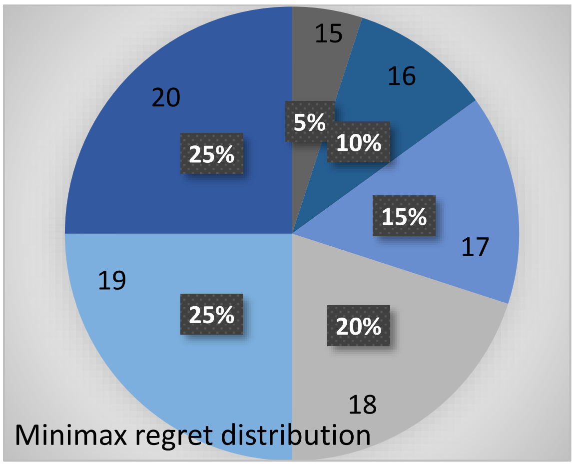


Figure 3: minimax regret strategy

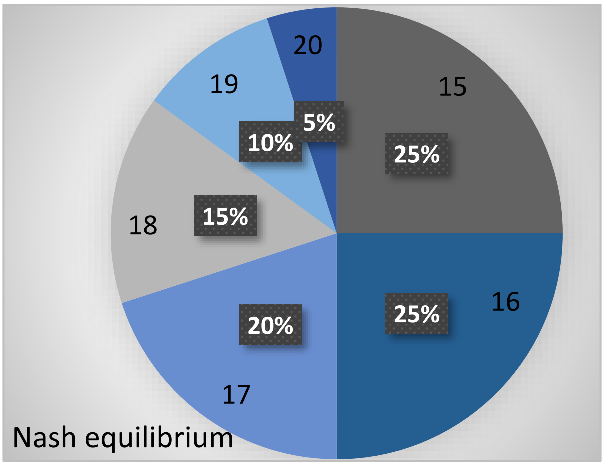


Figure 4: Nash Equilibrium

3. Minimax Regret in the Money Request Game

The pure strategy minimax regret concept goes as follows: in a normal-form game with N players i, pure strategy sets S_i and utility functions u_i, player i's regret by playing the pure strategy s_i when the opponents play the pure strategies s_{-i} is $r_i(s_i, s_{-i}) = \max_{\sigma_i \in S_i} u_i(\sigma_i, s_{-i}) - u_i(s_i, s_{-i})$. The maximal regret s_i leads to is $R_i(s_i) = \max_{s_{-i} \in S_{-i}} r_i(s_i, s_{-i})$. Player i's pure-strategy minimax regret is $\min_{s_i \in S_i} R_i(s_i)$ (see Halpern and Pass, 2012 and Renou and Schlag, 2010 for more details).

The regret a strategy leads to, given an opponent's strategy, is the difference between the best-reply payoff and the payoff obtained with the chosen strategy. For example, in the 11-20 money request game, if player 1 asks for 14 whereas player 2 asks for 17, player 1's best response is 16, and so player 1's regret, r(14,17), is 36-14=22. The maximal regret the amount 14 may lead to, R(14), is obtained when player 2 chooses 20, in which case the best response is 19 and the maximal regret assigned to 14 is 39-14=25.

The regret matrix for player 1 in the 11-20 money request game is given in matrix 2. The maximal regret assigned to each amount m, R(m), m from 11 to 20, is in bold in the matrix.

		Pl2									
		11	12	13	14	15	16	17	18	19	20
	11	9	0	21	22	23	24	25	26	27	**28**
	12	8	*19*	0	21	22	23	24	25	26	**27**
	13	7	18	*19*	0	21	22	23	24	25	**26**
	14	6	17	18	*19*	0	21	*22*	23	24	***25***
Pl1	15	5	16	17	18	19	0	21	22	23	**24**
	16	4	15	16	17	18	*19*	0	21	22	**23**
	17	*3*	14	15	16	17	18	*19*	0	21	**22**
	18	*2*	13	14	15	16	17	18	*19*	0	**21**
	19	1	12	13	14	15	16	17	18	***19***	0
	20	0	11	12	13	14	15	16	17	18	***19***

Matrix 2: player 1's regret matrix

This matrix brings into light the characteristics of the regrets. The regrets in italics, constant and equal to 19, express the fact that if both players request x, each player regrets not requesting x-1: he would receive the additional amount B-1 by doing so, B being the bonus equal to 20 in Arad and Rubinstein's game (AR's game). The regrets in the last column are the regret a player suffers from when he plays x different from 19 and the opponent plays 20. The lower the requested amount x, the more he suffers, because he suffers both from the loss of the bonus 20 and from the difference 19-x.

The matrix is quite regular in structure. On a same line x (the requested amount by player 1) the regrets are increasing in the opponent (player 2)'s amount y up to y=x, and they increase again in player 2's amount y from y=x+2 to y=20. Except for x=19, the maximal regret R(x) is always obtained in the last column, when the opponent plays 20. This is due to the fact that the regret in this column, except for 0, is B+19-x, whereas, in column y, with 11<y<20, it is B+y-1-x, which is lower by construction.

Garcia-Pola (2020) rightly observed that the pure-strategy minimax regret in this game is obtained for x=19 and x=20, and that applying the pure strategy minimax regret concept in an iterative way (Halpern and Pass, 2012) leads to x=19. By comparing minimax regret with level-*k* reasoning, he concluded that, in this game, minimax regret and level-1 reasoning lead to request the same amount 19.

In this paper we go further by turning to mixed-strategy minimax regret. We call x, respectively y, the amounts played by player 1, respectively player 2 (the opponent). The regrets in any line x are regularly increasing in the opponent's amount y because they are equal to y-1+B-x (except for y=x+1). Comparing adjacent lines x and x+1(for example the underlined lines 17 and 18), leads to observe that player 1's regrets are always lower in line x+1 than in line x, except if the other player plays x+1 (in this case, player 1's regret is null when he plays x and equal to 19 when he plays x+1). This fact expresses that a player regrets the unit of money he deliberately and systematically loses when he plays x instead of x+1, except if the other player fortunately plays x+1. Similar observations can be made by comparing two adjacent columns, y and y+1 (for example the squared columns 17 and 18). This time the regrets are systematically one unit larger in column y+1 than in column y, because they switch from B+y-1-x to B+y-x. The regrets in a same column, when the opponent plays y, are regularly decreasing in x (except for x=y-1). This follows from the fact that the regret B+y-1-x can be split into two parts: the regret player 1 suffers from when he plays 20, y-1+B-20 plus the regret he suffers from because he does not play 20, 20-x, except for x=y-1. So the regrets in column y are decreasing in x because 20-x is decreasing in x. This amount 20-x is often taken into account by the students, who wish to not move too far away from 20, the largest sure payoff they can get in this game.

This intuitively leads to the following expectation: a concept of regret should give a stronger probability to larger numbers. This expectation reveals to be right.

We work with Renou and Schlag's (2010) mixed notion of regret. The idea is to construct a mixed strategy that minimizes the regret, called z, regardless of the amount requested by the opponent. We call p_t player 1's probability of requesting t. This amounts to solving the optimization program:

$$\min_{z\, p_{11}p_{12}p_{13}p_{14}p_{15}p_{16}p_{17}p_{18}p_{19}p_{20}} z$$

$$\begin{aligned}
\text{u.c. } & 9p_{11} + 8p_{12} + 7p_{13} + 6p_{14} + 5p_{15} + 4p_{16} + 3p_{17} + 2p_{18} + 1p_{19} + 0p_{20} \leq z \\
& 0p_{11} + 19p_{12} + 18p_{13} + 17p_{14} + 16p_{15} + 15p_{16} + 14p_{17} + 13p_{18} + 12p_{19} + 11p_{20} \leq z \\
& 21p_{11} + 0p_{12} + 19p_{13} + 18p_{14} + 17p_{15} + 16p_{16} + 15p_{17} + 14p_{18} + 13p_{19} + 12p_{20} \leq z \\
& 22p_{11} + 21p_{12} + 0p_{13} + 19p_{14} + 18p_{15} + 17p_{16} + 16p_{17} + 15p_{18} + 14p_{19} + 13p_{20} \leq z \\
& 23p_{11} + 22p_{12} + 21p_{13} + 0p_{14} + 19p_{15} + 18p_{16} + 17p_{17} + 16p_{18} + 15p_{19} + 14p_{20} \leq z \\
& 24p_{11} + 23p_{12} + 22p_{13} + 21p_{14} + 0p_{15} + 19p_{16} + 18p_{17} + 17p_{18} + 16p_{19} + 15p_{20} \leq z \\
& 25p_{11} + 24p_{12} + 23p_{13} + 22p_{14} + 21p_{15} + 0p_{16} + 19p_{17} + 18p_{18} + 17p_{19} + 16p_{20} \leq z \\
& 26p_{11} + 25p_{12} + 24p_{13} + 23p_{14} + 22p_{15} + 21p_{16} + 0p_{17} + 19p_{18} + 18p_{19} + 17p_{20} \leq z \\
& 27p_{11} + 26p_{12} + 25p_{13} + 24p_{14} + 23p_{15} + 22p_{16} + 21p_{17} + 0p_{18} + 19p_{19} + 18p_{20} \leq z \\
& 28p_{11} + 27p_{12} + 26p_{13} + 25p_{14} + 24p_{15} + 23p_{16} + 22p_{17} + 21p_{18} + 0p_{19} + 19p_{20} \leq z \\
& p_{11} + p_{12} + p_{13} + p_{14} + p_{15} + p_{16} + p_{17} + p_{18} + p_{19} + p_{20} = 1 \\
& 0 \leq p_t \quad \text{t from 11 to 20}
\end{aligned} \tag{1}$$

Solving this program leads to the probabilities $\mathrm{p_i} = 0$, i from 11 to 14, $\mathrm{p}_{15} = \frac{1}{20}$, $\mathrm{p}_{16} = \frac{2}{20}$, $\mathrm{p}_{17} = \frac{3}{20}$, $\mathrm{p}_{18} = \frac{4}{20}$, $\mathrm{p}_{19} = \frac{5}{20}$, $\mathrm{p}_{20} = \frac{5}{20}$. The minimax regret is z=315/20=15.75.

This result generalizes to an 11-T money request game, with a bonus B, with B≥T>11+n, where n is the integer defined by: n(n+1)/2 <B<(n+1)(n+2)/2.

Proposition 1[1] *In the 11-T money request game with bonus B, B≥T>11+n, n being the integer defined by* $\frac{n(n+1)}{2} < B < \frac{(n+1)(n+2)}{2}$, *the minimax regret probabilities are given by:*
p_i=0 for i from 11 to T-n-1
$p_{T-n+i} = \frac{i+1}{B}$, *i from 0 to n-1*
$p_T = 1 - \frac{n(n+1)}{2B}$
The minimax regret, z, is equal to $B + \frac{n(n+1)(n+2)}{6B} - (n+1)$
Proof *See Appendix A*

p_T (p_{20} in AR's game) is just the complement to 1 of the sum $\sum_{i=T-n}^{T-1} p_i$. In AR's game, p_T= p_{T-1}= 5/20. By construction, in the general 11-T money request game with bonus B, the probability

[1] If B=n(n+1)/2 there is a family of minimax regret distributions, among them the distribution given in proposition 1.

assigned to the highest amount (that corresponds to the level-0 reasoning) is lower than or equal to the probability assigned to the amount T-1 (the level-1 reasoning amount).

Proposition 1 shows that p_i is linearly increasing in i, with i from T-n to T-1, with p_{T-n}=1/B, p_{T-1}=n/B and p_{i+1}-p_i=1/B. This result is in line with previous expectations: a player always regrets not playing y-1, where y is the amount played by the opponent, but this regret is decreasing in the amount x he plays, because y-1+B-x is decreasing in x.

Proposition 1 goes beyond Garcia-Pola's (2020) result. Garcia-Pola (2020) showed that the iterative pure-strategy minimax reasoning fits with level-1 reasoning which consists in requesting the amount 19. Proposition 1 shows that the mixed-strategy minimax regret probabilities fit with level-*k* reasoning, from level-1 to level-*n*, when each additional depth of reasoning is achieved by fewer persons. So, in the 11-20 money request game, the minimax regret probabilities fit with the following level-*k* behavior: 25% of the persons do a level-1 reasoning (hence play 19), 20% do a level-2 reasoning (i.e. play 18), 15% do a level-3 reasoning (i.e. play 17), 10% do a level-4 reasoning (i.e. play 16) and 5% do a level-5 reasoning (i.e. play 15). So, despite minimax regret has nothing to do with level-*k* reasoning, it selects a similar behavior in the 11-T money request game with bonus B, at least if one assumes that the percentage of players able to do a level-*k* reasoning in decreasing in *k*. Given that p_i=0 for i<T-*n*, this result, to be compatible with level-*k* reasoning, requires that nobody is able to do a level-(*n+1*) (or deeper) reasoning. B≥T>11+n and B>n(n+1)/2 imply n≥5, so this requirement is not very restrictive, in that in reality people seldomly do more than a level-4 reasoning (see for example Crawford, 2013). To put it more exactly, even if able to make more than 4 steps of level-*k* reasoning, people often refrain from doing so in that they fear that the other players are not able to do so.[2]

4. Minimax regret, Nash equilibrium and expected payoff

A funny result is the strange link between minimax regret and mixed Nash equilibrium. Usually, there is no obvious link between both concepts, except that they are quite different (see for example Umbhauer, 2020). But in this special game there is a strong link between both concepts, given in proposition 2.

Proposition 2 *In the 11-T money request game with bonus B (and B≥T>11+n with n the integer checking* $\frac{n(n+1)}{2} < B < \frac{(n+1)(n+2)}{2}$*), the played actions are the same in the mixed-strategy Nash equilibrium and with the mixed-strategy minimax regret. But the probabilities of the played amounts are reversed, that is to say:* $q_i = p_{2T-n-i}$ *for i from T-n to T, where* q_i*, respectively* p_i*, is the probability assigned to the amount i by the mixed-strategy Nash equilibrium, respectively the mixed-strategy minimax regret behavior.*

Proof. For a played amount i, the payoff $\mathrm{i}(1-\mathrm{q}_{i+1})+(\mathrm{i}+\mathrm{B})\mathrm{q}_{i+1}=\mathrm{i}+\mathrm{Bq}_{i+1}$ has to be equal to the payoff obtained with T, i.e. T. So q_i= (T-i+1)/B for i from T to T-n+1 and $q_{T-n} =$

[2] So it is theory of mind, the ability to enter into the other's reasoning, that often leads to stop the reasoning latest at level-4.

$1-\frac{n(n+1)}{2B}$, with $\frac{n(n+1)}{2} < B < \frac{(n+1)(n+2)}{2}$. It automatically follows that the amounts i, with i<T-n , lead to a lower payoff, hence q_i=0.

For example, in AR's game the Nash equilibrium probabilities are $q_i = 0\ for\ i = 11,12,13,14, q_{15} = \frac{5}{20}, q_{16} = \frac{5}{20}, q_{17} = \frac{4}{20}, q_{18} = \frac{3}{20}, q_{19} = \frac{2}{20}, q_{20} = \frac{1}{20}$.

So, whereas the minimax regret probabilities linearly increase from p_{T-n}=1/B to p_{T-1}=n/B (p_T being the complement to 1), the mixed-strategy Nash equilibrium probabilities linearly decrease from q_{T-n+1}=n/B to q_T=1/B (q_{T-n} being the complement to 1). The Nash equilibrium probabilities, given in Figure 4 for the 11-20 money request game with bonus 20, are much more difficult to justify from a behavioral point of view than the minimax regret ones. In particular they are not in line with level-*k* reasoning.

This surprising link is due to the structure of the money request game. We illustrate this fact in the 11-20 money request game with bonus 20. The minimax regret equations (1) checked with equality are recalled below (equations (2)). It can be observed that these equations are the mixed-strategy Nash equilibrium equations that ensure that player 2 gets the same payoff with all his strategies in the zero-sum game in matrix 3. So player 1's probabilities p_i in the minimax regret philosophy become player 1's mixed-strategy Nash equilibrium probabilities in the game in matrix 3.

$$\begin{aligned}
&19p_{15} + 18p_{16} + 17p_{17} + 16p_{18} + 15p_{19} + 14p_{20} = z\\
&0p_{15} + 19p_{16} + 18p_{17} + 17p_{18} + 16p_{19} + 15p_{20} = z \qquad (2)\\
&21p_{15} + 0p_{16} + 19p_{17} + 18p_{18} + 17p_{19} + 16p_{20} = z\\
&22p_{15} + 21p_{16} + 0p_{17} + 19p_{18} + 18p_{19} + 17p_{20} = z\\
&23p_{15} + 22p_{16} + 21p_{17} + 0p_{18} + 19p_{19} + 18p_{20} = z\\
&24p_{15} + 23p_{16} + 22p_{17} + 21p_{18} + 0p_{19} + 19p_{20} = z\\
&p_{15} + p_{16} + p_{17} + p_{18} + p_{19} + p_{20} = 1
\end{aligned}$$

		Pl2					
		15	*16*	*17*	*18*	*19*	*20*
	15	(-19,19)	(0 , 0)	(-21,21)	(-22,22)	(-23,23)	(-24,24)
	16	(-18,18)	(-19,19)	(0 , 0)	(-21,21)	(-22,22)	(-23,23)
Pl1	*17*	(-17,17)	(-18,18)	(-19,19)	(0 , 0)	(-21,21)	(-22,22)
	18	(-16,16)	(-17,17)	(-18,18)	(-19,19)	(0 , 0)	(-21,21)
	19	(-15,15)	(-16,16)	(-17,17)	(-18,18)	(-19,19)	(0 , 0)
	20	(-14,14)	(-15,15)	(-16,16)	(-17,17)	(-18,18)	(-19,19)

Matrix 3

The Karush Kuhn Tucker equations (see appendix A for the general case from which they are extracted) that follow from the minimization program become the equations 3 (after eliminating the multipliers equal to 0) :

$$\begin{aligned}
&19\lambda_{15} + 0\lambda_{16} + 21\lambda_{17} + 22\lambda_{18} + 23\lambda_{19} + 24\lambda_{20} + \lambda = 0\\
&18\lambda_{15} + 19\lambda_{16} + 0\lambda_{17} + 21\lambda_{18} + 22\lambda_{19} + 23\lambda_{20} + \lambda = 0\\
&17\lambda_{15} + 18\lambda_{16} + 19\lambda_{17} + 0\lambda_{18} + 21\lambda_{19} + 22\lambda_{20} + \lambda = 0 \qquad (3)\\
&16\lambda_{15} + 17\lambda_{16} + 18\lambda_{17} + 19\lambda_{18} + 0\lambda_{19} + 21\lambda_{20} + \lambda = 0\\
&15\lambda_{15} + 16\lambda_{16} + 17\lambda_{17} + 18\lambda_{18} + 19\lambda_{19} + 0\lambda_{20} + \lambda = 0\\
&14\lambda_{15} + 15\lambda_{16} + 16\lambda_{17} + 17\lambda_{18} + 18\lambda_{19} + 19\lambda_{20} + \lambda = 0\\
&1 - \sum_{i=15}^{20} \lambda_i = 0
\end{aligned}$$

These equations ensure that player 1 gets the same payoff with all his strategies in the zero-sum game in matrix 3. So the KKT multipliers λ_i become player 2's mixed Nash equilibrium probabilities in the zero-sum game in matrix 3.

Yet that there is a strong link between the zero-sum game in matrix 3 and the real played game, if rewritten in the reverse way, as in matrix 4:

		Pl2					
		20	19	18	17	16	15
	20	(20,20)	(20,39)	(20,18)	(20,17)	(20,16)	(20,15)
	19	(39,20)	(19,19)	(19,38)	(19,17)	(19,16)	(19,15)
Pl1	18	(18,20)	(38,19)	(18,18)	(18,37)	(18,16)	(18,15)
	17	(17,20)	(17,19)	(37,18)	(17,17)	(17,36)	(17,15)
	16	(16,20)	(16,19)	(16,18)	(36,17)	(16,16)	(16,35)
	15	(15,20)	(15,19)	(15,18)	(15,17)	(35,16)	(15,15)

Matrix 4

As a matter of fact, equalizing player 2's payoffs in two adjacent columns in matrix 4 leads exactly to the same calculi than equalizing the same two columns in matrix 3, because both equalities exploit the differences 1 and 19 present in both matrices at the same places.

For example, equalizing player 2's payoffs in the two first columns in both matrices leads to the equations:

$20p_{20} + 20p_{19} + 20p_{18} + 20p_{17} + 20p_{16} + 20p_{15} = 39p_{20} + 19p_{19} + 19p_{18} + 19p_{17} + 19p_{16} + 19p_{15}$ in matrix 4 (we call p_i the probability player 1 assigns to amount i)

and

$19p_{15} + 18p_{16} + 17p_{17} + 16p_{18} + 15p_{19} + 14p_{20} = 0p_{15} + 19p_{16} + 18p_{17} + 17p_{18} + 16p_{19} + 15p_{20}$ in matrix 3

These equations lead to $p_{20} = \frac{1}{20}$ in matrix 4 and $p_{15} = \frac{1}{20}$ in matrix 3.

And so we get the reversed probabilities.

Last but not least, mixed-strategy minimax regret, when both players play the minimax regret strategies, leads in AR's game to the mean payoff 21.5. This payoff is larger than the mixed-strategy Nash equilibrium mean payoff 20.

This result generalizes to any 11-T money request game, with bonus B≥T>*n*+11, *n* being defined by $\frac{n(n+1)}{2} < B < \frac{(n+1)(n+2)}{2}$.

Proposition 3 *In any 11-T money request game with bonus B≥T>n+11, n being defined by* $\frac{n(n+1)}{2} < B < \frac{(n+1)(n+2)}{2}$*, the mixed-strategy minimax regret payoff is equal to* $T + n - \frac{n(n+1)(n+2)}{3B}$*. This payoff is between* $T + \frac{n-4}{3}$ *and* $T + \frac{n}{3}$*, so it is strictly larger than T given that n is larger than 4. It is therefore larger than the Nash equilibrium payoff which, by construction, is always equal to T, regardless of n.*

Proof *See appendix B*

For example, for T=100, B=100, n=13, the expected payoff is 103.9. When we stay in the spirit of Arad and Rubinstein's game, so if we set B=T, the difference between the Nash equilibrium payoffs and the mixed-strategy minimax regret payoffs, relative to T, will stay low. As a matter of fact, given that n(n+1)/2<B, we have n/3<$\sqrt{2B}/3$, and, given that the expected payoff is lower than T+n/3, the expected payoff is lower than $T+\sqrt{2T}/3$. So the difference between both payoffs indeed increases in T, but the difference between both payoffs relative to T, $\frac{\sqrt{2T}}{3T}$, is decreasing in T. Things are different if B can be much larger than T (with T>11+n). In that case, the extra payoff can come close to (T-11)/3, which can be quite large.

5. Conclusion, behavioral comments

Arad and Rubinstein's 11-20 money request game is much richer than expected by the authors in that it triggers a more nuanced behavior than level-*k* reasoning. The following comments rely on the students' explanations of their strategy and on an analysis of regrets.

First of all, many students, namely the 8.3% who play 11, do not try to maximize their payoff. Some of them just do not want to offer the opponent the opportunity to get the bonus: the fear of not getting something (here the bonus) often triggers the wish that others do not get it either. And playing 11 is the only way to keep the opponent from getting the bonus. Yet some of the students that play 11 also want to minimize the maximal possible difference (in their disadvantage) between their payoff and the opponent's one. If a player plays 11, the opponent gets at best 9 units more, by contrast to the 19 possible units if he plays another amount.

In a less extreme (and quite opposite) way, many students who play 19, respectively 20, observe that they get more than their opponent in all situations except for one (when the opponent plays 18, respectively 19). That is to say, a student who plays 19 gets a larger payoff than the opponent when the opponent plays 20 or a number from 11 to 17, and he gets less than the opponent only if the latter plays 18, so he is better off than the opponent in 8 configurations (among 10). The same is true for a player who plays 20. And it can be observed that, from 18 to 11, the lower the number a student plays, the lower is the number of configurations in which he gets more than his opponent: with 18 he is better off than his opponent in 7 configurations, with 11 he is better off than his opponent in 1 configuration. With the same aim, students who play 20 sometimes want to counter a level-*k* reasoning in order to get more than the opponent: for example, they observe that their opponent surely expects them to play 19, hence plays 18, so that, at the end, they get 20 and the opponent gets 18. The fact that students sometimes more weight the difference in payoffs than the payoff they get is surely linked with their everyday life. Many of them pass competitive examinations where the only objective is to achieve better scores than the others.

What about the other students, the most numerous, who first focused on their own payoff?

Let us start with the students who play 19 and 20.

Are the students who play 20 really level-0 players? Most of the students who do a level-1, level-2, level-3 and even level-4 reasoning start their reasoning with a level-0 player who plays 20. And what is fine in this game is the fact that level-0 players playing 20 indeed exist (13.4%). This contrasts with other games, namely the guessing game, where many players do a level-*k* reasoning, starting with a level-0 behavior that in fact is not often observed in the population. Yet the existence of level-0 players in the 11-20 request game is due to the fact that playing 20 is a clever behavior. Playing 20 leads to the sure and large payoff 20. In the classroom experiment, 20 leads to the 4th best expected payoff (see figure 5) and, by contrast to 17, 18 and 19 who lead to better mean payoffs, 20 leads to the sure payoff 20, regardless of the met opponent. Moreover, some students who play 20 are perfectly able to do a level-*k* reasoning but explain that they do not want to do it in that they fear to not stop at the good level. So they are at least as clever as higher order level-*k* reasoning players. This strongly contrasts with other games. In the guessing game for example, level-0 players often play the mean of the interval of integers because they are not able to make the necessary calculi to become level-1 players.

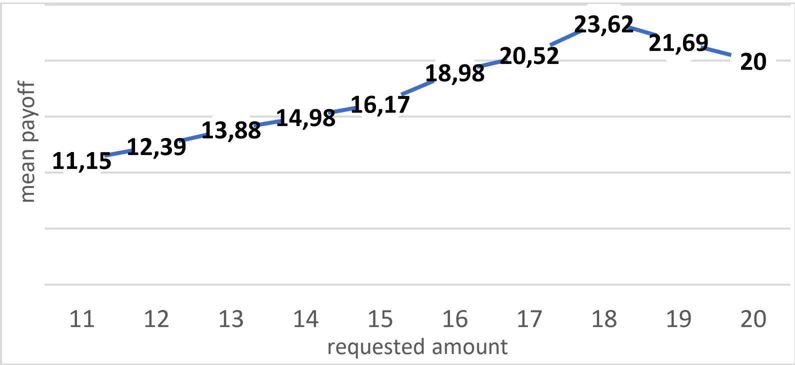


Figure 5: mean payoff associated to each requested amount in the Strasbourg university's classroom experiment

A similar observation can be made for many students playing 19. If some of them are pure level-1 players, many of them explain their behavior in this way: " by playing 19, I get 19 for sure, which is not far from 20, and I keep the opportunity to meet a student who plays 20 and so to get 39". Again, they have the cognitive skills to do a higher order level-*k* reasoning, but they refrain from doing so for payoff reasons. And they are right: they get the second best mean payoff, and they get for sure 19. Once again, in other games, namely guessing games, level-1 players usually do not win the game (they get 0), whereas in this game, they get a very large payoff, close to the best mean payoff.

Li and Rong (2018) exploit the cautious side of 19 and 20; they study the role of risk aversion in Arad and Rubinstein's game and they show how risk aversion increases the Nash equilibrium probabilities of 19 and 20.

In fact, there is an important difference between the 11-20 money request game and the guessing game. In a guessing game, to be the winner (to get 1), one has to do one additional step of reasoning in comparison with the depth of reasoning of the others. Otherwise one gets 0, the losers' payoff. For example, let us keep the students' distribution in table 1 and let us imagine a guessing game where the winning player is the one who is closest to the mean requested amount minus 1. Given the students' distribution, the mean requested number is 17.2 and the number closest to 17.2 -1 is 16. So only the students playing 16 share the payoff 1 and all the other students get 0. Clearly, in such a game, clever students would not stick to 20 and 19. Yet of course this guessing game is completely different (and iterative dominance would lead to a pure strategy Nash equilibrium where everybody plays 11).

In fact, if 20 is the level-0 amount for most students, some students choose 15 and 16 as the level-0 behavior, in that 15.5 is the mean of [11,20]. Among the students playing 15, some are level-0 players (they play a number in between the extreme upper values, 19 and 20, and the extreme lower values 11 and 12) and some are level-1 players who expect that the other student plays 16. And half of the students playing 14 are level-1 students in that that they expect that the other students play the mean, they fix at 15. Finally half of the few persons playing 13 do a level-2 reasoning starting at 15. So in some way, 15 and 16 appear as the "usual non clever statistical level-0 behavior".

Yet, despite the above remarks, level-*k* reasoning is clearly a main component of reasoning in the 11-20 money request game, especially among the numerous students who play 17 and 18. Most of the students playing 18 exclusively do a level-2 reasoning, and 60% of the students playing 17 exclusively do a level-3 reasoning. Moreover, even if far from the majority, some of the students playing 19 exclusively do a level-1 reasoning and nearly half of the students playing 16 exclusively do a level-4 reasoning. So, as claimed by Arad & Rubinstein, the 11-20 request game indeed triggers level-*k* reasoning.

But what about minimax regret motivations? According to Garcia-Pola (2020) there is seldom a minimax regret motivation in the players' behavior even if iterative pure-strategy minimax regret mimics level-1 reasoning. This has to be moderated. First, working with the general mixed approach of minimax regret allows to exploit all the regrets in matrix 2, and not only the maximal regrets (the regrets in bold). Second, students often talk about regrets. For example, when the students play 19, they say that at most they regret the additional unit they could win by playing 20, but that in exchange they have the opportunity to get 39 when the opponent plays 20. In the same way, when they play 16, 17, 18 they often add to the level-k reasoning (when they do it) the fact that at most they lose 4, 3 or 2 in comparison to 20. So they do not calculate the regrets by comparing the best reply payoffs to their payoffs, but they express regrets by comparing the sure payoff x they get when playing x, with 20, the sure payoff they could get by playing 20. Yet, the minimax regret when the opponent's plays y, which is B+y-1-x, can be written (20-x)+B+y-1-20. Hence comparing the minimax regret of two strategies x and x', when the opponent plays y, amounts to comparing 20-x and 20-x', the amounts students take into account when they talk about regrets. It derives from this fact that the way students express regrets is not disconnected from the notion of minimax regret. Third, what else than the difference between 20 and the requested amount x, can explain that the number of students doing a level-*k* reasoning is decreasing in *k*? Admittedly, in contrast to the classroom experiment at the Strasbourg university, this number is not always decreasing. In Arad and Rubinstein's experiment, the percentage of students doing a level-*k* reasoning is increasing in *k* up to *k*=3 (amount 17), but then it falls drastically. In Li and Rong's experiment, the percentage is growing up to *k*=2 (amount 18) before decreasing. Yet in this very easy game, adding a level of reasoning just amounts to diminishing the requested amount by 1, so switching from the level-*k* amount to the level-(*k*+1) amount requires almost no cognitive skills (everybody is able to subtract 1). And it is difficult to argue that students fear that the other students are unable to go one step further (because everybody knows that everybody can subtract 1, at least in a university environment). In facts, if few persons do a level-*k* reasoning with *k* larger than 4, it is mostly due to the fact that they do not want to lose more than 5 in comparison to the sure payoff 20, when unluckily they do not catch the bonus. So regret is a component of behavior.

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Appendix A

We solve

$$\min_{z\ p_{11}\dots p_T} z$$

$$\text{u.c. } \sum_{i=11}^{T}(T-i)p_i \leq z$$

$$0p_{11} + \sum_{i=12}^{T}(B+11-i)p_i \leq z$$

$$\sum_{i=11}^{j-2}(B+j-1-i)p_i + 0p_{j-1} + \sum_{i=j}^{T}(B+j-1-i)p_i \leq z \quad \text{j from 13 to T} \quad (1)$$

$$\sum_{i=11}^{T} p_i = 1$$

$$p_i \geq 0 \quad \text{i from 11 to T}$$

We suppose that only p_i, with i≥ T-n, is stricty positive, with n defined by $\frac{n(n+1)}{2} < B < \frac{(n+1)(n+2)}{2}$.

Given the structure of the regret matrix, this implies that the regrets associated to the columns j, with j<T-n, are strictly lower than z. This follows from the structure of the lines. In line k, the regrets are increasing in j, with j from 11 to k. So, given that we set p_i= 0 for i from 11 to T-n-

1, the only lines that matter are those with k≥T-n. In all those lines, the regrets are increasing in the column j, with j from 11 to T-n, which induces that the regret associated to column j, with j from 11 to T-n-1 is strictly lower than z.

We turn to the Karush Kuhn Tucker (KKT) function, $z + \sum_{j=11}^{T} \lambda_j (Regret(j) - z) - \sum_{i=11}^{T} \mu_i p_i + \lambda(\sum_{t=11}^{T} p_t - 1)$, where Regret (j) is the regret associated to column j.

Given the assumption on the optimal p_i and the consequences on the regrets, the multipliers λ_j for j from 11 to T-n-1 have to be null, the multipliers λ_j for j from T-n to T have to be positive or null, the multipliers μ_i have to be null for i from T-n to T, and they have to be positive or null for i from 11 to T-n-1.

The derivative of the KKT function in z leads to

$1 - \sum_{j=11}^{T} \lambda_j = 0$, hence $1 - \sum_{j=T-n}^{T} \lambda_j = 0$

More generally the derivatives in p_i for i from 11 to T are:

$(T-11)\lambda_{11} + 0\lambda_{12} + \sum_{j=13}^{T}(B+j-12)\lambda_j - \mu_{11} + \lambda = 0$ for i=11

$(T-i)\lambda_{11} + \sum_{j=12}^{i}(B-1+j-i)\lambda_j + 0\lambda_{i+1} + \sum_{j=i+2}^{T}(B-1+j-i)\lambda_j - \mu_i + \lambda = 0$ for i from 12 to T-2

$\lambda_{11} + \sum_{j=12}^{T-1}(B+j-T)\lambda_j + 0\lambda_T - \mu_{T-1} + \lambda = 0$ for i=T-1

$\sum_{j=12}^{T}(B-1+j-T)\lambda_j - \mu_T + \lambda = 0$ for i=T

Subtracting the equations 2 by 2, starting from the last two ones, leads to:

$$1 - \mu_{T-1} = B\lambda_T - \mu_T$$

And $1 + B\lambda_{i+2} - \mu_i = B\lambda_{i+1} - \mu_{i+1}$ for i from 11 to T-2

μ_i=0 for i from T-*n* to T, and *n* is at least equal to 5, given that B≥T>11+*n* and n(n+1)/2<B<(n+1)(n+2)/2. So we get $\lambda_T = \frac{1}{B}, \lambda_{T-1} = \frac{2}{B}$ and more generally $\lambda_j = \frac{(T+1-j)}{B}$ for j from T-n+1 to T, with $\lambda_{T-n+1} = \frac{n}{B}$. And $\lambda_{T-n} = 1 - \sum_{j=T-n+1}^{T} \lambda_j = 1 - \frac{n(n+1)}{2B}$.

Then we have $1 + B\lambda_{T-n+1} - \mu_{T-n-1} = B\lambda_{T-n} - \mu_{T-n}$

i.e. $1 + n - \mu_{T-n-1} = B - n(n+1)/2$

So $\mu_{T-n-1} = 1 + n - B + \frac{n(n+1)}{2} = \frac{(n+1)(n+2)}{2} - B > 0$ by definition of *n*.

Then we have $1 + B\lambda_{T-n} - \mu_{T-n-2} = B\lambda_{T-n-1} - \mu_{T-n-1}$ i.e.

$\mu_{T-n-2} = 1 + B\lambda_{T-n} + \mu_{T-n-1} = 2 + n$ and $\mu_i = 1 + \mu_{i+1} = T - i$ for i from 11 to T-n-3, because λ_j =0 for j from 11 to T-*n*-1.

We now turn back to the equations

$$\sum_{i=11}^{j-2}(B+j-1-i)p_i + 0p_{j-1} + \sum_{i=j}^{T}(B+j-1-i)p_i = z \quad \text{for j from T-n to T}^3.$$

We want p_i= 0 for i from 11 to T-n-1. Substracting the equations 2 by 2, starting from the first two ones, leads to : $1 + Bp_{j-1} = Bp_j$ for j from T-n to T-1.

Given that we want p_i=0 for i from 11 to T-n-1, it follows $p_{T-n} = \frac{1}{B}$ and more generally $p_{T-n+i} = \frac{i+1}{B}$ for i from 0 to n-1 and $p_T = 1 - \frac{n(n+1)}{2B}$.

[3] If T-n=12 we have to add the equation $0p_{11} + \sum_{i=12}^{T}(B+11-i)p_i \le z$ but this does not change the results.

Given the structure of the equations (1), the nullity of p_i for i<T-n immediately ensures that $\sum_{i=11}^{j-2}(B+j-1-i)p_i + 0p_{j-1} + \sum_{i=j}^{T}(B+j-1-i)p_i < z$ for j<T-n, and therefore justifies the nullity of the λ_j, for j from 11 to T-n-1.

So, given the convexity of the optimization problem, $p_i = 0$ for i from 0 to T-n-1, $p_{T-n+i} = \frac{i+1}{B}$ for i from 0 to n-1 and $p_T = 1 - \frac{n(n+1)}{2B}$ is the solution of the optimization program.

Given that $\sum_{i=T-n}^{T}(B+T-n-1-i)p_i = z$, we have

$$z = \sum_{i=1}^{n} \frac{(B-i)i}{B} + (B-n-1)\left(1 - \frac{n(n+1)}{2B}\right)$$

$$= B - \sum_{i=1}^{n} \frac{i^2}{B} + \frac{n(n+1)^2}{2B} - (n+1) = B + \frac{n(n+1)(n+2)}{6B} - (n+1)$$

By construction this regret is lower than B-1 (the regret obtained with the pure strategies B and B-1); it can be checked that z is strictly lower than B-1 given the definition of *n*.

Appendix B

We have $z = B - (n+1) + \frac{n(n+1)(n+2)}{6B}$

The expected payoff can be calculated as follows. For each number chosen by the opponent the mean payoff obtained by a player is the best reply payoff to the opponent's amount minus z, by construction of z.

As a matter of facts, when player 2 plays an amount y in the support of the minimax strategy, then player 1, when he plays the amount x (in the support of the minimax regret strategy), gets x= (y-1+B) – $r_1(x,y)$

Hence, given that player 1 plays x with probability p_x, his mean payoff when player 2 plays y is $\sum_{x=T-n}^{T} p_x(y-1+B-r_1(x,y)) = y-1+B-\sum_{x=T-n}^{T} p_x r_1(x,y) = y-1+B-z$

by construction of z. So, given that the opponent chooses each amount y with probability p_y, the mean payoff of a player is simply:

$\sum_{y=T-n}^{T} p_y(y-1+B-z) = \left(\sum_{y=T-n}^{T} p_y(y-1+B)\right) - z$

It follows from above that the expected minimax regret payoff is just the mean expected best reply payoff minus z. So we get:

$$\frac{1}{B}.(B+T-n-1) + \frac{2}{B}.(B+T-n+1-1) + \cdots + \frac{n}{B}.(B+T-n-2+n) + \frac{\left(B - \frac{n(n+1)}{2}\right)}{B}.(B+T+n-n-1) - z$$

$$= (B+T-n-2-z) + \sum_{i=1}^{n}\frac{i^2}{B} + \frac{(n+1)\left(B - \frac{n(n+1)}{2}\right)}{B} = T + n - \frac{n(n+1)(n+2)}{3B}$$

Given that n(n+1)/2<B<(n+1)(n+2)/2, this payoff is between T+(n-4)/3 and T+n/3.