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An economic model of metapopulation dynamics

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Abstract

In this paper, we aim to model the impact of human activities on the wildlife habitat in a general equilibrium framework by embedding the Levins model (1969) of metapopulation dynamics into a Ramsey model (1928) with a pollution externality. In the long run, as in Levins (1969), two steady states coexist: a zero one with mass extinction and another one with positive wildlife when the migration rate of the metapopulation exceeds the rate of extinction. A green tax always increases the wildlife and lowers the consumption demand. It is welfare-improving if and only if agents overweight the wildlife. In the short run, we show that a sufficiently negative effect of wildlife habitat on consumption demand can lead to the emergence of a limit cycle near the positive steady state through a Hopf bifurcation. We show also that the negative pollution effect on wildlife habitat works as a destabilizing force in the economy by promoting limit cycles.

Keywords: metapopulation dynamics, pollution, Ramsey model, Hopf bifurcation.

JEL Classification: C61, E32, O44.

1 Introduction

In ecology, a metapopulation represents a spatially fragmented population of the same species. The concept of metapopulation was introduced in the ecological literature in 1969 by Richard Levins (Hanski and Gilpin, 1991). In his seminal contribution, Levins (1969) represents the natural space as a partition of patches of the same size, homogeneous inside, that can be occupied or not by a metapopulation. The share of occupied patches changes over time. Dynamics are driven by two exogenous forces: the migration rate and the extinction rate. According to Levins' (1969) formulation, there are two steady states: a zero share means a massive extinction while a positive share a preserved wildlife. Dynamics are quite simple: the zero steady state is unstable while the positive one is stable and positive if and only if the migration rate exceeds the extinction rate.

Since the emergence of life, planet Earth has experienced five mass extinctions. A mass extinction is conventionally defined as a change where more

than three-quarters of species disappear in a geologically short interval of time (Barnosky et al., 2011). Following Ceballos et al. (2015), a sixth mass extinction is under way due to human activities because of deforestation and pollution that imply climate change. Evidence suggests that both the migration and the extinction rate in the Levins' model (1969) depend on the pollution coming from human activities. Today, a plausible representation of metapopulation dynamics has to take in account the interplay between economic activities and pollution, and the effects of pollution on both the extinction rate and the migration rate. To the best of our knowledge, such an integrated framework does not yet exist in the literature. In this respect, we aim to fill the gap between economics and ecology. More simply and precisely, we embed the Levins' model (1969) into the Ramsey model (1928) augmented with a pollution externality resulting from production and affecting both the migration and the extinction rates.

Evidence suggests also that the consumption behavior is influenced by environmental quality. For instance, the literature has pointed out that consumers have a higher willingness to pay for green products (Roe et al., 2001; Kim and Han, 2010; or Biswas, 2016). Even if, to the best of our knowledge, there is no empirical evidence on the effects of wildlife habitat on consumption demand, the common sense suggests that a link exists. If the household likes to consume in a pleasant environment, a drop in wildlife entails a lower consumption. Conversely, a decrease in wildlife implies a drop in utility to be compensated by the household with a higher consumption demand. The ambiguous environmental effects on consumption demand have been already studied in the literature. Theorists have considered pollution or natural capital instead of wildlife in the utility function. For instance, Bosi and Desmarchelier (2016) have focused on the occurrence of limit cycles in a Ramsey economy where an Environmental Kuznets Curve (EKC)¹ appears at the steady state.² The present paper is not about the EKC, but one can expect that the effect of wildlife habitat on consumption demand affects the transitional dynamics of the Levins' model (1969).

In this paper, we study a continuous-time Ramsey model where a pollution externality, coming from production, impacts the evolution of a metapopulation. To simplify, we assimilate wildlife to a single metapopulation and we assume that the fraction of occupied patches (a measure of environmental health) affects the marginal utility of consumption. In addition, a green tax is introduced and levied on production at the firm level in order to finance depollution according to a balanced budget rule.

As in Levins (1969), two steady states coexist in the long run with and without wildlife. Wildlife is positive when the rate of migration exceeds the extinction rate. From an economic point of view, even if the green tax lowers both the capital intensity and the consumption demand at the steady state, the green tax always increases the wildlife with an ambiguous effect on welfare.

¹The EKC is an inverted U-shaped relation between income and pollution.

²More precisely, they have shown that a positive effect of pollution on consumption demand promotes the occurrence of a limit cycle through a Hopf bifurcation when the steady state lies on the upward-sloping branch of the EKC, while, along the downward-sloping branch, limit cycles arise if and only if pollution lowers consumption.

The tax is welfare-improving if households overweight wildlife with respect to consumption.

In the short run, because of the pollution effects, the interplay between the wildlife habitat and consumption demand leads to richer dynamics around the positive steady state than those observed by Levins (1969). Indeed, a sufficiently large impact of wildlife on consumption demand can promote the emergence of a limit cycle near the steady state through a Hopf bifurcation. Moreover, the larger the (negative) impact of pollution on wildlife habitat, the lower the effect of wildlife on consumption demand, at the origin of the limit cycle. In other terms, the negative pollution effect on wildlife undoubtedly plays a destabilizing role.

The rest of the paper is organized as follows. Section 2 introduces the model. Section 3 and 4 focus on the equilibrium system and its steady state. Section 5 studies the local dynamics. An example with isoelastic preferences is considered in section 6, while a numerical illustration is provided in section 7. Section 8 concludes. All the proof are gathered in the Appendix.

2 Model

We consider an economy with households, firms and a government. Households work, consume and like nature, firms produce and pollutes, the government taxes the firms to maintain the environment. Let us introduce the three ingredients of the model: a metapopulation dynamics à la Levins (1969), the economic fundamentals à la Ramsey (1928) and a simple pollution process.

2.1 Metapopulation

In ecology, a metapopulation represents a spatially fragmented population of the same species. To simplify the model, we assimilate wildlife to a single metapopulation. Following Levins (1969), we consider that space is represented by a partition of patches occupied or unoccupied by the metapopulation. Let q denotes the fraction of patches occupied at a given time. As in Levins (1969), the evolution of this share is simply given by:

$$\dot{q} = \varphi q(1 - q) - \beta q \tag{1}$$

At each time, any occupied patch can become unoccupied at the extinction rate β . The contribution to the change in the share of occupied patches is given by βq . Conversely, any unoccupied patch can become occupied at the migration rate φ . The migration pressure on the share $1 - q$ of unoccupied patches is given by φq . A simple analysis of (1) allow us to point out that there exist two distinct steady states:

$$q_0 = 0 \text{ and } q^* = 1 - \beta/\varphi$$

It follows that q_0 leads to wildlife mass extinction while q^* represents an equilibrium where wildlife is positive. Interestingly, $q^* > 0$ if and only if the migration rate φ exceeds the extinction rate β .

Human activities pollute and stress the wildlife habitat mainly through the climate change. To put it in other way, pollution accelerates the extinction rate. We also consider that a degraded environment renders more difficult the wildlife migration. In the sequel, P will denote the aggregate stock of pollution.

Assumption 1 *Pollution has a positive impact on the extinction rate and a negative impact on the migration rate:*

$$\beta \equiv \beta(P) \text{ and } \varphi \equiv \varphi(P) \quad (2)$$

such that $\beta'(P) > 0$ and $\varphi'(P) < 0$.

Assumption 1 captures the pressure put by humans on wildlife. The role of pollution is summed up by the following elasticities and their difference.

Definition 1 *The colonization rate is the difference between the migration and the extinction rate: $s = s(P) \equiv \varphi(P) - \beta(P)$. We introduce also the pollution elasticities of migration and extinction:*

$$\varepsilon_\varphi(P) \equiv \frac{P\varphi'(P)}{\varphi(P)} \text{ and } \varepsilon_\beta(P) \equiv \frac{P\beta'(P)}{\beta(P)}$$

and the pollution impact on colonization

$$d(P) \equiv \varepsilon_\varphi(P) - \varepsilon_\beta(P) < 0$$

Notice that the pollution impact on colonization is negative because, according to Assumption 1, $\varepsilon_\varphi(P) < \varepsilon_\beta(P)$.

2.2 Firms

The firm j chooses the amount of capital K_j and labor L_j to maximize the profit. In addition, the government levies a proportional tax $\tau \in (0, 1)$ on polluting production $F(K_j, L_j)$ of firm j to finance the maintenance of natural resource.

Assumption 2 *The production function $F: \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ is C^2 , homogeneous of degree one, strictly increasing and concave. Inada conditions hold.*

Let $K \equiv \sum_{j=1}^J K_j$ be the aggregate capital stock and $L \equiv \sum_{j=1}^J L_j$ be the aggregate labor demand. $k \equiv K/L$ is the capital intensity in the economy. The price of product is normalized to one. r and w denote the real interest rate and the real wage. These prices are taken as given by the firms.

Proposition 2 (firm) *The aggregate production is given by $Y = F(K, L)$. Profit maximization yields*

$$r = (1 - \tau) \rho(k) \text{ and } w = (1 - \tau) \omega(k) \quad (3)$$

with $\rho(k) \equiv f'(k)$ and $\omega(k) \equiv f(k) - kf'(k)$.

We introduce the capital share in total disposable income and the elasticity of capital-labor substitution:

$$\alpha(k) \equiv \frac{rk}{(1-\tau)f(k)} = \frac{kf'(k)}{f(k)} \text{ and } \sigma(k) = \alpha(k) \frac{\omega(k)}{k\omega'(k)}$$

In addition, we determine the elasticities of factor prices:

$$\frac{k\rho'(k)}{\rho(k)} = -\frac{1-\alpha(k)}{\sigma(k)} \text{ and } \frac{k\omega'(k)}{\omega(k)} = \frac{\alpha(k)}{\sigma(k)}$$

2.3 Households

The representative household earns a capital income rh where h denotes the individual wealth at time t and a labor income wl where $l = 1$ (inelastic labor supply). Thus, the household consumes and saves her income according to the budget constraint:

$$c + \dot{h} \leq (r - \delta)h + w \quad (4)$$

where \dot{h} denotes the time-derivative of wealth. The gross investment includes the capital depreciation at the rate δ .

We assume that wildlife habitat enters the household's utility function $u(c, q)$ with $u_q > 0$. We suppose that it affects the marginal utility of consumption ($u_{cq} \neq 0$). Indeed, intuition suggests that wildlife plays a role in consumption demand. If wildlife raises the consumption demand, then wildlife and consumption are complement ($u_{cq} > 0$): this happens when households like to consume in a pleasant environment, in presence, for instance, of a large biodiversity. Conversely, if wildlife lowers consumption demand, then wildlife and consumption are substitutable ($u_{cq} < 0$): in this case, the household compensates the utility loss due to a lower wildlife by pushing her consumption demand.

Assumption 3 *Preferences are rationalized by a non-separable felicity function $u(c, q)$. First and second-order restrictions hold on the sign of derivatives: $u_c > 0$, $u_q > 0$ and $u_{cc} < 0$, jointly with the limit conditions: $\lim_{c \rightarrow 0} u_c = \infty$ and $\lim_{c \rightarrow \infty} u_c = 0$.*

Let us introduce the first and second-order elasticities of felicity:

$$(\varepsilon_c, \varepsilon_q) \equiv \left(\frac{cu_c}{u}, \frac{qu_q}{u} \right) \quad (5)$$

and

$$\begin{bmatrix} \varepsilon_{cc} & \varepsilon_{cq} \\ \varepsilon_{qc} & \varepsilon_{qq} \end{bmatrix} \equiv \begin{bmatrix} \frac{cu_{cc}}{u_c} & \frac{qu_{cq}}{u_c} \\ \frac{cu_{qc}}{u_q} & \frac{qu_{qq}}{u_q} \end{bmatrix} \quad (6)$$

$-1/\varepsilon_{cc}$ represents the intertemporal elasticity of substitution in consumption while ε_{cq} captures the effect of wildlife on the marginal utility of consumption. Typically, if $\varepsilon_{cq} > 0$ (< 0), then wildlife and consumption are complement (substitute) for households.

In a Ramsey model, the representative household maximizes an intertemporal utility functional:

$$\int_0^{\infty} e^{-\theta t} u(c, q) dt \quad (7)$$

under the budget constraint (4) where $\theta > 0$ denotes the rate of time preference. Let μ denote the multiplier associated to the budget constraint.

Proposition 3 (household) *The first-order conditions of the consumer's program are given by the shadow price of consumption*

$$\mu = u_c(c, q) \quad (8)$$

the intertemporal consumption smoothing (Euler equation)

$$\dot{\mu} = \mu(\theta + \delta - r) \quad (9)$$

and the budget constraint (4), now binding:

$$\dot{h} = (r - \delta)h + w - c \quad (10)$$

jointly with the transversality condition $\lim_{t \rightarrow \infty} e^{-\theta t} \mu(t) h(t) = 0$.

Applying the Implicit Function Theorem to the static relation $\mu = u_c(c, q)$, we obtain the consumption demand $c \equiv c(\mu, q)$ with elasticities

$$\frac{\mu}{c} \frac{dc}{d\mu} = \frac{1}{\varepsilon_{cc}} < 0 \quad \text{and} \quad \frac{q}{c} \frac{dc}{dq} = -\frac{\varepsilon_{cq}}{\varepsilon_{cc}} \quad (11)$$

2.4 Government

The government uses all the tax revenues to finance depollution expenditures (M) according to a balanced budget rule:

$$M = \tau F(K, L) \quad (12)$$

2.5 Pollution

The aggregate stock of pollution P is a pure externality coming from production (Y). The government takes care of depollution through the abatement expenditures M . The pollution accumulation follows a linear process:

$$\dot{P} = -aP + bY - \gamma M \quad (13)$$

$a \geq 0$, $b \geq 0$ and $\gamma \geq 0$ capture respectively the natural rate of pollution absorption, the environmental impact of production and the pollution abatement efficiency. We observe that, without human activities (Y and M), pollution is reabsorbed by nature according to the law $P(t) = P_0 e^{-at}$.

3 Equilibrium

At equilibrium, all the markets (good, capital and labor) clear. The aggregate wealth is equal to the aggregate capital: $hL = K$. Hence, $h = k$. Without loss of generality, we normalize the population of workers: $L = 1$.

Proposition 4 (equilibrium) *Equilibrium dynamics are driven by four equations: consumption smoothing, resource constraint, pollution accumulation, metapopulation dynamics.*

$$\dot{\mu} = f_1(\mu, k, P, q) \equiv \mu [\theta + \delta - (1 - \tau) \rho(k)] \quad (14)$$

$$\dot{k} = f_2(\mu, k, P, q) \equiv (1 - \tau) f(k) - \delta k - c(\mu, q) \quad (15)$$

$$\dot{P} = f_3(\mu, k, P, q) \equiv -aP + (b - \gamma\tau) f(k) \quad (16)$$

$$\dot{q} = f_4(\mu, k, P, q) \equiv \varphi(P) q (1 - q) - \beta(P) q \quad (17)$$

and the transversality condition $\lim_{t \rightarrow \infty} e^{-\theta t} \mu(t) k(t) = 0$.

4 Steady state

In the following, we leave aside the trivial steady state $q_0 = 0$. In the absence of wildlife, mankind disappears and our economic analysis loses its interest.

Proposition 5 (steady state) *The steady state with a positive wildlife is given by*

$$\rho(k^*) = \frac{\theta + \delta}{1 - \tau} \quad (18)$$

$$c^* = \left[\frac{\theta + \delta}{\alpha(k^*)} - \delta \right] k^* \quad (19)$$

$$P^* = \frac{\theta + \delta}{a\alpha(k^*)} \frac{b - \gamma\tau}{1 - \tau} k^* \quad (20)$$

$$q^* = 1 - \frac{\beta(P^*)}{\varphi(P^*)} > 0 \quad (21)$$

In order to have a positive pollution level at the steady state, we introduce the following restriction.

Assumption 4 $b > \gamma\tau$.

Proposition 6 (uniqueness) *Under Assumption 4, there exists a unique steady state with positive wildlife ($\beta < \varphi$).*

Let us show how τ affects this positive steady state.

Proposition 7 (comparative statics) *The long-run effects of taxation on the positive steady state are captured by the fiscal elasticities:*

$$\begin{aligned} \frac{\tau}{k^*} \frac{\partial k^*}{\partial \tau} &= -\frac{\tau}{1-\tau} \frac{\sigma(k^*)}{1-\alpha(k^*)} < 0 \\ \frac{\tau}{c^*} \frac{\partial c^*}{\partial \tau} &= -\frac{\tau}{1-\tau} \frac{k^*}{c^*} \left[\frac{\theta + \delta}{\alpha(k^*)} + \frac{\theta \sigma(k^*)}{1-\alpha(k^*)} \right] < 0 \end{aligned} \quad (22)$$

$$\begin{aligned} \frac{\tau}{P^*} \frac{\partial P^*}{\partial \tau} &= -\frac{\gamma \tau}{b - \gamma \tau} - \sigma(k^*) \frac{\tau}{1-\tau} \frac{\alpha(k^*)}{1-\alpha(k^*)} < 0 \\ \frac{\tau}{q^*} \frac{\partial q^*}{\partial \tau} &= d(P^*) \frac{1-q^*}{q^*} \frac{\tau}{P^*} \frac{\partial P^*}{\partial \tau} > 0 \end{aligned} \quad (23)$$

Let us provide some intuition. Since the green tax is levied on the production level, a higher green-tax rate reduces the production level and the capital intensity in turn. A drop in the production level entails also in the long run: (1) a lower disposable income and, thus, a lower consumption, and (2) a weaker pollution level according to the process (13). Under Assumption 1, the drop in pollution lowers the extinction rate and jointly increases the migration rate, leading to a richer wildlife in the long run.

Focus now on the impact of the green tax on welfare. Because of the representative agent, the welfare functional sums up to her utility functional:

$$W(c, q) = \int_0^\infty e^{-\theta t} u(c, q) dt$$

Let us introduce a critical value for the social propensity to wildlife.

$$E \equiv -\frac{\frac{\tau}{c^*} \frac{\partial c^*}{\partial \tau}}{\frac{\tau}{q^*} \frac{\partial q^*}{\partial \tau}} > 0 \quad (24)$$

Proposition 8 (welfare) *The green-tax rate is welfare-improving in the long run if and only if the relative preference for wildlife is sufficiently large (that is, $\varepsilon_q/\varepsilon_c > E$, where the elasticities ε_c and ε_q are given by (5)).*

Let us provide some economic intuition about this result. $\varepsilon_q/\varepsilon_c$ depends on the slope of the indifference curve in the (c, q) -plane and describes how the household weights wildlife with respect to consumption. Inequality $\varepsilon_q/\varepsilon_c > E$ holds when households display ecological preference (that is, a higher ratio $\varepsilon_q/\varepsilon_c$). Unsurprisingly, a higher green-tax rate has two opposite effects on utility: (1) a lower consumption level (inequality (22)), which reduces the household's utility, and (2) a richer wildlife habitat (inequality (23)), which increases her utility. Thus, if the representative household overweights nature with respect to consumption, the positive effect (2) dominates the negative effect (1) and the green tax turns out to be welfare-improving.

Corollary 9 (welfare) *The critical value is explicitly given by*

$$E = -\frac{s(P^*)}{\beta(P^*)d(P^*)} \frac{[\theta(1-\alpha(k^*)) [1-\sigma(k^*)] + \delta [1-\alpha(k^*)]]}{(\theta + \delta [1-\alpha(k^*)]) \left(\alpha(k^*) \sigma(k^*) + [1-\alpha(k^*)] \frac{1-\tau}{\tau} \frac{\gamma\tau}{b-\gamma\tau} \right)} \quad (25)$$

and, in the case of a Cobb-Douglas technology, by

$$E = -\frac{s(P^*)}{\beta(P^*)d(P^*)} \left[\alpha + (1-\alpha) \frac{1-\tau}{\tau} \frac{\gamma\tau}{b-\gamma\tau} \right]^{-1} \quad (26)$$

5 Local dynamics

To study the multiplicity of equilibria (local indeterminacy) and the occurrence of local bifurcations around the unique positive steady state, we focus on the dynamic system.

Proposition 10 *The Jacobian matrix of system (14)-(17) is given by*

$$J = \begin{bmatrix} 0 & \frac{\mu^* (1-\alpha)(\theta+\delta)}{k^* \sigma} & 0 & 0 \\ -\frac{k^*}{\mu^*} \frac{1}{\varepsilon_{cc}} \frac{\theta+(1-\alpha)\delta}{\alpha} & \theta & 0 & \frac{k^*}{q^*} \frac{\varepsilon_{cq}}{\varepsilon_{cc}} \frac{\theta+(1-\alpha)\delta}{\alpha} \\ 0 & \frac{(b-\gamma\tau)(\theta+\delta)}{1-\tau} & -a & 0 \\ 0 & 0 & \frac{\beta}{\varphi} \frac{ds}{P^*} & -s \end{bmatrix} \quad (27)$$

where $\alpha = \alpha(k^*)$, $\sigma = \sigma(k^*)$, $\varepsilon_{cc} = \varepsilon_{cc}(c^*, q^*)$, $\varepsilon_{cq} = \varepsilon_{cq}(c^*, q^*)$, $\beta = \beta(P^*)$, $\varphi = \varphi(P^*)$, $d = d(P^*)$, and μ , k , P and q are evaluated at the steady state.

For simplicity, we focus on the Cobb-Douglas technology. In order to study the local dynamics (local indeterminacy and local bifurcations), we apply a methodology developed by Bosi and Desmarchelier (2017) and based on the sums of principal minors of the Jacobian matrix.

Lemma 11 *In the case of a Cobb-Douglas technology, the trace, the sum of principal minors of order two, the sum of principal minors of order three, and the determinant of the Jacobian matrix (27) are given by*

$$T = \theta - a - s \quad (28)$$

$$S_2 = as - \theta(a+s) + \frac{D}{as} \quad (29)$$

$$S_3 = as\theta + D \frac{T-\theta}{as} + a\beta d [\theta + \delta(1-\alpha)] \frac{\varepsilon_{cq}}{\varepsilon_{cc}} \quad (30)$$

$$D = a(1-\alpha)(\theta+\delta) [\theta + (1-\alpha)\delta] \frac{s}{\alpha\varepsilon_{cc}} \quad (31)$$

The following proposition shows that a unique equilibrium trajectory exists in a neighborhood of the positive steady state.

Proposition 12 (local determinacy) *The equilibrium is locally unique around the positive steady state.*

Focus now on local bifurcations. From a mathematical point of view, a transcritical bifurcation occurs when $\beta = \varphi$. We do not care about this case because, from an economic point of view, the trivial steady state $q_0 = 0$ and the case where $\beta > \varphi$ (that is $q^* < 0$) are meaningless.

In order to study the occurrence of (limit) cycles through a Hopf bifurcation, we consider the main economic variable, that is the cross elasticity ε_{cq} capturing the wildlife impact on preferences and consumption demand. In this respect, let

$$\varepsilon_H \equiv \varepsilon_{cc} \frac{\frac{T}{2} \left(S_2 + \sqrt{S_2^2 - 4D} \right) - D \frac{T-\theta}{as} - as\theta}{a\beta d [\theta + (1-\alpha)\delta]} \quad (32)$$

As we will see, in the case of isoelastic preferences, the RHS of (32) does not depend on ε_{cq} and it is well defined.

Endogenous (limit) cycles arise when the discount rate exceeds the rate of natural absorption.

Assumption 5 $a < \theta$.

Proposition 13 (Hopf) *Under Assumption 5, a Hopf bifurcation around the positive steady state generically occurs at $\varepsilon_{cq} = \varepsilon_H$.*

$d < 0$ captures the negative impact of pollution on the wildlife habitat. Focus on the critical value (32). The larger the (negative) impact of pollution on wildlife diffusion, the lower the bifurcation value ε_H . In other terms, when $|d|$ becomes higher, endogenous (limit) cycles can take place.

Since the number of stable eigenvalues is odd (indeed $D < 0$ if $\beta < \varphi$ and $a < \theta$), then the dimension of the stable manifold goes from 3 to 1 or from 1 to 3 when the system undergoes a Hopf bifurcation. In the first case the bifurcation is supercritical and the limit cycle is stable, while in the second case the bifurcation is subcritical and this cycle is unstable.

The following proposition rules out the class of saddle-node bifurcations as well as any bifurcation of codimension two.

Proposition 14 *Under Assumption 5, around the positive steady state, any saddle-node, Bogdanov-Takens, Gavrilov-Guckenheimer or double-Hopf bifurcation is excluded.*

6 Isoelastic preferences

Metapopulation is driven by the following explicit migration and extinction rates:

$$\beta(P) = A_\beta P^{\varepsilon_\beta} \text{ and } \varphi(P) = A_\varphi P^{\varepsilon_\varphi}$$

with constant elasticities $\varepsilon_\varphi < 0$ and $\varepsilon_\beta > 0$. In this case, the colonization rate and the pollution impact on colonization are given by $s = A_\varphi P^{\varepsilon_\varphi} - A_\beta P^{\varepsilon_\beta}$ and $d = \varepsilon_\varphi - \varepsilon_\beta$.

We consider also a Cobb-Douglas technology: $f(k) = Ak^\alpha$, and isoelastic preferences:

$$u(c, q) = \frac{(cq^\eta)^{1-\varepsilon}}{1-\varepsilon}$$

η measures the propensity to wildlife while $1/\varepsilon$, the elasticity of intertemporal substitution of the composite good cq^η . Straightforward computations give $\varepsilon_c = 1 - \varepsilon$, $\varepsilon_q = \eta(1 - \varepsilon)$, $\varepsilon_{cc} = -\varepsilon$ and $\varepsilon_{cq} = \eta(1 - \varepsilon)$.

Corollary 15 (steady state) *The steady state is computed in terms of the fundamental parameters:*

$$\begin{aligned} k^* &= \left[\frac{\alpha A (1 - \tau)}{\theta + \delta} \right]^{\frac{1}{1-\alpha}} \\ c^* &= \frac{\theta + (1 - \alpha)\delta}{\alpha} k^* \\ P^* &= \frac{\theta + \delta}{a\alpha} \frac{b - \gamma\tau}{1 - \tau} k^* = (b - \gamma\tau) \frac{A}{a} \left[\frac{\alpha A (1 - \tau)}{\theta + \delta} \right]^{\frac{\alpha}{1-\alpha}} \\ q^* &= 1 - \frac{A_\beta}{A_\varphi} P^{*-d} \end{aligned} \quad (33)$$

with $\mu^* = c^{*-\varepsilon} q^{*\eta(1-\varepsilon)}$.

η is the main parameter because it captures the peculiarity of the model, that is the preference for wildlife.

Corollary 16 (welfare) *In the case of isoelastic preferences, the green-tax rate is welfare-improving in the long run if and only if $\eta > E$ where now*

$$E = \frac{1}{\varepsilon_\varphi - \varepsilon_\beta} \frac{1 - \frac{A_\varphi}{A_\beta} P^{*\varepsilon_\varphi - \varepsilon_\beta}}{\alpha + (1 - \alpha) \frac{1 - \tau}{\tau} \frac{\gamma\tau}{b - \gamma\tau}} \quad (34)$$

and P^* is given by (33).

We observe that neither P^* nor E depend on η . Thus, E is an explicit and well-defined critical value for η . Unsurprisingly, if agents overweight wildlife with respect to consumption ($\eta > E$), a higher green-tax rate raises the social welfare.

In the isoelastic case, even the right-hand side of (32) does not depend on η .

Corollary 17 (Hopf) *A limit cycle generically arises around the critical propensity to wildlife*

$$\eta_H = \frac{\varepsilon}{1 - \varepsilon} \frac{a\theta s(P^*) + \frac{D}{a} \frac{T - \theta}{s(P^*)} - \frac{T}{2} \left(S_2 + \sqrt{S_2^2 - 4D} \right)}{a\beta (\varepsilon_\varphi - \varepsilon_\beta) [\theta + (1 - \alpha)\delta]} \quad (35)$$

where

$$T = \theta - a - s(P^*) \quad (36)$$

$$S_2 = (a - \theta)s(P^*) - a\theta + \frac{D}{as(P^*)} \quad (37)$$

$$D = -(\theta + \delta)[\theta + (1 - \alpha)\delta] \frac{a(1 - \alpha)}{\varepsilon} s(P^*) \quad (38)$$

Thus, fixing the fundamental parameters, according to expression (35), we are able to compute the numerical value of the preference for wildlife (η_H) giving rise to endogenous cycles. In the next section, we calibrate and simulate the model, and we find this value.

7 Simulations

In order to illustrate the above theoretical results, we calibrate the model and provide simulations of dynamics using the original nonlinear system (14)-(17).³

We set the fundamental parameters as follows.

Parameter	A	A_β	A_φ	ε_β	ε_φ	α	θ	δ	τ	a	b	γ
Value	1	1	1	1	-1	0.33	0.01	0.025	0.002	0.003	0.0015	0.3

(39)

The parameters A , A_β , A_φ , ε_β and ε_φ are normalized. α is the usual value of capital share in total income. θ and δ take their standard quarterly values. τ represents the public air protection expenditures ((12) implies $\tau = M/Y$): according to the OECD Environmental Performance Reviews for France (2016, p. 149), the public air protection expenditures amount to less than 5 billion euros, that is less than 0.25% of France GDP. To be in accordance with these data, we simply set $\tau = 0.002$. Finally, a , b and γ are chosen to satisfy $a < \theta$ (Assumption 5) and the positivity of the steady state: $k^* = 28.385671$, $c^* = 2.3010$, $P^* = 0.90499041$ and $q^* = 0.18099236$.

We observe that these steady state values do not depend on η . Moreover, if $s > 0$ (positive steady state), the bifurcation value $\eta_H > 0$ if and only if $\varepsilon > 1$, that is iff $\varepsilon_{cq} < 0$. In other terms, the existence of a limit cycle around the positive steady state is ensured when consumption and wildlife are substitutable goods. To grasp this point, let the economy be at the steady state today and assume an exogenous increase in the pollution level P . Under Assumption 1, this implies a lower migration rate, a higher extinction rate and a weaker wildlife habitat q in turn. Under Assumption 3, the drop in q lowers the household's utility. To compensate this loss, because of her substitutable preferences, the household increases the consumption, while reducing her saving which lowers the capital stock tomorrow and the pollution level at the end. Thus, when consumption and wildlife are sufficiently substitutable, a higher pollution today entails a lower pollution tomorrow giving rise to endogenous cycles.

³We use the MATCONT package for MATLAB.

Thus, we fix the elasticity of intertemporal substitution $1/\varepsilon = 1/2 < 1$ to have limit cycles. According to expression (35) and calibration (39), we find the bifurcation value of ecological preference $\eta_H = 3.322$. At $\eta = \eta_H$, we obtain $\mu^* = 55.238$ and, replacing in the Jacobian matrix (27), we get the eigenvalues: $\lambda_1 = -0.205636$, $\lambda_2 = 0.0126424$ and $\lambda_3 = -0.0148104i = -\lambda_4$.

Coherently, we find two nonreal and conjugated eigenvalues with zero real part (λ_3 and λ_4), that is the signature of the occurrence of a Hopf bifurcation. MATCONT computes also the first Lyapunov coefficient at the Hopf critical value: $l_1 = 1.278750 * 10^{-5} > 0$. A positive Lyapunov coefficient means that the Hopf bifurcation is subcritical and the limit cycle is unstable. MATCONT generates Figure 1 directly from the original nonlinear system (14)-(17) using the current calibration and projecting a 4D cycle on the 3D subspace of quantities $\{(k, P, q)\}$: a limit cycles arises around the positive steady state.

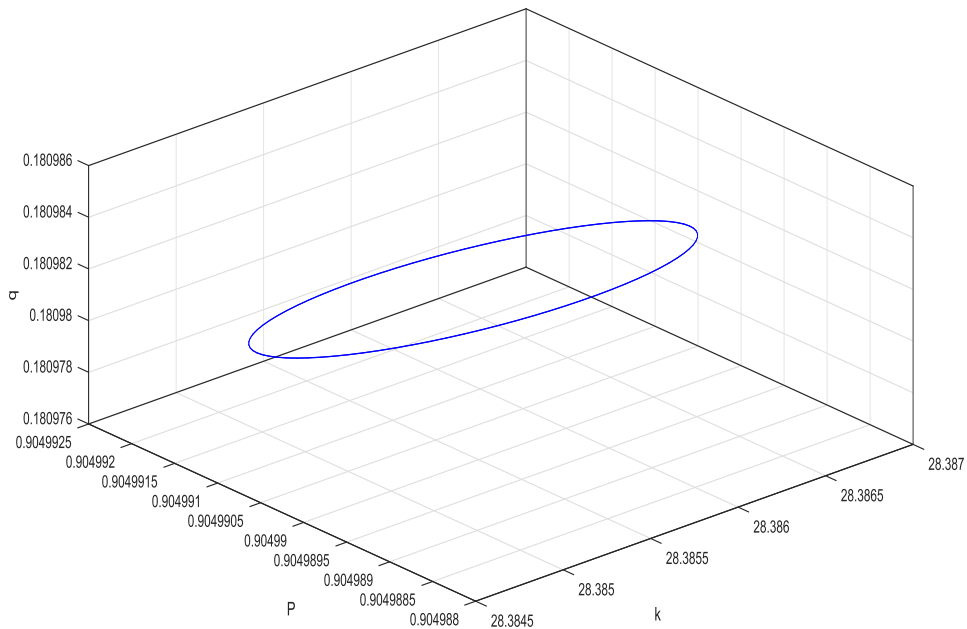


Fig. 1. The unstable limit cycle

8 Conclusion

In this paper, we have embedded the Levins model (1969) of metapopulation dynamics into the Ramsey model (1928) in order to capture the effects of economic activities on the wildlife habitat. This interdisciplinary approach gives interesting results both concerning the long and the short run.

In the long run, as in Levins (1969), two steady states coexist: a zero one with mass extinction and another one with positive wildlife when the migration rate of the metapopulation exceeds its extinction rate. A simple exercise of

comparative statics shows that, even if the green tax lowers both the capital intensity and the consumption level at the steady state, it improves the welfare if the representative household overweights wildlife with respect to consumption.

In the short run, because of the role of pollution, the interaction between wildlife and consumption leads to richer dynamics than the ones described by Levins (1969). In particular, we observe that a negative impact of wildlife on consumption demand can give rise to the emergence of a limit cycle through a Hopf bifurcation near the positive steady state. Finally, we have shown that the negative pollution effect on wildlife works as a destabilizing force in the economy: the larger the (negative) impact of pollution on wildlife habitat, the lower the effect of wildlife on consumption demand at the origin of limit cycles.

9 Appendix

Proof of Proposition 2

The profit maximization $\max_{K_j, L_j} [F(K_j, L_j) - rK_j - wL_j - \tau F(K_j, L_j)]$ entails the following first-order conditions:

$$r = (1 - \tau) f'(k_j) \text{ and } w = (1 - \tau) [f(k_j) - k_j f'(k_j)]$$

where $k_j \equiv K_j/L_j$ is the capital intensity and $f(k_j) \equiv F(k_j, 1)$ the average productivity of the firm j . All the firms share the same technology and address the same demand for capital. ■

Proof of Proposition 3

The agent maximizes the intertemporal utility function (7) under the budget constraint (4). Setting the Hamiltonian $H = e^{-\theta t} u(c, q) + \lambda [(r - \delta)h + w - c]$, deriving the first-order conditions $\partial H/\partial c = 0$, $\partial H/\partial h = -\lambda$ and $\partial H/\partial \mu = \dot{h}$, and defining $\mu \equiv \lambda e^{\theta t}$, we get (8), (9) and (10). ■

Proof of Proposition 4

(3) and (9) gives (14). Since $h = k$, the equilibrium budget constraints becomes a resource constraint: $\dot{k} = (1 - \tau) f(k) - \delta k - c$ with $c = c(\mu, q)$. Since $L = 1$, the process (13) yields (16) in intensive terms. (2) and (1) imply (17). ■

Proof of Proposition 5

Any steady state solves the following system of equations: $\dot{\mu} = \dot{k} = \dot{P} = \dot{q} = 0$. (14) and (17) gives (18) and (21) respectively. Moreover,

$$c^* = (1 - \tau) f(k^*) - \delta k^* = \left[(1 - \tau) \frac{\rho(k^*)}{\alpha(k^*)} - \delta \right] k^* \quad (40)$$

$$P^* = \frac{b - \gamma\tau}{a} f(k^*) = \frac{b - \gamma\tau}{a} \frac{\rho(k^*)}{\alpha(k^*)} k^* \quad (41)$$

yields (19) and (20) respectively. ■

Proof of Proposition 6

According to Assumption 2 and equation (18), the capital intensity k^* of Modified Golden Rule is unique. Replacing k^* in (19) and (20), we get the

unique values for c^* and P^* . Substituting P^* in (21), we obtain the uniqueness of q^* . The uniqueness of c^* and q^* entails also that of $\mu^* = u_c(c^*, q^*) > 0$. ■

Proof of Proposition 7

To obtain the fiscal elasticities, we differentiate (18), (21), (40) and (41) with respect to τ , k , c , P and q , and use the steady state properties. ■

Proof of Proposition 8

We compute the welfare functional at the steady state: $W(c^*, q^*) = u(c^*, q^*)/\theta$. The tax derivative of welfare is given by

$$\frac{\partial W}{\partial \tau} = \frac{1}{\theta} \left(u_c \frac{\partial c^*}{\partial \tau} + u_q \frac{\partial q^*}{\partial \tau} \right)$$

and $\partial W/\partial \tau > 0$ if and only if $\varepsilon_q/\varepsilon_c > E$. ■

Proof of Corollary 9

Replacing (19), (21), (22) and (23) in (24), we get (25). In the case of Cobb-Douglas technology, $\sigma(k^*) = 1$ and $\alpha(k^*)$ becomes a constant. Hence, (25) implies (26). ■

Proof of Proposition 10

The Jacobian matrix of system (14)-(17) is defined as

$$J \equiv \begin{bmatrix} \frac{\partial f_1}{\partial \mu} & \frac{\partial f_1}{\partial k} & \frac{\partial f_1}{\partial P} & \frac{\partial f_1}{\partial q} \\ \frac{\partial f_2}{\partial \mu} & \frac{\partial f_2}{\partial k} & \frac{\partial f_2}{\partial P} & \frac{\partial f_2}{\partial q} \\ \frac{\partial f_3}{\partial \mu} & \frac{\partial f_3}{\partial k} & \frac{\partial f_3}{\partial P} & \frac{\partial f_3}{\partial q} \\ \frac{\partial f_4}{\partial \mu} & \frac{\partial f_4}{\partial k} & \frac{\partial f_4}{\partial P} & \frac{\partial f_4}{\partial q} \end{bmatrix}$$

■

Proof of Lemma 11

In the case of a Cobb-Douglas technology, α is constant and $\sigma = 1$. Computing the sums of principal minors of order 1 to 4 (see Bosi and Desmarchelier (2007)) and using (20) and (21), we find (28), (29), (30) and (31). ■

Proof of Proposition 12

Three variables (k , P and q) of system (14)-(17) are predetermined, while one (μ) is a jump variable. In this case, local indeterminacy requires four stable eigenvalues. Thus, $D > 0$ is a necessary (but not sufficient) condition for local indeterminacy. The existence of a positive steady state ($q^* > 0$) requires $\varphi > \beta$ which entails in turn $D < 0$. This rules out the local indeterminacy. ■

Proof of Proposition 13

According to Bosi and Desmarchelier (2017), a Hopf bifurcation generically occurs in a 4D system of autonomous ODE if and only if:

$$S_2 = \frac{S_3}{T} + \frac{T}{S_3}D \tag{42}$$

and $S_3/T > 0$. (42) is satisfied if and only if

$$\frac{S_3}{T} = \frac{1}{2} \left(S_2 \pm \sqrt{S_2^2 - 4D} \right) \tag{43}$$

that is $\varepsilon_{cq} = \varepsilon_0$ or $\varepsilon_{cq} = \varepsilon_H$, where

$$\varepsilon_0 \equiv \varepsilon_{cc} \frac{\frac{T}{2} \left(S_2 - \sqrt{S_2^2 - 4D} \right) - D \frac{T-\theta}{as} - as\theta}{a\beta d [\theta + (1-\alpha)\delta]}$$

Moreover, since $a < \theta$ and $\beta < \varphi$ (existence of a positive steady state), according to (43), we have $D < 0$ and, therefore,

$$\begin{aligned} \frac{S_3(\varepsilon_0)}{T} &= \frac{1}{2} \left(S_2 - \sqrt{S_2^2 - 4D} \right) < 0 \\ \frac{S_3(\varepsilon_H)}{T} &= \frac{1}{2} \left(S_2 + \sqrt{S_2^2 - 4D} \right) > 0 \end{aligned}$$

■

Proof of Proposition 14

We observe that $D < 0$ because $a < \theta$ and $\beta < \varphi$ (positive steady state) and we apply Propositions 13, 15, 16 and 17 in Bosi and Desmarchelier (2017) respectively. ■

Proof of Corollary 16

We know that the green-tax rate is welfare-improving in the long run iff $\varepsilon_q/\varepsilon_c > E$. In the case of isoelastic preferences, $\varepsilon_q/\varepsilon_c = \eta$ and E is given by (34). ■

Proof of Proposition 17

In terms of propensity to wildlife, the Hopf bifurcation value becomes $\eta_H = \varepsilon_H/(1-\varepsilon)$. ε_H is given by (32) with, now, the explicit moments (36), (37) and (38). ■

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