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Second price all-pay auctions, how much money do players get or lose?

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Abstract

The paper studies second price all-pay auctions - wars of attrition - in a new way, based on class room experiments and Kosfeld, Droste and Voorneveld's (2002) best reply matching equilibrium. Two players fight over a prize of value V , have a budget M , submit bids lower or equal to M ; both pay the lowest bid and the prize goes to the highest bidder. The behaviour probability distributions in the class room experiments are strikingly different from the mixed Nash equilibrium. They fit with best reply matching or generalized best reply matching, an ordinal logic according to which, if bid A is the best response to bid B , and if B is played with probability p , then A is also played with probability p . In the mixed Nash equilibrium, the expected payoff is never negative and close to 0. In the best reply and generalized best reply matching equilibria, players may lose money, up to $1/12^{\text{th}}$ of the budget when M is large in comparison to V , but they can also get a lot of money, especially if V is large. The study leads to examine possible bifurcations in the bidding behaviour and gives some insights into how to regulate games to avoid pathological gambling with a huge waste of money.

Keywords : second price all-pay auction – war of attrition – best reply matching – Nash equilibrium - classroom experiment

JEL classification: C72, D44

1. Introduction

The paper studies second price all-pay auctions in a new way, based on class room experiments and Kosfeld, Droste and Voorneveld's best reply matching equilibrium (2002). Whereas a lot has been said on first price all-pay auctions, there are only few papers with experiments on second price all-pay auctions - equivalent to wars of attrition - (see Hörisch and Kirchkamp (2010) and Dechevaux, Kovenock and Sheremeta (2015) for experiments with this class of games). The second price all-pay auction studied in this paper goes as follows: two players fight over a prize of value V , have a budget M , and simultaneously submit bids lower or equal to M . Both pay the lowest bid and the prize goes to the highest bidder; in case of a tie, each player gets the prize with probability $1/2$.

The mixed Nash equilibrium distribution of this game has a special shape, with a mass point on M and decreasing probabilities from bid 0 to a given threshold bid. This shape is strikingly

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different from the distributions' shapes obtained in the class room experiments (an experiment with 116 students, another with 109 students). The students bid 0 with a high probability, assign very low probabilities to small bids different from 0, and significant probabilities to M and other high bids. The distributions are so different that we can't conclude on overbidding or underbidding in comparison with the Nash equilibrium behaviour. This observation is partly shared by Hörisch and Kirchkamp (2010): whereas overbidding is regularly observed in first price all-pay auction experiments (see for example Gneezy and Smorodinsky (2006) and Lugovskyy et al. (2010)), Hörisch and Kirchkamp (2010) establish that underbidding prevails in sequential war of attrition experiments.

In our class room experiments, the students' behaviour fits with best reply (and generalized best reply) matching, a behaviour studied by Kosfeld et al. (2002), according to which, if bid A is the best response to bid B , and if B is played with probability p , then A is also played with probability p . Mixed Nash equilibria and best reply matching equilibria follow a different logic: whereas Nash equilibrium probabilities are calculated so as to equalize the payoffs of the bids played at equilibrium, best reply matching probabilities just aim to match best responses, each bid being played as often as the bid to which it is a best reply.

Yet with the best reply matching logic, second price all-pay auctions may become dangerous. In the mixed Nash equilibrium, the expected payoffs are never negative: they are null (in the continuous game) or slightly positive (in the discrete game). This follows from the way the mixed Nash equilibrium is calculated: each played bid leads to the same payoff, necessarily higher or equal to the payoff obtained with bid 0 (which is never negative). This is no longer true with a best reply matching logic, which may lead to play high bids too often. M , in particular, may play a bad focal role, in that nobody can bid more. Even if it is large, much larger than V , M can be seen as the bid to choose if one wants to win the prize, and possibly – but not surely – pay a low amount. We show in the paper that players can lose on average $1/12^{\text{th}}$ of their budget when M is very large in comparison to V . But we also show that players can get a huge amount of money, especially if V is large, when turning to generalized best reply matching, which provides a higher degree of flexibility in choosing best responses in case of several best responses to a given strategy.

Section 2 gives the pure and mixed Nash equilibria for continuous second price all-pay auction games. We establish the special shape of the mixed Nash equilibrium probability distribution, when the players are risk neutral and risk averse. We also link mixed Nash equilibria in the continuous game to mixed Nash equilibria in the discrete game, where M , V and the bids are integers. In section 3, we present two class room experiments and comment on the students' behaviour. In section 4, we present Kosfeld et al.'s best reply matching equilibrium and Umbhauer(2007)'s generalized best reply matching equilibrium, which build on Bernheim (1984) and Pearce's (1984) notion of rationalizability, and we compare the mixed Nash equilibrium philosophy with the best reply matching philosophy. We namely focus on the meaning of probabilities in both concepts. In section 5, we establish the proximity between the student's behaviour and best reply, and generalized best reply, matching equilibria. We also bring into light that some best responses are more focal than others, namely M , V and 0, and how these focal values impact the equilibrium payoffs. In section 6 we examine the players' payoffs with generalized best reply matching, the role of the ratio M/V and the possible switch from some focal values to other focal values. As a by-

product we show how generalized best reply matching can bring closer two classes of games, first price all-pay auctions and second price all-pay auctions. We namely show that these two auctions have a common generalized best reply matching equilibrium, that leads to a positive payoff close to $V/6$ (when V is large) in the second price all-pay auction. In section 7, we give and comment on the best reply matching equilibrium for any values of V and M . In section 8, we mainly focus on the loss of payoffs when M is very large compared to V . We establish that the mean loss is $1/12^{\text{th}}$ of the budget M and that the loss may even rise up to $1/4^{\text{th}}$ of M . Section 9 concludes in a contrasted way. It comes back on the role of M , the incentives to limit the ratio M/V , the possibility of bifurcations in the players' behaviour and on the possibility to delete M .

2. Nash equilibria in continuous and discrete second price all-pay auctions

Two players have a budget M . They fight over a prize of value V . Each player i submits a bid b_i , $i=1,2$ lower or equal to M . The prize goes to the highest bidder but both bidders pay the lowest bid. In case of a tie, the prize goes to each bidder with probability $1/2$. Throughout the paper, we suppose $M > V/2$, and most often we restrict attention to the case $M \geq V$. V , M and the rules of the game are common knowledge.

The second price all-pay auction is often compared to a war of attrition in continuous time, where each player has to choose a time t in the interval $[0,1]$ to leave the game (1 plays the role of M); staying in the game is costly (the cost increases in time) but, as soon as one player leaves the game, the game stops and the other gets the prize (this amounts to saying that if player i bids less than player j , player j gets the prize and both pay b_i) (see for example Hendricks, Weiss and Wilson (1988) for the war of attrition with the interval of time).

When $M \geq V$, this game is known to have a lot of intuitive asymmetric Nash equilibria, one player bidding nothing (0), the other bidding V or more. In some way, if the first player fears to lose money, whereas the second player is a hothead, the first player bids 0, whereas the second can afford to bid M (even if M is much larger than V), given that he will pay nothing thanks to the cautious behaviour of the first player. Second price all-pay auctions are rightly seen as dangerous games with great opportunities: when a hothead meets another one, both hotheads lose a lot of money, but if he meets a cautious one, then he gets V without paying anything. And if a cautious player meets another cautious one, each player gets the expected amount $V/2$ without paying anything.

Things become less intuitive when turning to the unique symmetric Nash equilibrium (NE) of this game, which is a mixed NE. The second price all-pay auction is equivalent to a war of attrition, so the mixed NE has the same structure than the one already given in Hendricks, Weiss and Wilson (1988).

Result 1 (folk result)

All the bids in $[M-V/2, M[$ are weakly dominated by M .

The symmetric mixed Nash equilibrium in the continuous game is given by: b is played with probability $f(b)db$, with $f(b) = e^{-b/V}/V$ for b in $[0, M-V/2]$, M is a mass point played with

probability $f(M) = 1 - F(M - V/2) = e^{1/2 - M/V}$, and b in $]M - V/2, M[$ is played with probability 0, where $f(b)$ and $F(b)$ are the density probability distribution and the cumulative probability distribution. The net expected payoff (expected payoff minus M) is equal to 0 at equilibrium.

Proof see Appendix 1

Corollary of result 1:

The symmetric mixed Nash equilibrium in the continuous game without a limit budget is given by $f(b) = e^{-b/V}/V$ for b in $[0, +\infty[$, $f(b)$ being the density probability distribution.

Proof see Appendix 2

Let us comment on these equilibria. Figures 1a and 1b give the general form of the probability distribution, for $V=3$ and $M=6$ (Figure 1a), for $V=15$ and $M=30$ (Figure 1b).

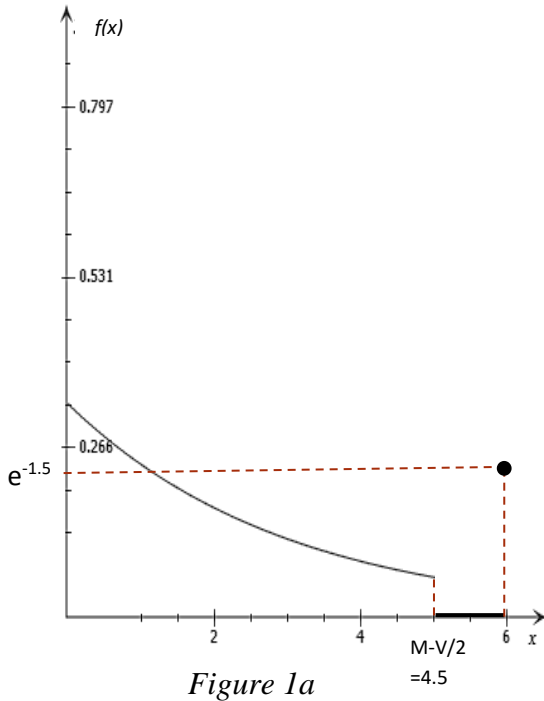


Figure 1a

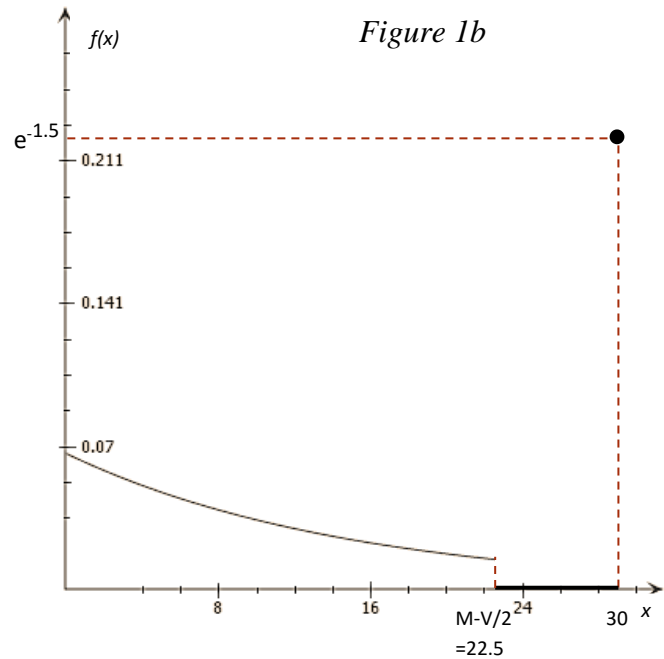


Figure 1b

Let us talk about the probability $e^{1/2 - M/V}$ assigned to M . The fact that M is a mass point is rather intuitive, in that M is a focal point with a special property. Given that nobody can play more than M , given that many players bid less, a player is sure, when he bids M , to get the prize with a high probability and to most often pay less than V (especially if $M - V/2 < V$). In this case, bidding M leads to a negative payoff only if the opponent plays M too.

The probability $e^{1/2 - M/V}$ assigned to M leads to three remarks.

First, it is decreasing in M for a fixed value of V , which ensures a continuity between the game with a budget M going to infinity (V being a fixed amount) and the model without limit budget. In fact, whether the players have or not a limit budget M , they bid b in $[0, M - V/2]$ with the same probability $f(b) = e^{-b/V}/V$. If there is a limit budget M , then $f(M) = \int_{M - V/2}^{\infty} f(b) db$, where $f(b) = e^{-b/V}/V$ is the probability assigned to b in the model without limit budget. So M , when it exists, focuses the probabilities assigned to each bid higher than

$M-V/2$ in the model without limit budget. It follows that there is no discontinuity between the equilibrium obtained with a budget M going to infinity and the equilibrium obtained in the game without limit budget.

Second, $e^{1/2-M/V}$ may be large, especially when M is close to V , but it fast decreases when the ratio M/V grows.

Third, the link between the probabilities assigned to M and 0 is far from being intuitive. There is no mass point on bid 0 , and $f(0)db = db/V$, so doesn't depend on M . Moreover, when V and M become large, but M/V remains constant, $f(0)$ goes to 0 whereas $f(M)$ remains constant. So for example, in Figures 1a and 1b, $f(M) = e^{-1.5} = 0.223$ because $6/3=30/15=2$, but $f(0)=0.333$ for $V=3$ and $M=6$ and $f(0)=0.067$ for $V=15$ and $M=30$.

Let us talk about the probabilities assigned to the other bids, from 0 to $M-V/2$ (from 0 to ∞ when there is no limit M). As already observed, with or without limit budget, each bid b in $[0, M-V/2]$ ($[0, \infty[$), is played with probability $f(b)db = (e^{-b/V}/V)db$. So the density function decreases in b , and the curves become flatter when V increases. It seems rather intuitive to bid 0 with a higher probability, in that bidding 0 never leads to lose money. But it is not so obvious to justify the decreasing probabilities.

Let us also observe that to get the above equilibrium, we implicitly supposed that the agents are risk neutral given that the utility is assumed to be equal to the amount of gotten money. So we can call it the risk neutral Nash equilibrium. Following Hörisch and Kirchkamp (2010), we can opt for the utility function $U(x)=e^{-rM}-e^{-rx}$ to express risk aversion, r being the player's degree of risk aversion.

Proposition 1

All the bids in $[M-V/2, M[$ are weakly dominated by M .

With the utility function $U(x)=e^{-rM}-e^{-rx}$, the symmetric mixed Nash equilibrium in the continuous game is given by: b is played with probability $f(b)db$, with $f(b) = \frac{re^{-rb/V}}{1-e^{-rV}}$ for b in $[0, M-V/2]$, M is a mass point played with probability $f(M)=1-F(M-V/2) = e^{-r(M-V/2)}$, and b in $]M-V/2, M[$ is played with probability 0 , where $f(b)$ and $F(b)$ are the density probability distribution and the cumulative probability distribution. The net expected payoff is equal to 0 at equilibrium.

$F(M)$ decreases in r . When $r \rightarrow 0$, i.e. when the degree of risk aversion goes to 0 , we get back the risk neutral probability distributions.

Proof: see Appendix 3

Let us choose $r=1$. We give in Figure 2a the risk neutral density functions $f(b) = e^{-b/3}/3$ and $f(b) = e^{-b/15}/15$ (respectively for $V=3$ and $V=15$) and the risk averse density functions $f(b) = \frac{e^{-rb/V}}{1-e^{-rV}}$ and $f(b) = \frac{e^{-rb/V}}{1-e^{-rV}}$ (respectively for $V=3$ and $V=15$). In Figure 2b we draw the functions $f(b) = \frac{e^{-b/3}}{1-e^{-3}}$ and $f(b) = \frac{e^{-b/15}}{1-e^{-15}}$ for b in $[0.75, 1.25]$ in order to show the difference between the two functions.

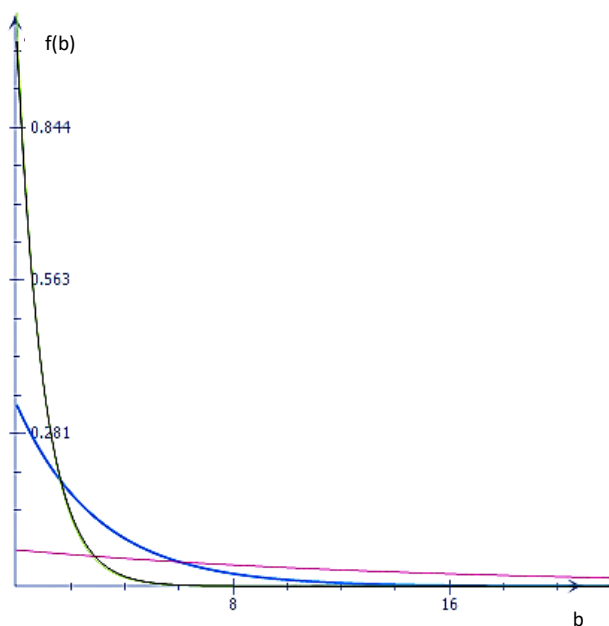


Figure 2a : The two steepest curves (black and green), which seem almost identical,

represent $f(b) = \frac{-b}{e^{1-e^{-3}} - 1 - e^{-3}}$ and $f(b) = \frac{-b}{e^{1-e^{-15}} - 1 - e^{-15}}$.

The flattest (pink) curve represents $f(b) = e^{-b/15}/15$. The in between (blue) curve represents $f(b) = e^{-b/3}/3$.

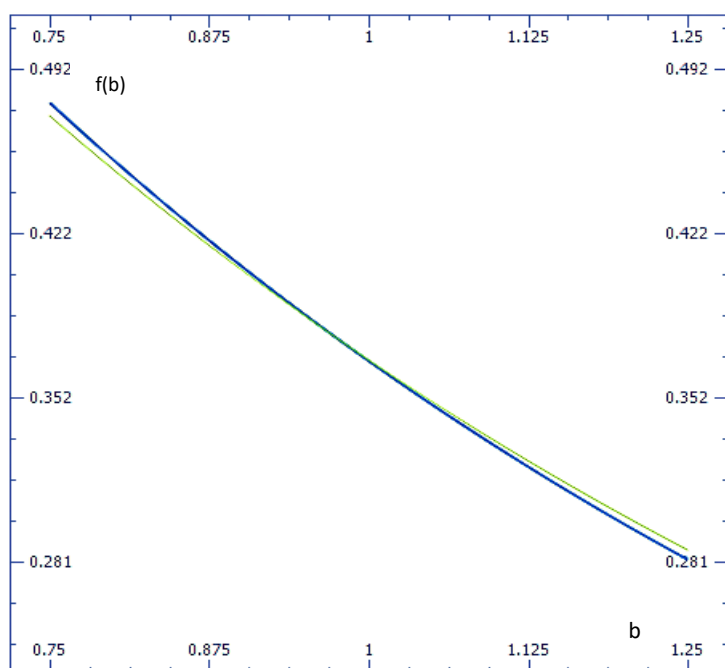


Figure 2b : the blue curve, respectively the green curve, represents the function $f(b) =$

$\frac{-b}{e^{1-e^{-3}} - 1 - e^{-3}}$, respectively the

function $f(b) = \frac{-b}{e^{1-e^{-15}} - 1 - e^{-15}}$ for bids in $[0.75, 1.25]$.

We observe that $\frac{-b}{e^{1-e^{-15}} - 1 - e^{-15}}$ is lower than $\frac{-b}{e^{1-e^{-3}} - 1 - e^{-3}}$ for $b < 0.975$, and higher for $b > 0.975$.

In Figure 2a, we clearly observe that risk aversion leads to more dichotomy in the probabilities, given that the probability distributions decrease very fast and assign a probability close to 0 to bids higher than 5. This contrasts with the much smoother curves obtained for risk neutral players. For $V=3$, respectively $V=15$, risk aversion, in comparison with risk neutrality, leads to assign a higher probability to bids lower than 1.599, respectively lower than 2.901, and a lower probability to higher bids. Let us also observe that $f(M)$ is much lower when the player is risk averse: for $V=3$ and $M=6$ we get $f(M)=0.0088$ instead of 0.223 when the player is risk neutral, and for $V=15$ and $M=30$ we get $f(M)=1.692 \times 10^{-10}$ instead of 0.223 when the player is risk neutral. So risk aversion, without changing the nature of the distribution – we still have a decreasing curve from 0 to $M-V/2$, a null probability for bids in

$]M-V/2, M[$, and a mass point on M - strongly changes the values of the probabilities assigned to each bid and focuses most of the probability on low bids. This is a rather expected result, given that only bid 0 is a bid without risk.

We also observe that, despite the curves $f(b) = \frac{e^{-b/3}}{1-e^{-3}}$ and $f(b) = \frac{e^{-b/15}}{1-e^{-15}}$ are different (see Figure 2b), they are very similar (by contrast to the curves $f(b) = e^{-b/3}/3$ and $f(b) = e^{-b/15}/15$). This is due to the fact that $f(b)$ fast goes to $f(b) = e^{-b}$ when V grows, given that $-e^{-V}$ fast goes to 0 (being added to 1). It follows that $\int_0^1 f(b)db$ goes to 0.6321, so that bids from 0 to 1 cumulate almost 2/3 of the probability. So, to summarize, working with utility functions that take risk into account redirects the probability to very low bids, even if it does not change the nature of the distribution functions.

Finally, let us now draw attention to the fact that in our class room experiments, second price all-pay auctions are discrete games: M , V and b are integers, so the bid increment is equal to 1. As far as we know, few has been said on the mixed NE in discrete second price all-pay auctions. It namely matters to know if the mixed NE in the discrete game converges to the mixed NE in the continuous game. Yet this convergence is in no way automatic (see Umbhauer 2017).

Result 2 (out of Umbhauer 2017)

V , M and the bids are integers.

When V is an odd integer, then the discrete mixed Nash equilibrium goes to the continuous mixed Nash equilibrium for large values of V .

When V is an even integer, then the discrete mixed Nash equilibrium doesn't converge to the continuous mixed Nash equilibrium for large values of V . Only the sum of the discrete probabilities of two adjacent bids goes to the sum of the continuous probabilities of the same two bids.

Proof see Umbhauer (2017)

We illustrate this result with two examples out of Umbhauer 2017. For $V=9$ and $M=12$, we get the discrete Nash equilibrium and the continuous Nash equilibrium in Figure 3a.

We observe that the discrete Nash equilibrium probabilities go to the continuous Nash equilibrium probabilities (when multiplying the continuous NE probabilities $f(i)$, for i from 0 to 7 ($=M-V/2-0.5$) by $(1-f(12))/(\sum_{i=0}^7 f(i))$ to take into account that each bid in $[0, M-V/2]$ is played with probability $f(b)db$ in the continuous game (see Umbhauer 2017)).

For $V=8$ and $M=12$, we get the probabilities in Figure 3b. We clearly observe that in that case the discrete probabilities do not converge to the continuous ones (multiplied by $(1-f(12))/(\sum_{i=0}^7 f(i))$ for the same reason than above) and that they obey a yoyo phenomenon, one probability being much lower than the corresponding continuous probability, the adjacent one being much larger. Yet we could show that the discrete and continuous probabilities converge when summing the probabilities two by two.

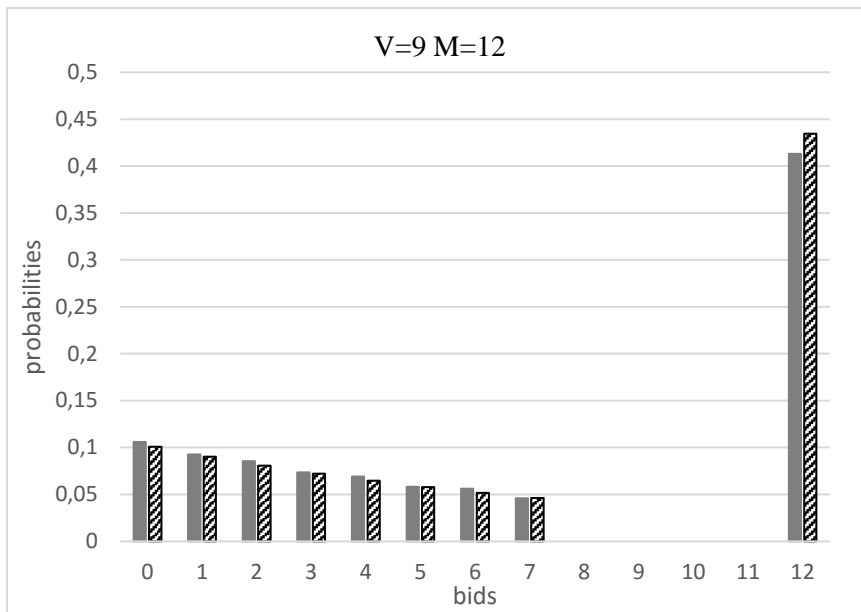


Figure 3a : The full columns are the discrete Nash equilibrium probabilities, the shaded columns are the adjusted continuous Nash equilibrium probabilities

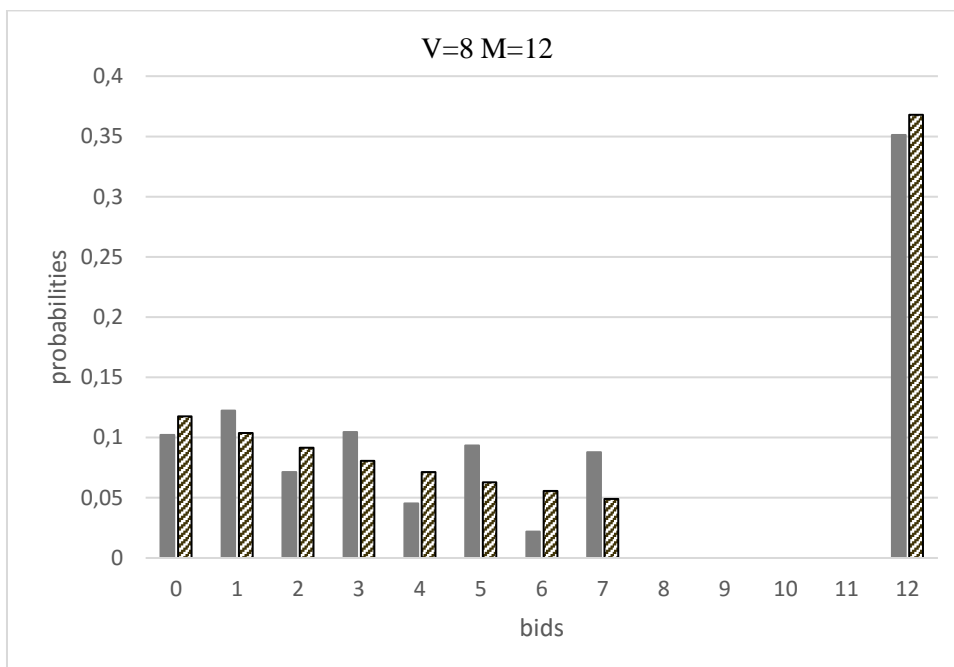


Figure 3b : The full columns are the discrete Nash equilibrium probabilities, the shaded columns are the adjusted continuous Nash equilibrium probabilities.

It follows that, in our class room experiments, we prefer working with odd values of V , in order to get mixed Nash equilibria that are close to the ones obtained in result 1.

3. Class room experiments

In the first classroom experiment¹, 116 L3 students, i.e. undergraduate students in their third year of training, played the second price all-pay auction game in matrix 1 (Game 1), with $V=3$, $M=5$ and the possible bids 0, 1, 2, 3, 4 and 5. In the second classroom experiment, 109 L3 students played the second price all-pay auction game in matrix 2 (Game 2), with $V=30$, $M=60$ and the possible bids 0, 10, 20, 30, 40, 50 and 60. This second game is isomorphic to

¹ This experiment has also been partly studied in Umbhauer (2016).

the game with $V=3$, $M=6$, the bids being 0, 1, 2, 3, 4, 5 and 6 (the payoffs have just to be divided by 10) –so we work with an odd V , despite $V=30$.

		Player 2					
		0	1	2	3	4	5
Player 1	0	(6.5,6.5)	(5, 8)	(5, 8)	(5, 8)	(5, 8)	(5, 8)
	1	(8, 5)	(5.5,5.5)	(4, 7)	(4, 7)	(4, 7)	(4, 7)
	2	(8, 5)	(7, 4)	(4.5,4.5)	(3, 6)	(3, 6)	(3, 6)
	3	(8, 5)	(7, 4)	(6, 3)	(3.5,3.5)	(2, 5)	(2, 5)
	4	(8, 5)	(7, 4)	(6, 3)	(5, 2)	(2.5,2.5)	(1, 4)
	5	(8, 5)	(7, 4)	(6, 3)	(5, 2)	(4, 1)	(1.5,1.5)

Matrix 1 : Game 1, $V=3$, $M=5$

		Player 2						
		0	10	20	30	40	50	60
Player1	0	(75, 75)	(60, 90)	(60, 90)	(60, 90)	(60, 90)	(60, 90)	(60, 90)
	10	(90, 60)	(65, 65)	(50, 80)	(50, 80)	(50, 80)	(50, 80)	(50, 80)
	20	(90, 60)	(80, 50)	(55, 55)	(40, 70)	(40, 70)	(40, 70)	(40, 70)
	30	(90, 60)	(80, 50)	(70, 40)	(45,45)	(30, 60)	(30, 60)	(30, 60)
	40	(90, 60)	(80, 50)	(70, 40)	(60, 30)	(35, 35)	(20,50)	(20,50)
	50	(90, 60)	(80, 50)	(70, 40)	(60, 30)	(50, 20)	(25,25)	(10,40)
	60	(90, 60)	(80, 50)	(70, 40)	(60, 30)	(50, 20)	(40,10)	(15,15)

Matrix 2: Game 2, $V=30$, $M=60$

The games were played during game theory lectures and all the students knew what is a normal form game. So they had no difficulty to understand the two games, and the meaning of the normal forms in matrix 1 and matrix 2. And the students had the matrices in front of them while choosing their bid. Let us add that the first game was played by students trained in Nash equilibria and dominance. By contrast, the second game was played by novice students with no training in Nash equilibria and dominance.

The students' way to play is given in table 1.

Game 1 $V=3$, $M=5$ Bids	Nash equilibrium probabilities (percentage)	Students: frequencies of the bids	Game 2 $V=30$, $M=60$ bids	Nash equilibrium probabilities (percentage)	Students: frequencies of the bids
0	28.3%	37.9%	0	27.7%	33%
1	19.5%	9.5%	10	20.48%	5.5%
2	15.4%	1.7%	20	14.06%	2.8%
3	9.2%	20.7%	30	11.1%	21.1%
4	0	15.5%	40	6.66%	4.6%
5	27.6%	14.7%	50	0%	5.5%
			60	20%	27.5%

Table 1

The students' way to play is also reproduced in Figures 4a and 4b, and juxtaposed with the mixed Nash equilibria in Figures 5a and 5b; the probabilities are replaced by an equivalent number of students.

The students' distributions do not fit with the mixed NE distributions. The main difference concerns low bids different from 0. The probabilities assigned to low bids (1 and 2 in Game 1, 10 and 20 in Game 2) by the NE are much higher than the frequencies with which the students play these bids. Whereas bids 1 and 2 in Game 1 are played with probability 34.9% in the NE, they are only played with probability 11.2% by the students. And whereas the low bids 10 and 20 in Game 2 are played with probability 34.54% in the NE, they are only played with probability 8.3% by the students. Students fear that if they play low bids different from 0, the opponent bids more and makes money whereas they lose their money. In some way, bidding a low amount is seen as a way to encourage the opponent to bid more, even if there is no sequentiality in this game.

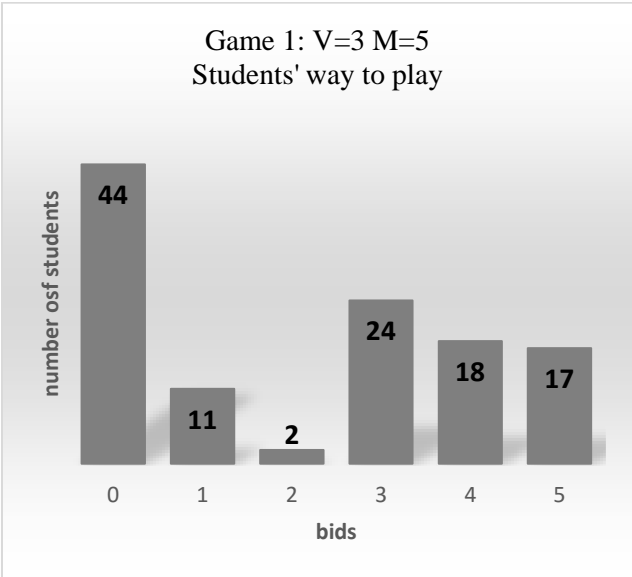


Figure 4a

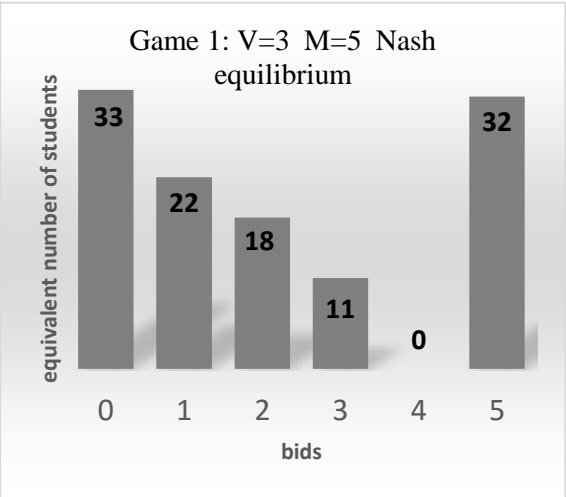


Figure 5a

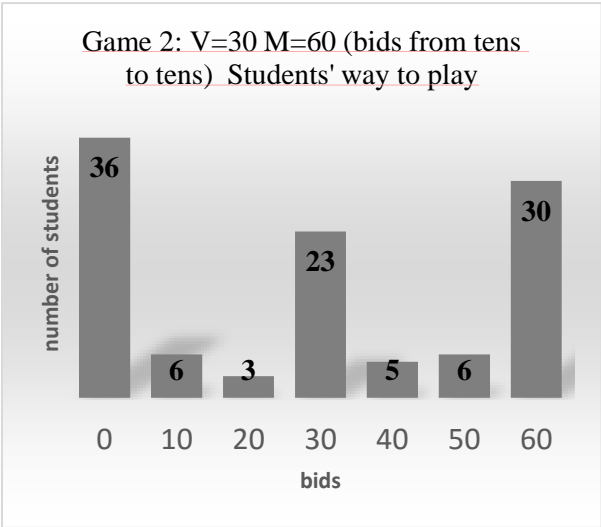


Figure 4b

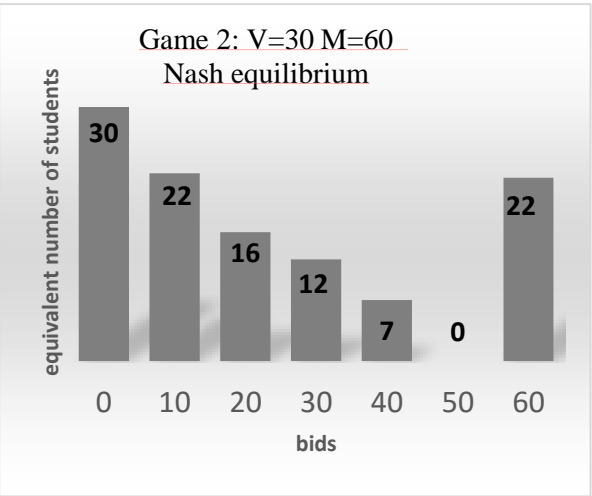


Figure 5b

Another difference concerns the probability assigned to the value V of the prize (3 in Game 1, 30 in Game 2). Students bid this value much more often than in the NE (20.7% versus 9.2% in the first game, 21.1% versus 11.1% in the second game).

We can also observe that the students more often bid 0 than in the NE (37.9% versus 28.3% in Game 1, 33% versus 27.7% in Game 2), though the difference in the probabilities in the second game is less significant.

The way students play bids higher than V is different in the two games. Whereas 30.2% of them play both bids 4 and 5 almost with the same probabilities in Game 1 (in contrast to the NE that assigns probability 0 to bid 4 and 27.6% to bid 5), students, like the NE, assign a small probability to bids 40 and 50 (10.1% for the students, 6.66% in the NE) in Game 2, and a large probability to bid 60 (27.5% for the students, 20% in the NE).

By putting these observations together, it is obvious that the students' probabilities are different from the Nash equilibrium distribution, and - this matters more- that the shapes of the students' distributions are quite different from the mixed NE one.

Well, in our opinion, these differences simply highlight that the philosophy of a mixed NE doesn't fit with the way to play of real players. Let us justify our point of view by turning to best reply matching.

4. Philosophy of mixed Nash equilibria and philosophy of best reply matching.

We first recall Kosfeld et al.'s (2002) Best Reply Matching (BRM) equilibrium.

Definition 1 (Kosfeld & al. 2002): Normal form Best Reply Matching equilibrium
 Let $G=(\mathcal{N}, S_i, \succ_i, i \in \mathcal{N})$ be a game in normal form (\mathcal{N} is the set of players, S_i is player i 's set of pure strategies). A mixed strategy p is a BRM equilibrium if for every player $i \in \mathcal{N}$ and for every pure strategy $s_i \in S_i$:

$$p_i(s_i) = \sum_{s_{-i} \in B_i^{-1}(s_i)} \frac{1}{\text{Card } B_i(s_{-i})} p_{-i}(s_{-i})$$

where $B_i(s_{-i})$ is the set of player i 's best replies to the strategies s_{-i} played by the other agents.

In a BRM equilibrium, the probability assigned to a pure strategy by player i is linked to the probability assigned to the opponents' strategies to which this pure strategy is a best reply: if player i 's opponents play s_{-i} with probability $p_{-i}(s_{-i})$, and if the set of player i 's best responses to s_{-i} is the subset of pure strategies $B_i(s_{-i})$, then each strategy of this subset is played with the probability $p_{-i}(s_{-i})$ divided by the cardinal of $B_i(s_{-i})$.

This criterion builds on the notion of rationalizability developed by Bernheim (1984) and Pearce (1984), according to which a strategy s_i is rationalizable if it is a best response to at least one profile s_{-i} played by the other players. Kosfeld et al. (2002) observe that, if the opponents often play s_{-i} , then s_i often becomes the best response, so should often be played, for actions and responses to be consistent. In other words, if, for example A_1 –respectively B_1 – is player 1's best response to player 2's action A_2 –respectively B_2 –, and A_2 –respectively B_2 –

is player 2's best response to player 1's action A_1 –respectively B_1 -, then a BRM equilibrium can lead player 1 to play A_1 and B_1 with probabilities $1/3$ and $2/3$, and player 2 to play A_2 and B_2 with the same probabilities $1/3$ and $2/3$. So player 1 plays A_1 as often as player 2 plays the action A_2 to which A_1 is the best response, and she plays B_1 as often as player 2 plays the action B_2 to which B_1 is the best response. And vice versa for player 2. Kosfeld et al., in some way, rationalize the probabilities of a player by the other players' probabilities. And this kind of behavior is far from a mixed NE behavior.

Let us illustrate the concepts on the normal form game in matrix 3.

		Player 2	
		A_2	B_2
Player 1	A_1	$(2.9, 1)$	$(5, 5)$
	B_1	$(3, 3)$	$(1, 2.9)$

Matrix 3

We call p and $1-p$, respectively q and $1-q$, the probabilities assigned to A_1 and B_1 , respectively to A_2 and B_2 . The mixed NE leads to equalize the payoffs obtained with A_1 and B_1 , otherwise player 1 would only play the action leading to the highest payoff. So we get $2.9q+5(1-q)=3q+1-q$, i.e. $q=40/41$, i.e. we get a condition on player 2's probabilities. We also have to equalize the payoffs obtained by player 2 with A_2 and B_2 , $p+3(1-p)=5p+2.9(1-p)$, so we get $p=1/41$, a condition on player 1's probabilities. This may seem quite strange. First, a player's probabilities have no impact on his own payoff. They only ensure that the opponent is indifferent between his actions in the NE support. So, when player 1 plays A_1 with probability $1/41$ and B_1 with probability $40/41$, these probabilities mean nothing for herself. She could play A_1 and B_1 each with probability $1/2$, given that, due to player 2's probabilities, she is indifferent between A_1 and B_1 . Second, if she chooses the probabilities $1/41$ and $40/41$, it is only to help player 2 to become indifferent between his two actions. This namely explains why she plays B_1 with a probability close to 1, despite B_1 is not interesting for her when considering the range of payoffs (2.9 and 5 for A_1 , 3 and 1 for B_1). As a matter of facts, she plays B_1 with a probability close to 1 and A_1 with a probability close to 0 because player 2's payoffs- when he plays A_2 and B_2 - are close when she plays B_1 (he gets 3 with A_2 and 2.9 with B_2) whereas they are quite different when she plays A_1 (he gets 1 with A_2 and 5 with B_2). And vice versa for player 2.

Well, we simply think that real players do not choose probabilities in this way. In real life, when somebody plays A with probability $1/41$ and B with probability $40/41$, it is because she thinks that B is much more often her best response than A, 40 times more often, which justifies that she plays B with probability $40/41$. In real life, probabilities –at least often- simply translate the frequency with which an action is seen to be a best response, and, as a consequence, the frequency with which we are ready to play it. And this is what is done in the BRM equilibrium. Let us define it for the game in matrix 3:

- A_1 is player 1's best response to B_2 , so has to be played as often as B_2 , i.e. $p = 1-q$
- B_1 is player 1's best response to A_2 , so has to be played as often as A_2 , i.e. $1-p = q$
- A_2 is player 2's best response to B_1 , so has to be played as often as B_1 , i.e. $q = 1-p$
- B_2 is player 2's best response to A_1 , so has to be played as often as A_1 , i.e. $1-q = p$

And $0 \leq p \leq 1$, $0 \leq q \leq 1$. So, for the studied game, we get an infinite number of BRM equilibria, characterized by the fact that player 1 plays A as often as player 2 plays B, and plays B as often as player 2 plays A.

Let us make 3 observations:

- First, the BRM way to define probabilities allows to cope with the asymmetric pure strategy NE. Given that A_1 is the best response to B_2 and B_2 is the best response to A_1 , the BRM equilibrium allows player 1 to play A_1 with probability 1 and player 2 to play B_2 with probability 1, given that A_1 is player 1's best response as often as player 2 plays B_2 and vice versa. A similar reasoning holds for the profile (B_1, A_2) . So, in this game, the pure strategy NE are also BRM equilibria.
- Second, it is important to note that the expected payoff in the mixed NE (here 121/41) may be higher or lower than the expected payoff in the BRM equilibrium. In the studied game, as long as we choose p between 1/41 and 20/41, the players get more with the NE than with the BRM equilibrium, but for $p < 1/41$ and $p > 20/41$ the players get more with the BRM equilibrium. What matters is the philosophy leading to the payoffs. In the mixed NE, each player wants to get the same payoff with each of the played strategies (i.e. the strategies in the support of the mixed NE). This is not convincing (even if mathematically logical). Especially if the support of the mixed NE is the whole set of pure strategies, have you ever seen a player who says: "let's try to put probabilities on my strategies so that the opponents get the same payoff with all their pure strategies"?²
Real behavior is less sophisticated (and less strange): players simply try to behave at best. In the BRM equilibrium, people choose an action with a high probability if it is the best response to other actions, which are also played with a high probability. Players simply try to be consistent with the way other players are playing, adapting their probability to play an action to the probabilities with which the others play the strategies to which this action is a best reply. This way to deal with probabilities has no link with the mixed NE way to deal with probabilities.
- Third, let us observe that, in the studied game, the mixed NE is also a BRM equilibrium (because $p=1-q=1/41$) - most often mixed NE are not BRM equilibria. Yet the justification of this special BRM equilibrium is not the mixed NE one. Whereas player 1 and player 2, in the NE, calculate the probabilities by equalizing for both players the payoffs obtained with A and B, in the BRM equilibrium, player 1 plays A with probability 1/41 because it is her best response to B_2 which is also played with probability 1/41 and she plays B_1 with probability 40/41 because it is her best response to A_2 , which is played with the same probability 40/41 (and the symmetric explanation holds for player 2). So both actions are played because each is a best response, and not because they lead to the same payoff. In some degree, even when the BRM equilibrium is a mixed equilibrium, players reason in a pure strategy way: in our example the aim is to play A when the other plays B, and to play B when he plays A. This is not the case in a mixed NE, where each player best reacts to the mixed strategies of the others.

² We don't say that this way of doing is always meaningless. If, by doing so, the payoff of the opponent is always low regardless of what he plays, and if the game is a zero sum game (so a player is better off when his opponent is worse off) then behaving in such a way may be strategically meaningful. But, in a usual game like the one in matrix 3, this behaviour is quite strange.

Let us add a generalization of the BRM concept. When there are several best replies to a profile s_{-i} , we think that there is no reason to demand that each best reply is played with the same probability, so we think that is reasonable to generalize Kosfeld et al.'s criterion by allowing to play the different best replies with different probabilities as follows:

Definition 2 (Umbhauer 2007): Generalized Best Reply Matching equilibrium
 Let $G=(\mathcal{N}, S_i, \succ_i, i \in \mathcal{N})$ be a game in normal form. A mixed strategy p is a Generalized BRM (GBRM) equilibrium if for every player $i \in \mathcal{N}$ and for every pure strategy $s_i \in S_i$:

$$p_i(s_i) = \sum_{s_{-i} \in B_i^{-1}(s_i)} \delta_{s_i} p_{-i}(s_{-i})$$

with $\delta_{s_i} \in [0, 1]$ for any s_i belonging to $B_i(s_{-i})$ and $\sum_{s_i \in B_i(s_{-i})} \delta_{s_i} = 1$.

Pure Nash equilibria, by contrast to mixed ones, are automatically GBRM equilibria (out of Umbhauer 2016): if player 1 plays A and player 2 plays B in a pure strategy Nash equilibrium –so they play the actions with probability 1-, player 1 plays A as often as the opponent plays the action B to which A is a –perhaps among several- best reply, and player 2 plays B as often as player 1 plays the action A to which B is a –perhaps among several- best reply.

5. Best reply matching and generalized best reply matching in second price all-pay auctions, real way to play and focal points

We look for the BRM equilibria in both games studied by the students. We first work with Game 1³. To do so, we write the best reply matrix 4a where b_i means that player i 's action is a best reply to the opponent's action, $i=1,2$.

		Player 2					
		q ₀	q ₁	q ₂	q ₃	q ₄	q ₅
		0	1	2	3	4	5
Pl.1	p ₀	0	b ₂	b ₂	b ₁ b ₂	b ₁ b ₂	b ₁ b ₂
	p ₁	1	b ₁		b ₂	b ₂	b ₂
	p ₂	2	b ₁	b ₁		b₂	b ₂
	p ₃	3	b ₁ b ₂	b ₁	b ₁		b ₂
	p ₄	4	b ₁ b ₂	b₁	b ₁	b ₁	
	p ₅	5	b ₁ b ₂	b ₁	b ₁	b ₁	

Matrix 4a

		Player 2							
		q ₀	q ₁₀	q ₂₀	q ₃₀	q ₄₀	q ₅₀	q ₆₀	
		0	10	20	30	40	50	60	
Pl.1	p ₀	0		b ₂	b ₂	b ₁ b ₂	b ₁ b ₂	b ₁ b ₂	b ₁ b ₂
	p ₁₀	10	b ₁		b ₂	b ₂	b ₂	b ₂	b ₂
	p ₂₀	20	b ₁	b ₁		b ₂	b ₂	b ₂	b ₂
	p ₃₀	30	b ₁ b ₂	b ₁	b ₁		b ₂	b ₂	b ₂
	p ₄₀	40	b ₁ b ₂	b ₁	b ₁	b ₁			
	p ₅₀	50	b ₁ b ₂	b ₁	b ₁	b ₁			
	p ₆₀	60	b ₁ b ₂	b ₁	b ₁	b ₁			

Matrix 4b

For example, the bold b_1 in italics means that bid 4 is one of player 1's best replies to player 2's bid 1, and the bold b_2 in italics means that bid 3 is one of player 2's best replies to player 1's bid 2.

In all the studied games, we write p_i , respectively q_i , the probability assigned to bid i by player 1, respectively by player 2 : in Game 1, i goes from 0 to 5.

So we get the system of equations:

³ The results linked to the Game 1 are partly out of Umbhauer 2016.

$$\begin{aligned}
p_0 &= q_3/3 + q_4 + q_5 & q_0 &= p_3/3 + p_4 + p_5 \\
p_1 &= q_0/5 & q_1 &= p_0/5 \\
p_2 &= q_0/5 + q_1/4 & q_2 &= p_0/5 + p_1/4 \\
p_3 &= q_0/5 + q_1/4 + q_2/3 & q_3 &= p_0/5 + p_1/4 + p_2/3 \\
p_4 &= q_0/5 + q_1/4 + q_2/3 + q_3/3 & q_4 &= p_0/5 + p_1/4 + p_2/3 + p_3/3 \\
p_5 &= q_0/5 + q_1/4 + q_2/3 + q_3/3 = p_4 & q_5 &= p_0/5 + p_1/4 + p_2/3 + p_3/3 = q_4 \\
p_0 + p_1 + p_2 + p_3 + p_4 + p_5 &= 1 & q_0 + q_1 + q_2 + q_3 + q_4 + q_5 &= 1
\end{aligned}$$

The solution of this system of equations is :

$$p_0 = q_0 = 180/481 = 37.4\%, \quad p_1 = q_1 = p_0/5 = 7.5\%, \quad p_2 = q_2 = p_0/4 = 9.4\%, \quad p_3 = q_3 = p_0/3 = 12.5\%, \quad p_4 = p_5 = q_4 = q_5 = 4p_0/9 = 16.6\%$$

These probabilities, reproduced in table 2a are far from the Nash equilibrium ones and they fit much better with the students' probabilities, except p_2 (higher) and p_3 (lower). This proximity is due to the fact that BRM exploits main facts also observed by the students, namely that bids 1 and 2 are seldom best responses. As a matter of facts, bidding 1 and 2 seldom leads to win the prize (namely if the opponent bids 3, 4 or 5) and, if you don't win, you lose money with these bids (so it is better to bid 0). In fact, bid 1 is a best response only if the opponent bids 0 (and in this case, bids 2, 3, 4, 5 are also best responses), bid 2 is a best response only if the opponent bids 0 or 1 and in these two cases, bids 3, 4 and 5 are also best responses. By contrast, 0, 3, 4 and 5 are often best responses (bid 0 is the unique best response to bids 4 and 5 and one best response to bid 3, bid 3 is a best response to bids 0, 1 and 2, bids 4 and 5 are best responses to bids 0, 1, 2 and 3).

V=3, M=5 bids	Nash equilibrium probabilities	Students : frequencies of the bids	BRM equilibrium
0	28.3%	37.9%	37.4%
1	19.5%	9.5%	7.5%
2	15.4%	1.7%	9.4%
3	9.2%	20.7%	12.5%
4	0	15.5%	16.6%
5	27.6%	14.7%	16.6%

Table 2a

V=30, M=60 bids	Nash equilibrium probabilities	Students: frequencies of the bids	BRM equilibrium
0	27.7%	33%	39.15%
10	20.48%	5.5%	6.5%
20	14.06%	2.8%	7.8%
30	11.1%	21.1%	9.8%
40	6.66%	4.6%	12.25%
50	0%	5.5%	12.25%
60	20%	27.5%	12.25%

Table 2b

The BRM equilibrium in the second game, whose best reply matrix is given in matrix 4b, is given by: $p_0 = q_0 = 240/613 = 39.15\%$, $p_{10} = q_{10} = p_0/6 = 40/613 = 6.5\%$, $p_{20} = q_{20} = p_0/5 = 48/613 = 7.8\%$, $p_{30} = q_{30} = p_0/4 = 60/613 = 9.8\%$, $p_{40} = p_{50} = p_{60} = q_{40} = q_{50} = q_{60} = 5p_0/16 = 75/613 = 12.25\%$.

(given the system of equations:

$$\begin{aligned}
p_0 &= q_{30}/4 + q_{40} + q_{50} + q_{60} & q_0 &= p_{30}/4 + p_{40} + p_{50} + p_{60} \\
p_{10} &= q_0/6 & q_{10} &= p_0/6 \\
p_{20} &= q_0/6 + q_{10}/5 & q_{20} &= p_0/6 + p_{10}/5 \\
p_{30} &= q_0/6 + q_{10}/5 + q_{20}/4 & q_{30} &= p_0/6 + p_{10}/5 + p_{20}/4 \\
p_{40} &= q_0/6 + q_{10}/5 + q_{20}/4 + q_{30}/4 & q_{40} &= p_0/6 + p_{10}/5 + p_{20}/4 + p_{30}/4 \\
p_{50} &= q_0/6 + q_{10}/5 + q_{20}/4 + q_{30}/4 = p_{40} & q_{50} &= p_0/6 + p_{10}/5 + p_{20}/4 + p_{30}/4 = q_{40} \\
p_{60} &= q_0/6 + q_{10}/5 + q_{20}/4 + q_{30}/4 = p_{40} & q_{60} &= p_0/6 + p_{10}/5 + p_{20}/4 + p_{30}/4 = q_{40} \\
p_0 + p_{10} + p_{20} + p_{30} + p_{40} + p_{50} + p_{60} &= 1 & q_0 + q_{10} + q_{20} + q_{30} + q_{40} + q_{50} + q_{60} &= 1
\end{aligned}$$

Well, this time, the BRM equilibrium is different from the NE and from the students' behavior. This may be due to the fact that students, especially when the number of bids grows and when there are several best replies, do not play all best replies with the same probability, and may even choose to play only some of them, as allowed by GBRM. Focal bids may be preferred each time they are best replies. So, observe that bids 40, 50 and 60 are best replies to the bids 0, 10, 20 and 30, which explains that they are each played with the same probability 12.5% in the BRM equilibrium. With the GBRM concept, a player can choose to play more often some best responses than others, provided that the sum of the probabilities is the same, i.e. $12.5 \times 3 = 37.5\%$. And we can observe that the sum of probabilities assigned to 40, 50 and 60 by the students is 37.6%.

To develop this point, let us suppose that students more focus on threshold values, i.e., in this game, 30 (the value of the prize), 60 (the maximal bid and budget) and 0 (the cautious bid). So suppose that, each time the best responses include one or several of these bids, the players only play these bids. For example, when player 1 bids 10, player 2 only best replies with bid 30 and bid 60, despite bids 20, 40 and 50 are also best responses. This leads to the GBRM matrix 5 (consider only the b_1 and b_2 (underlined and not underlined), the B_1 and B_2 are used in a further study).

		Player 2							
		q ₀	q ₁₀	q ₂₀	q ₃₀	q ₄₀	q ₅₀	q ₆₀	
		0	10	20	30	40	50	60	
Player 1	p ₀	0	B_2		<u>$b_1 b_2$</u>	<u>b_1</u>	<u>b_1</u>	<u>$b_1 b_2$</u>	
	p ₁₀	10	B_1		<u>b_2</u>			b_2	
	p ₂₀	20		B_1		<u>b_2</u>		b_2	
	p ₃₀	30	<u>$b_1 b_2$</u>	<u>b_1</u>	<u>b_1</u>		B_2	B_2	b_2
	p ₄₀	40	<u>b_2</u>			B_1			
	p ₅₀	50	<u>b_2</u>			B_1			
	p ₆₀	60	<u>$b_1 b_2$</u>	b_1	b_1	b_1			

Matrix 5

The set of equations becomes:

$$\begin{aligned}
 p_0 &= q_{30}/2 + q_{40} + q_{50} + q_{60} & q_0 &= p_{30}/2 + p_{40} + p_{50} + p_{60} \\
 p_{10} &= 0 & q_{10} &= 0 \\
 p_{20} &= 0 & q_{20} &= 0 \\
 p_{30} &= q_0/2 + q_{10}/2 + q_{20}/2 & q_{30} &= p_0/2 + p_{10}/2 + p_{20}/2 \\
 p_{40} &= 0 & q_{40} &= 0 \\
 p_{50} &= 0 & q_{50} &= 0 \\
 p_{60} &= q_0/2 + q_{10}/2 + q_{20}/2 + q_{30}/2 & q_{60} &= p_0/2 + p_{10}/2 + p_{20}/2 + p_{30}/2 \\
 p_0 + p_{10} + p_{20} + p_{30} + p_{40} + p_{50} + p_{60} &= 1 & q_0 + q_{10} + q_{20} + q_{30} + q_{40} + q_{50} + q_0 &= 1
 \end{aligned}$$

which simplifies to :

$$\begin{aligned}
 p_0 &= q_{30}/2 + q_{60} & q_0 &= p_{30}/2 + p_{60} \\
 p_{30} &= q_0/2 & q_{30} &= p_0/2 \\
 p_{60} &= q_0/2 + q_{30}/2 & q_{60} &= p_0/2 + p_{30}/2 \\
 p_0 + p_{30} + p_{60} &= 1 & q_0 + q_{30} + q_{60} &= 1
 \end{aligned}$$

whose solution is $p_0 = q_0 = 4/9 = 44.4\%$ $p_{30} = q_{30} = p_0/2 = 22.2\%$, $p_{60} = q_{60} = 3p_0/4 = 33.3\%$ $p_{10} = q_{10} = p_{20} = q_{20} = p_{40} = q_{40} = p_{50} = q_{50} = 0$.

So we get a 3 peak distribution which is similar to the students' one as regards the shape (highest peak on 0, second highest peak on 60 and lowest peak on 30) (see Figure 6a which gives the probabilities in number of students and table 3).

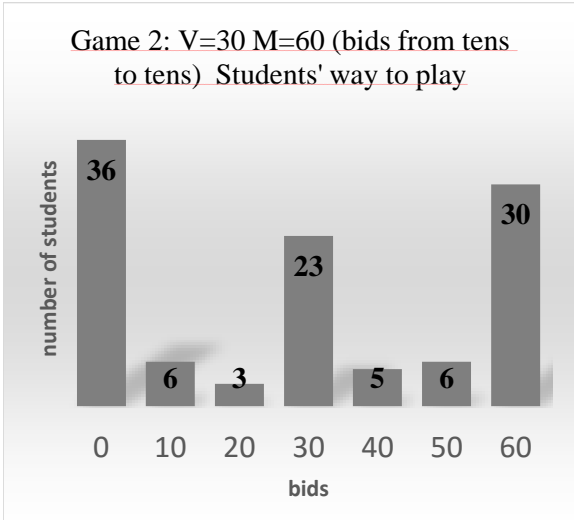


Figure 5b

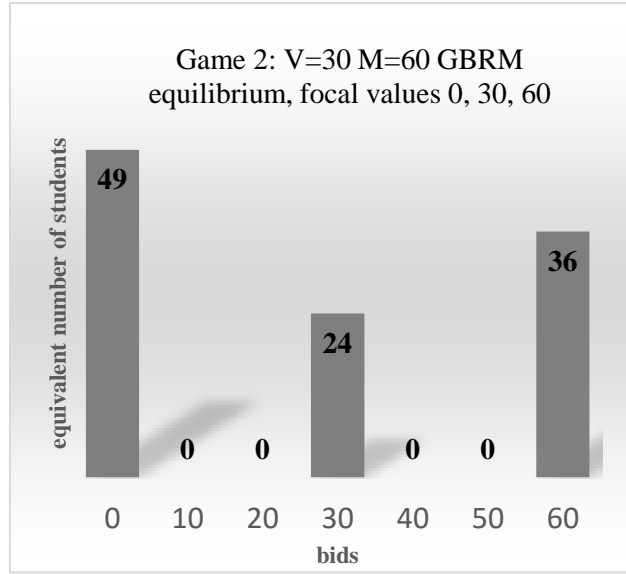


Figure 6a

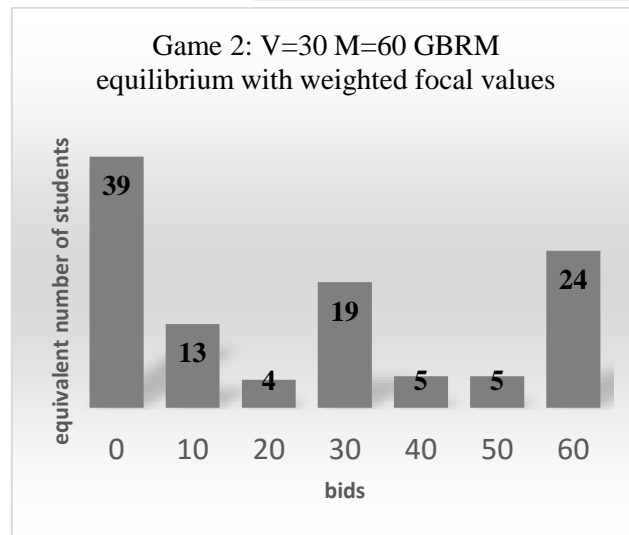


Figure 6b

We can even come closer to the students' probabilities, by not completely excluding 10, 20, 40 and 50 from the played best responses. So we may suppose that players, when selecting best responses, focus on 0, 30 and 60 but also play 10 as the closest best response to 0, 20 as the closest best response to 10, and 40 and 50 as best responses to 30 (we add the B_1 and B_2 in matrix 5). So we get the system of equations:

$$\begin{aligned}
 p_0 &= q_{30}/4 + q_{40} + q_{50} + q_{60} & q_0 &= p_{30}/4 + p_{40} + p_{50} + p_{60} \\
 p_{10} &= q_0/3 & q_{10} &= p_0/3 \\
 p_{20} &= q_{10}/3 & q_{20} &= p_{10}/3 \\
 p_{30} &= q_0/3 + q_{10}/3 + q_{20}/2 & q_{30} &= p_0/3 + p_{10}/3 + p_{20}/2 \\
 p_{40} &= q_{30}/4 = p_{50} & q_{40} &= p_{30}/4 = q_{50} \\
 p_{60} &= q_0/3 + q_{10}/3 + q_{20}/2 + q_{30}/4 & q_{60} &= p_0/3 + p_{10}/3 + p_{20}/2 + p_{30}/4 \\
 p_0 + p_{10} + p_{20} + p_{30} + p_{40} + p_{50} + p_{60} &= 1 & q_0 + q_{10} + q_{20} + q_{30} + q_{40} + q_{50} + q_{60} &= 1
 \end{aligned}$$

whose solution is : $p_0=q_0=72/203=35.5\%$, $p_{10}=q_{10}=24/203=11.8\%$, $p_{20}=q_{20}=8/203=3.9\%$, $p_{30}=q_{30}=36/203=17.7\%$, $p_{40}=p_{50}=q_{40}=q_{50}=9/203=4.45\%$ and $p_{60}=q_{60}=45/203=22.2\%$.

These values are illustrated in Figure 6b and in table 3. It derives that, without special assumptions, we can come close to the students' distribution both in probabilities (see table 3) and in shape (see Figure 6b and Figure 5b).

V=30, M=60 bids	Nash equilibrium	Students	GBRM equilibrium with bids 0, 30 and 60	GBRM equilibrium, with weighted focus on 0, 30 and 60
0	27.7%	33%	44.5%	35.5%
10	20.48%	5.5%	0	11.8%
20	14.06%	2.8%	0	3.9%
30	11.1%	21.1%	22.2%	17.7%
40	6.66%	4.6%	0	4.45%
50	0%	5.5%	0	4.45%
60	20%	27.5%	33.3%	22.2%

Table 3

6. Generalized best reply matching, evolution in focal values, and a way to bring closer first price and second price all-pay auctions

Let us be more general. We first come back to a game where students only focus on 0, V and M, with $M \geq V$ (observe that more than 4/5 of the students only play these bids in Game 2); observe that for any bid in $[0, M]$, at least one of these three bids is a best response to it. So we get the game in matrix 6a, and the best reply matrix 6b ('...' represent the other bids and probabilities).

		Player 2				
		0	...	V	...	M
0	P11	(M+V/2, M+V/2)		(M, M+V)		(M, M+V)
	...					
M	P11	(M+V, M)		(M-V/2, M-V/2)		(M-V, M)
	...					
M	P11	(M+V, M)		(M, M-V)		(V/2, V/2)
	...					

Matrix 6a

		Player 2				
		q ₀	...	q _V	...	q _M
p ₀	0			b ₁ b ₂	b ₁	b ₁ b ₂
	...			b ₂		b ₂
p _V	V	b ₁ b ₂	b ₁			b ₂
	...	b ₂				
p _M	M	b ₁ b ₂	b ₁	b ₁		
	...					

Matrix 6b

For any values of V and M, given that p_i and q_i , i from 1 to $M-1$, $i \neq V$, are equal to 0 (given that the associated bids are never chosen as best responses), the GBRM equations become:

$$\begin{aligned}
 p_0 &= q_V/2 + q_M & q_0 &= p_V/2 + p_M \\
 p_V &= q_0/2 & q_V &= p_0/2 \\
 p_M &= q_0/2 + q_V/2 & q_M &= p_0/2 + p_V/2 \\
 p_0 + p_V + p_M &= 1 & q_0 + q_V + q_M &= 1
 \end{aligned}$$

It follows $p_0 = q_0 = 4/9$, $p_V = q_V = 2/9$, $p_M = q_M = 3/9$.

First observe that these probabilities exploit true facts: bid 0 is more played than M because it is the unique best response to M (so it is played at least as much as M) but it is also one among the 2 best responses to V (0 and M are best responses), which explains the higher probability on bid 0. M is more played than V because each time V is a best response, M is a best response too, and M is also a best response to V (this explains that M is played with a

higher probability than V). These facts may easily be observed by a real player, so he can play in accordance with the hierarchy $p_0 > p_M > p_V$.

Second observe that the probabilities do not depend on the values of V and M (what matters is that $M+V > M+V/2$, $M > M-V/2$, $M > V/2$ and $M > M-V$, which is true for all values of M and V , with $M \geq V/2$). In contrast to the mixed NE concept, the BRM equilibrium is an ordinal concept. It takes into account differences in payoffs but not the value of the payoffs. This is not typical to BRM given that other criteria (like, for example, the pure strategy Nash equilibrium) also only focus on differences in payoffs. But unfortunately, this may imply bad payoffs. As a matter of facts, the GBRM equilibrium net payoff⁴ (payoff minus M) is equal to: $4/9 \times 4V/18 + 2/9(4V/9 - V/9 - 3V/9) + 3/9(4V/9 + 3(V/2 - M)/9) = 24.5V/81 - M/9$. It follows that the net payoff is negative as soon as $M > 24.5V/9$. This flaw doesn't appear with the mixed NE, which is a cardinal concept that equalizes the payoffs obtained with the bids in the equilibrium support. Given that bid 0 is in the mixed NE support, and given that bid 0 leads to a positive net payoff, the mixed NE net payoff is always positive, regardless of the values of M and V .

Yet this doesn't mean that GBRM is necessarily dangerous. Up to now, we focused on the values 0, V and M just to come close to our students in the second experiment, who seemed to prefer 0, V and M as best responses. Observe that our students did not lose money even if their net payoff was barely positive (their payoff is $60.026 > 60$), namely because M/V was not too large: in the experiment, $M=2V < 24.5V/9$. So it is possible that the students most focused on 0, $V=30$ and $M=60$, because they estimated that the possible loss with M was not large enough to prevent them from bidding M . In other terms, M was not felt as being risky. It may be that the students would have behaved differently for other values of V , M , and the ratio M/V . It may be that there exists a kind of bifurcation in the focal bids chosen as best replies when the values of M , V and M/V change and exceed threshold values. For example, let us suppose, as allowed by GBRM, that, if M/V becomes large, players only best respond with 0 and V ; this is possible because either 0 or V belong to the best responses to each possible bid (0 is a best response to all bids higher or equal to V , and V is a best response to each bid lower than V), as illustrated in matrix 5 (see the underlined b_1 and b_2). In that case, the system of GBRM equations, after deleting all the null probabilities, reduces to $q_0 = p_V$ and $q_V = p_0$, each player bidding 0 as often as the opponent bids V and vice versa. It is interesting to note that the symmetric GBRM equilibrium behaviour, which consists to bid 0 and V with probability $1/2$, leads to a net payoff $V/4$, which is never negative and can be much larger than the mixed NE net payoff (for $V=30$ and $M=60$, $V/4=7.5$ whereas the mixed NE net payoff is only 4.157).

To summarize, if real players, like the students in our experiments, behave in accordance with GBRM, that allows to combine BRM and a focus on some focal actions, then it becomes crucial to know if players sufficiently promptly bifurcate in their choice of best replies to avoid losing money. For example, they do not lose money by focusing on 0, V and M as best responses, as long as $M/V < 24.5/9$, but they lose money if $M/V > 24.5/9$. So, if they switch

⁴ By 'net payoff' we mean 'expected net payoff', both for BRM (or GBRM) equilibria and mixed Nash equilibria. For ease of notations, we will proceed so throughout the paper.

from a game where $M/V=2$ to a game where $M/V=4$, to avoid losing money they can switch from the focal values 0, V and M, to the focal values 0 and V. But usually a change in focal values requires time; so players may lose money before changing these values. Another way to cope with this result is to say that second price all-pay auction games should be regulated. A regulatory authority could set an upper bound for M/V , to avoid that players lose a lot of money. This upper bound should take into account the following remark. In an auction game, there is a – sometimes forgotten - third player: the organizer of the auction, who offers the prize and gets the money. In fact the true game can be seen as a pure conflict game with, on the one side, the organizer, on the other side the two players. The players get V and pay a total amount K, the organizer gets K and loses V. So a fair equilibrium should be one where the net payoff of each player is 0, or at least close to 0. So, if players adopt a behaviour close to the one above, regulating M could lead to set $M=2.5V$.

GBRM yields other interesting results. One among them is that GBRM allows second price all-pay auctions to come close to first price all-pay auctions. So, let us suppose that, perhaps because M is large in comparison to V ($M \gg 24.5V/9$) and because the players fear the behaviour of the opponent, they always only choose, as a best response, the lowest possible one : so, in the second price all-pay auction, bid $i+1$ is chosen as the best response to bid i , i from 0 to $V-1$, and bid 0 is chosen as the best response to all bids higher or equal to V. Given that $p_i=q_i=0$, for i from $V+1$ to M, the GBRM equations become:

$$p_0=q_V \quad q_0=p_V$$

$$p_i=q_{i-1} \quad q_i=p_{i-1} \quad i \text{ from } 1 \text{ to } V \quad \sum_{i=0}^V p_i = \sum_{i=0}^V q_i = 1$$

It derives that the symmetric GBRM equilibrium assigns the same probability $1/(V+1)$ to each bid from 0 to V and a null probability to the higher bids.

Well, the only –but crucial- difference between a first price all-pay auction and a second price all-pay auction is that in the first price all-pay auction each player pays *his* bid. So the structure of the best reply matrix is of course different.

We represent the first price all-pay auction for $V=30$ and $M=60$ in matrix 7a.

		Player 2						
		0	10	20	30	40	50	60
Player 1	0	(75, 75)	(60, 80)	(60, 70)	(60, 60)	(60, 50)	(60, 40)	(60, 30)
	10	(80, 60)	(65, 65)	(50, 70)	(50, 60)	(50, 50)	(50, 40)	(50, 30)
	20	(70, 60)	(70, 50)	(55, 55)	(40, 60)	(40, 50)	(40, 40)	(40, 30)
	30	(60, 60)	(60, 50)	(60, 40)	(45, 45)	(30, 50)	(30, 40)	(30, 30)
	40	(50, 60)	(50, 50)	(50, 40)	(50, 30)	(35, 35)	(20, 40)	(20, 30)
	50	(40, 60)	(40, 50)	(40, 40)	(40, 30)	(40, 20)	(25, 25)	(10, 30)
	60	(30, 60)	(30, 50)	(30, 40)	(30, 30)	(30, 20)	(30, 10)	(15, 15)

Matrix 7a

We recall in matrix 7b the best reply matrix for the second price all-pay auction with $V=30$ and $M=60$; matrix 7c is the best reply matrix for the first price all-pay auction with $V=30$ and $M=60$. The bold and underlined best responses are the best responses selected by our GBRM equilibrium.

We namely observe, in the first price all-pay auction, that bid $i+1$ is the only best response to bid i , i from 0 to $V-2$, that bid 0 is the only best reply to all bids higher or equal to V , and that bid 0 and bid V are the only best responses to bid $V-1$. So observe that, if we choose V as the best response to $V-1$, we get exactly the same GBRM equations and so the same symmetric GBRM equilibrium as above for this very different game.

		Player 2						
		q ₀	q ₁₀	q ₂₀	q ₃₀	q ₄₀	q ₅₀	q ₆₀
		0	10	20	30	40	50	60
Pl.1	p ₀	0	<u>b₂</u>	b ₂	<u>b₁b₂</u>	<u>b₁b₂</u>	<u>b₁b₂</u>	<u>b₁b₂</u>
	p ₁₀	10	<u>b₁</u>	<u>b₂</u>	b ₂	b ₂	b ₂	b ₂
	p ₂₀	20	b ₁	<u>b₁</u>	<u>b₂</u>	b ₂	b ₂	b ₂
	p ₃₀	30	<u>b₁b₂</u>	b ₁	<u>b₁</u>		b ₂	b ₂
	p ₄₀	40	<u>b₁b₂</u>	b ₁	b ₁	b ₁		
	p ₅₀	50	<u>b₁b₂</u>	b ₁	b ₁	b ₁		
	p ₆₀	60	<u>b₁b₂</u>	b ₁	b ₁	b ₁		

Matrix 7b

		Player 2						
		q ₀	q ₁₀	q ₂₀	q ₃₀	q ₄₀	q ₅₀	q ₆₀
		0	10	20	30	40	50	60
Pl.1	p ₀	0	<u>b₂</u>	b ₁	<u>b₁</u>	<u>b₁</u>	<u>b₁</u>	<u>b₁</u>
	p ₁₀	10	<u>b₁</u>	<u>b₂</u>				
	p ₂₀	20	b ₂	<u>b₁</u>	<u>b₂</u>			
	p ₃₀	30	<u>b₂</u>		<u>b₁</u>			
	p ₄₀	40	<u>b₂</u>					
	p ₅₀	50	<u>b₂</u>					
	p ₆₀	60	<u>b₂</u>					

Matrix 7c

Even if real players, in the second price all-pay auction surely not often behave as above (especially if M is not too large), we find it always interesting to get some robustness, i.e. to get a consistent behaviour that holds in different contexts. Moreover, the obtained equilibrium is a very easy one, easy to learn, which is also a good thing.

And what is more, it leads to a null payoff in the first price all-pay auction, and to a positive payoff in the second price all-pay auction.

Proposition 2

The GBRM equilibrium that assigns the probability $1/(V+1)$ to each bid from 0 to V leads to a null net payoff in the first price all-pay auction. It leads to the net payoff $(V^2+2V)/(6V+6)$ in the second price all-pay auction. Moreover, in the second price all-pay auction, the net payoff obtained with each played bid is positive.

Proof see Appendix 4

It follows from proposition 2 that, by playing this GBRM equilibrium in the second price all-pay auction, the players play in a way that is as safe as the Nash equilibrium way, given that they get a positive net payoff with each played bid. And the net payoff is much larger than the NE one, especially if V is large. In our game, for $V=30$ and $M=60$ (isomorphic to $V=3$ and $M=6$ the payoffs being multiplied by 10) we get the net payoff 6.25, which is positive and larger than the NE net payoff.

Let us finally observe that the above GBRM is fair in the first price all-pay auction (null net payoff for the players and the organizer) but to the benefit of the players in the second price all-pay auction.

7. Best reply matching equilibrium for any values of V and M, evolution of payoffs and probabilities

In sections 7 and 8, we come back to the BRM equilibrium, so we suppose that each best response is played with the same probability.

Proposition 4 gives the symmetric BRM equilibrium in the second price all-pay auction game for any integer values V and M, $M \geq V$.

Proposition 3

For $M \geq V$, the symmetric BRM equilibrium is given by:

$$q_0 = \frac{1}{2 + \sum_{i=0}^{V-1} \frac{1}{M-i} - \frac{1}{(M-V+1)^2}}$$

$$q_i = q_0/(M-i+1), \quad i \text{ from } 1 \text{ to } V$$

$$q_i = q_0(M-V+2)/(M-V+1)^2, \quad i \text{ from } V+1 \text{ to } M$$

where q_i is the probability to play bid i , i from 0 to M.

It follows that: $\sum_{i=V+1}^M q_i < q_0 < \sum_{i=V}^M q_i$

Proof see Appendix 5

Let us comment on this equilibrium.

We first compare the shape of the BRM equilibrium distribution and the shape of the mixed (risk neutral) NE distribution.

In the BRM equilibrium, q_i is increasing in i for i from 1 to $V+1$, and is constant from $V+1$ to M , a result in sharp contrast with the mixed NE probabilities that are decreasing from 1 to $M-V/2$ and null from $M-V/2$ to M (excluded). BRM clearly takes into account that a higher bid is more often a best reply than a lower one (different from 0), in that each bid b (different from 0) is a best reply to all the bids lower than b , if $b \leq V$, and a best reply to all bids from 0 to V if b is higher than V . And bid 0, in contrast to the other low bids, has a special status in that it is a best reply to all bids from V to M .

Clearly, the Nash and the BRM distributions have no common points, except the fact that the probability to bid 0 is higher than the probabilities to bid i , i from 1 to V , both in the BRM equilibrium and in the NE. The strong differences and the few similarities are highlighted in Figures 7a and 7b, which give the BRM equilibrium and the mixed NE for $V=9$ and $M=12$.

Taking risk aversion into account doesn't change the fact that the NE probabilities decrease from bid 0 to bid 7, so do not fit with the BRM probabilities; the main impact of risk aversion is that the continuous NE probability to bid 12 is very small and that the probabilities⁵ fast decrease from bid 0 to bid 7 (see Figure 7c).

Second, let us talk about payoffs. As regards the payoffs in the BRM equilibrium and in the mixed NE, for M not too far from V , the BRM equilibrium payoffs may be higher than the NE ones. For example, for $V=3$ and $M=5$, the Nash equilibrium payoff is $1589.5/293 = 5.425$, whereas the BRM payoff is 5.457.

⁵ We here take the adjusted continuous Nash equilibrium probabilities (i.e. the continuous probabilities $f(b)$ multiplied by $(1-f(12))/\sum_{i=0}^7 f(i)$)

$$\begin{aligned}
 & (= 180(6.5 \times 180 + 5(481 - 180)) + 36(8 \times 180 + 5.5 \times 36 + 4(481 - 180 - 36)) + 45(8 \times 180 + 7 \times 36 + \\
 & 4.5 \times 45 + 3 \times 60 + 3 \times 80 + 3 \times 80) + 60(8 \times 180 + 7 \times 36 + 6 \times 45 + 3.5 \times 60 + 2 \times 80 + 2 \times 80) + 80(8 \times 180 \\
 & + 7 \times 36 + 6 \times 45 + 5 \times 60 + 2.5 \times 80 + 1 \times 80) + 80(8 \times 180 + 7 \times 36 + 6 \times 45 + 5 \times 60 + 4 \times 80 + 1.5 \times 80) / 481^2 = \\
 & 1262620.5 / 481^2 = 5.457).
 \end{aligned}$$

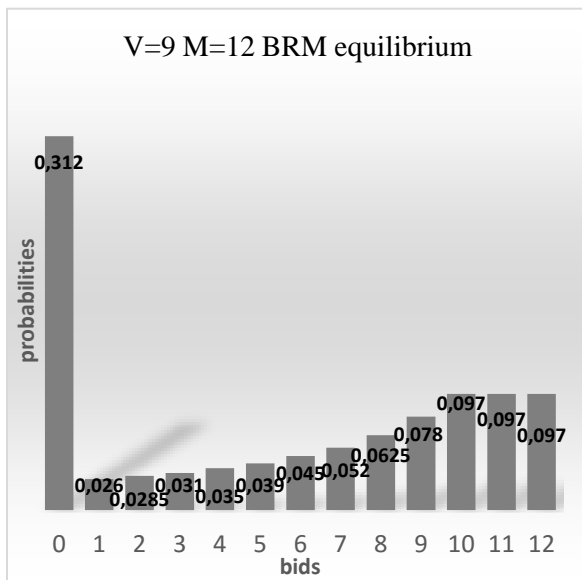


Figure 7a

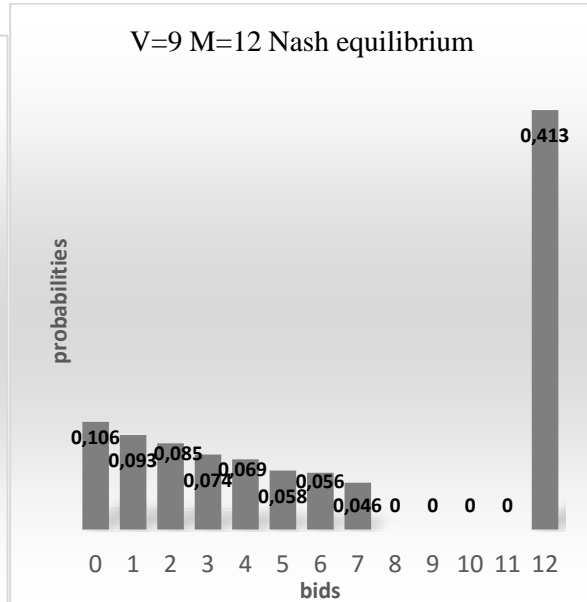


Figure 7b

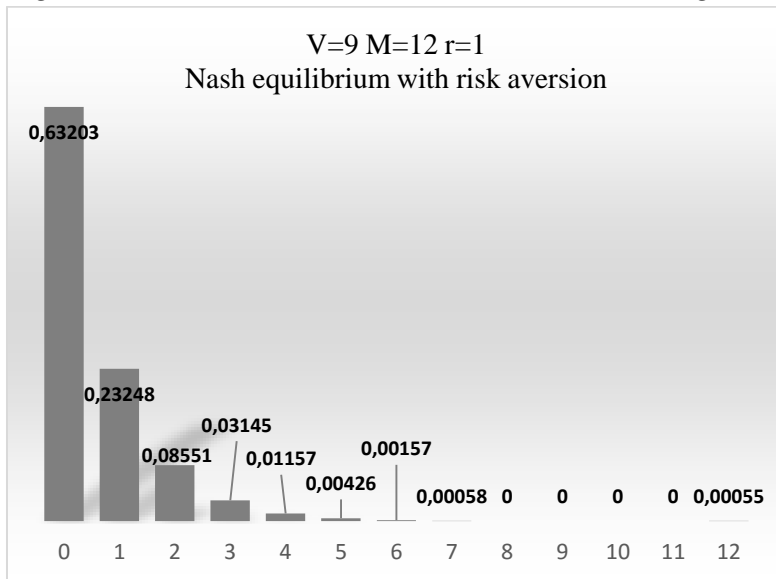


Figure 7c

For $V=9$ and $M=12$, the BRM equilibrium payoff is 13.528, whereas the Nash equilibrium payoff is only 12.477.

For $V=30$ and $M=60$ (and bids from tens to tens) the results are reversed: the Nash equilibrium payoff is 64.157 and the BRM payoff is slightly lower ($24082745/613^2 = 64.089$).

Proposition 4

We call $Eg(i)$ the payoff obtained with bid i in the BRM equilibrium. We get:

$$Eg(i+1)-Eg(i)= Vq_i/2+Vq_{i+1}/2-\sum_{j=i+1}^M q_j \quad i \text{ from } 0 \text{ to } M-1.$$

$Eg(i+1)-Eg(i)$ is increasing in i , i from 1 to $M-1$.

For $M > 2V$, $Eg(i)$ decreases from bid 1 to bid $M-V$, then increases up to M .

Proof see Appendix 6

It derives from proposition 4 that the general form of the payoff function goes as follows. Most often $Eg(1) > Eg(0)$ due to the fact that $Vq_0/2 + Vq_1/2 - \sum_{j=1}^M q_j > 0$ (because of the large value of q_0). Then, generally, at least if M is not too close to V , the payoff function decreases (due to the fact that $Vq_1/2 + Vq_2/2 - \sum_{j=2}^M q_j < 0$) for a while; and, when $Eg(i+1) - Eg(i)$ becomes positive, the payoff function increases up to bid M . We represent the (net) payoff function for $V=9$ and $M=12$ in Figure 8.

Let us comment on these payoffs. If we compare the payoffs in Figure 8 and the probabilities assigned to the bids in Figure 7a, we may feel uncomfortable in that the evolution of the payoffs doesn't follow the evolution of the probabilities. For example, q_1 is the smallest probability but bid 1 yields a better net payoff than many other bids; and q_i increases from 1 to 7 whereas the net payoffs decrease in the same time. Yet let us first observe that the player, regardless of his chosen bid, never gets a negative net payoff by playing in this way and that he always gets a net payoff which is higher than the NE net payoff. So, his lowest net payoff, obtained for bid 7, is equal to 0.704 which is higher than 0 and higher than the Nash equilibrium net payoff 0.477. So, of course, the BRM equilibrium is not stabilized as regards the Nash logic (the only best reply to the above payoff profile is to bid 12), but, on the one hand, it leads to a higher payoff than the NE and, on the other hand, it is consistent in another way: the probability of a bid b is the probability of the bids to which b is a best response (divided by the number of best responses). This justifies the low probability on bid 1: bid 1 is only a best reply to bid 0, and, in that case, all the bids from 2 to 12 are also best replies.

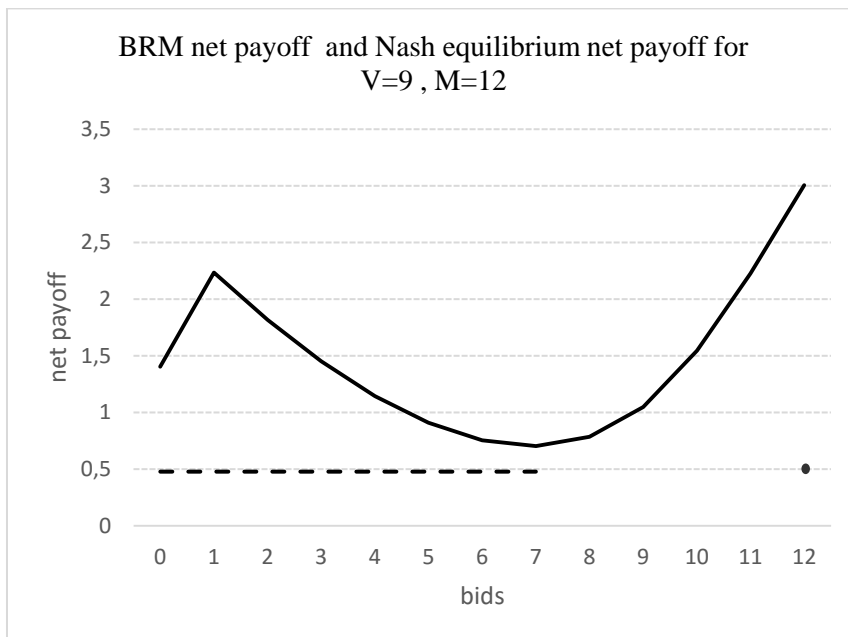


Figure 8: the BRM net payoffs are given by the full curve whereas the (risk neutral) NE net payoffs are given by the dashed line.

Third, Figures 7a and 7b highlight that it is difficult to speak about overbidding or underbidding, when comparing BRM behaviour and Nash behaviour, because the structure of behaviours is completely different. The only thing we can say is that, with BRM, for $M > 2V$, bids higher than the value of the prize focus more probability than the bids from 1 to V . As a matter of fact, given that $q_i = q_V + q_V / (M - V + 1)$, for i from $V + 1$ to M , we get

$$\sum_{i=1}^V q_i < V q_V < \sum_{i=V+1}^M q_i = (M - V) q_V + (M - V) q_V / (M - V + 1)$$

because $(M - 2V) q_V + (M - V) q_V / (M - V + 1) > 0$

Fourth, one can observe that weak dominance has no impact in the BRM equilibrium, in that all the bids from $V + 1$ to M are best responses to the same bids (from 0 to V). This isn't shocking given that the pure strategy Nash equilibria of the game also lead one player to bid 0 and the other player to bid any bid in $[V, M]$ (for $M \geq V$).

Fifth, let us draw attention to the weight assigned to bid 0. Due to the special status of bid 0 - bid 0 is the unique best reply to all the bids from $V + 1$ to M , and (only) one best reply to V -, it immediately follows that that $\sum_{i=V+1}^M q_i < q_0 < \sum_{i=1}^V q_i$. So, at least for $M > 2V$, less than 1/3 of the probability is assigned to the bids from 1 to V , and the remaining probability is almost equally shared among bid 0 on the one hand, and the bids from $V + 1$ to M on the other hand. So 0 is a mode of the distribution (we could say that there is a kind of mass point on 0). This remark generalizes as follows:

Proposition 5

The BRM equilibrium, for $M \geq V$, assigns to bid 0 a higher probability than the mixed (risk neutral) NE, as soon as $V \geq 4$.

Proof see Appendix 7

This result doesn't hold for the risk averse Nash equilibrium which also assigns high values to low bids (close to 0.632 for bids between 0 and 1 when V is large). But the BRM probabilities do not better converge to the risk averse Nash equilibrium probabilities: given that $\sum_{i=V+1}^M q_i < q_0 < \sum_{i=1}^V q_i$, in the BRM equilibrium, q_0 is lower than 1/2 and close to $\sum_{i=V}^M q_i$, something which isn't true in the risk averse Nash equilibrium.

8. Best reply matching equilibrium and money losses

We now only focus on the BRM equilibrium and the risk neutral Nash equilibrium. We first give a few additional results for two special values of the ratio M/V , $M/V=2$ and $M/V=1$.

Proposition 6

For $M=2V$ and V large, $q_0 \approx 1/(2 + \ln(2)) = 0.371$, $\sum_{i=V}^M q_i \approx 0,371$ and $\sum_{i=1}^{V-1} q_i \approx 0.258$.

Proof see Appendix 8

Let us compare these values with the NE ones. In the mixed NE, which goes to the NE in the continuous game, we have:

$$\int_V^{M-V/2} \frac{e^{-x/V}}{V} dx + e^{1/2-M/V} = [-e^{-x/V}]_V^{3V/2} + e^{-3/2} = e^{-1} = 0.368$$

So we get almost the same weight on the bids from V to M , but not distributed in the same way. In the NE, $M (=2V)$ is played with the probability $e^{-3/2} = 0.223$ and the weight $(0.368 - 0.223) = 0.146$ is shared among the bids from V to $M - V/2 = 3V/2$ in a decreasing way, the bids from $3V/2$ to $2V$ (excluded) being played with probability 0.

And the remaining probability $(1 - 0.368) = 0.632$ is shared among the bids from 0 to V in a decreasing way but without mass point on 0.

Proposition 7

For $M=V$, $q_0=q_M \approx 1/(1+\ln(V)+\gamma+1/(2V))$ where γ is Euler's constant 0.577, and $q_i=1/[(1+\ln(V)+\gamma+1/(2V))(V-i+1)]$, i from 1 to V

Proof see Appendix 8

Contrary to the case $M=2V$ and V large, where bid 0 on the one hand, the set of bids from V to M on the other hand, are each played with a probability close to 0.371, we observe that, for $M=V$, bid 0 and the set of bids from V to M (which reduces to the bid V), are each played with the probability $1/(1+\ln(V)+\gamma+1/(2V))$, which goes to 0 for V large. Yet this isn't strange, in that $M=V$ is a special case. If $M=V$, bid 0 is the best reply to only one bid, $M (=V)$, so is played as often as M is played (that is why $q_M=q_0$). M is the bid that is most often a best reply (it is a best reply to all other bids), so it is played with the highest probability, but the other bids are also often best replies (each bid is a best reply to all bids lower than it); for example, $M-1$ is a best reply to all bids except $M-1$ and M . This explains that the other bids also focus high probabilities, which explains that 0 and M are played with a probability decreasing in V . Yet observe that this probability decreases slowly (for $V=200$, M and 0 are still each played with probability 0.145 (so the other 199 bids share the probability 0.71), for $V=10000$, 0 and M still focus 18.5% of the probability).

Now we turn to the possible waste of money the BRM equilibrium may lead to, when M becomes large. More precisely, we focus on second price all-pay auction games with a very large budget ($M \rightarrow \infty$) compared to V ($M/V \rightarrow \infty$).

In that case q_0 goes to $1/2$, and all the other probabilities go to 0 (but are still increasing in i from 1 to $V+1$ and constant from $V+1$ to M), because $\sum_{i=1}^V q_i < Vq_V = Vq_0/(M-V+1) \rightarrow 0$ and $\sum_{i=V+1}^M q_i = q_0 (M-V+2)(M-V)/(M-V+1)^2 \rightarrow q_0$, so q_0 and $\sum_{i=V+1}^M q_i$ go to $1/2$.

It derives that, when M is large, the BRM probabilities are shared on bid 0 (probability $1/2$) and homogeneously shared on the set of bids from $V+1$ to M (probability $1/2$ on this set). We have a kind of bimodal distribution, $1/2$ on bid 0 and $1/2$ on a set (each bid in the set being played with the same probability).

Observe that this result is close to the pure strategy Nash equilibrium spirit (for each player bidding 0, there is a player bidding i , with i higher than V), and quite far from the mixed NE

(with no mass point on any bid and decreasing probabilities on $[0, +\infty[$ (probability $(1/V)db$ on 0)).

Yet this behaviour leads to a negative net payoff.

Proposition 8

For very large values of M , M large in comparison to V ($M/V \rightarrow \infty$, V is a constant), the mean loss of a player, at the BRM equilibrium, is equal to $1/12^{\text{th}}$ of his budget M . The main loss is obtained for the bid $M-V$: the player loses $1/4^{\text{th}}$ of M .

Proof see Appendix 9

So, for large values of M , a player may often suffer from the winner's curse. Playing high bids leads him to often win the prize, but he pays too much given that he often wins against players who bid too much, leading him to lose up to $1/4^{\text{th}}$ of his budget M . And in average, he loses $1/12^{\text{th}}$ of M , which is a huge amount of money. This leads again to our previous remark on regulation. The above result is established for V constant and M going to infinity, i.e. for very large ratios M/V . Well, to avoid a huge waste of money, it may be necessary to limit the ratio M/V . This is all the more necessary that the wasted money by the two players ($2M/12$) is not wasted for everybody: it is gotten by the organizer, and this of course seems unfair.

9. Concluding remarks

In this paper we draw attention to the real behaviour in second price all-pay auctions. This behaviour seems to better fit with best reply matching and generalized best reply matching than with the mixed Nash equilibrium.

We established that, by contrast to the mixed Nash behaviour, that never leads to win or lose money, a best reply matching (or generalized best reply) matching behaviour can lead to lose or get a huge amount of money. With best reply matching, when M and M/V become large, players lose $1/12^{\text{th}}$ of their budget, and even up to $1/4^{\text{th}}$ of the budget with some bids. With generalized best reply matching, that allows to combine best reply matching and focal points, all depends on the chosen focal points. We established for example that, if the bidders always focus on the lowest possible best response, they get a net payoff close to $V/6$ (when V is large). And if they only focus on 0 or V as best responses, each player gets a net payoff $V/4$, even if M/V is large.

So it is important to conclude that players can get or lose money, much more than in the Nash equilibrium, according to the best responses they focus on. We namely highlighted in the paper that if M/V becomes larger than a given threshold, then players should delete M from their focal values, something which may be difficult given the inertia linked to focal points.

Well, by contrast to continuous Nash equilibria that lead to a null net payoff, best reply matching and generalized best reply matching most often lead to unfair equilibria where either the players are making money, either the organizer is making money. And this is not necessarily a bad thing, first because it simply translates real behaviour, second because the winner or the loser is not always on the same side: the players and the organizer can win or lose. Yet the possible losses of the players advocate for some regulation of the second price

all-pay auctions: so $M=2.5V$ seems to be a good way to limit bids, in that, for $M=2.5V$, a natural focal behaviour combined to best reply matching does not lead the players to lose money, without allowing them to get high payoffs.

Yet let us make two last remarks. First observe that the players' losses of money we talked about ($1/12^{\text{th}}$ of M , $1/4^{\text{th}}$ of M) are linked to M , whereas the players' gains of money are linked to V ($V/6$, $V/4$). This leads us to the second remark. There is a difference between the existence of a limit budget, even if it goes to ∞ , and the non-existence of a limit budget. To establish the result in proposition 8, we need the existence of M . In addition, for M to be a focal point, M has to exist. And we have established that if players focuses on best responses lower or equal to V , they usually make money. So perhaps a good way to help the players is to delete M , so to set no limits for the bids. May be that the main role of M , which a priori protects the players (by forbidding them to bid too much), is to protect the organizer. That's an open question.

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Appendix 1 Proof of result 1 (out of Umbhauer 2016)

All bids between $M-V/2$ and M are weakly dominated by M (obvious result), so it is conjectured that the Nash equilibrium strategy is a density function $f(\cdot)$ on $[0, M-V/2]$ with a mass point on M .

Call $f_2(\cdot)$ player 2's equilibrium strategy. Suppose that player 1 plays b . She wins the auction each time player 2 bids less than b . So her (expected) payoff $Eg(b)$ is equal to:

$$Eg(b) = M + \int_0^b (V - x)f_2(x)dx - b\left(\int_b^{M-V/2} f_2(x)dx + f_2(M)\right)$$

We check that a player gets the same payoff with M and $M-V/2$, regardless of the opponents' equilibrium distribution.

$$Eg(M) = M + \int_0^{M-V/2} (V - x)f_2(x)dx + \left(\frac{V}{2} - M\right) f_2(M) = Eg(M-V/2)$$

$Eg(b)$ has to be constant for each b in $[0, M-V/2] \cup \{M\}$. So $Eg'(b) = 0$ for b in $[0, M-V/2]$.

$$\text{We get } (V-b)f_2(b) - F_2(M-V/2) + F_2(b) - f_2(M) + bf_2(b) = 0$$

where $F_2(\cdot)$ is the cumulative distribution of the density function $f_2(\cdot)$.

By construction $f_2(M) = 1 - F_2(M-V/2)$, so we get the differential equation $Vf_2(b) - 1 + F_2(b) = 0$ whose solution is $F_2(b) = 1 + Ke^{-b/V}$ where K is a constant determined as follows:

$F_2(0) = 0$ because there is no mass point on 0, so $1 + K = 0$, i.e. $K = -1$.

It follows $F_2(b) = 1 - e^{-b/V}$ for b in $[0, M-V/2]$, $f_2(M) = 1 - F_2(M-V/2) = e^{1/2 - M/V} (< 1)$,

$f_2(b) = e^{-b/V}/V$ for b in $[0, M-V/2]$ (and $f_2(b) = 0$ for b in $]M-V/2, M[$).

By symmetry, we get $f_1(b) = e^{-b/V}/V$ for b in $[0, M-V/2]$, $f_1(M) = 1 - F_1(M-V/2) = e^{1/2 - M/V}$ (and $f_1(b) = 0$ for b in $]M-V/2, M[$).

Given that $Eg(0)$ is equal to M , given that bid 0 is played at equilibrium, and given that a player gets the same payoff with each played bid, each player gets the (expected) payoff M , hence a net payoff equal to 0 at equilibrium.

Appendix 2 Proof of the Corollary of result 1

If there is no limit budget, player 1's payoff when she plays b is:

$$Eg(b) = M + \int_0^b (V - x)f_2(x)dx - b\left(\int_b^{+\infty} f_2(x)dx\right)$$

$Eg(b)$ has to be constant for each b in $[0, +\infty[$. So $Eg'(b) = 0$ for b in $[0, +\infty[$.

$$\text{We get } (V-b)f_2(b) - (1 - F_2(b)) + bf_2(b) = 0$$

i.e.: $Vf_2(b) - 1 + F_2(b) = 0$.

So we get the solution $f_2(b) = e^{-b/V}/V$ for b in $[0, +\infty[$. By symmetry, we get $f_1(b) = e^{-b/V}/V$ for b in $[0, +\infty[$.

Appendix 3 Proof of proposition 1

All bids between $M-V/2$ and M are still weakly dominated by M (nothing changes as regards the way the players get the prize and the amount they pay), so it is again conjectured that the Nash equilibrium strategy is a density function $f(\cdot)$ on $[0, M-V/2]$ with a mass point on M . So we start as in Appendix 1, but $Eg(b)$ now becomes:

$$Eg(b) = \int_0^b (e^{-rM} - e^{-r(M+V-x)})f_2(x)dx + (e^{-rM} - e^{-r(M-b)})\left(\int_b^{M-V/2} f_2(x)dx + f_2(M)\right)$$

It immediately follows that a player gets the same payoff with M and $M-V/2$, regardless of the opponents' equilibrium distribution.

$Eg(b)$ has to be constant for each b in $[0, M-V/2] \cup \{M\}$. So $Eg'(b) = 0$ for b in $[0, M-V/2]$. We get $(e^{-rM} - e^{-r(M+V-b)})f_2(b) - re^{-r(M-b)}(F_2(M-V/2) - F_2(b) + f_2(M)) - (e^{-rM} - e^{-r(M-b)})f_2(b) = 0$
i.e $f_2(b)(1 - e^{-rV}) + rF_2(b) - r = 0$

The solution of this differential equation is: $F_2(b) = 1 - e^{-\frac{rb}{1-e^{-rV}}}$ so $f_2(b) = \frac{re^{-\frac{rb}{1-e^{-rV}}}}{1-e^{-rV}}$ for b in $[0, M-V/2]$ and M is a mass point played with probability $f_2(M) = 1 - F_2(M-V/2) = e^{-\frac{r(M-V/2)}{1-e^{-rV}}}$ (and $f_2(b) = 0$ for b in $]M-V/2, M[$).

By symmetry, we get $f_1(b) = \frac{re^{-\frac{-rb}{1-e^{-rV}}}}{1-e^{-rV}}$ for b in $]0, M-V/2[$, $f_1(M) = e^{-\frac{r(M-V/2)}{1-e^{-rV}}}$ and $f_1(b) = 0$ for b in $]M-V/2, M[$.

So we set $f(M) = e^{-\frac{r(M-V/2)}{1-e^{-rV}}}$. We get $f'_r(M) = (M-V/2) \frac{(-1+e^{-rV}(1+rV))}{(1-e^{-rV})^2} e^{-\frac{r(M-V/2)}{1-e^{-rV}}} < 0$ for any $r > 0$.

It follows that $f(M)$ decreases with risk aversion.

Appendix 4 Proof of proposition 2

We first calculate the net payoff in the first price all-pay auction. It is well-known that the discrete NE leads to play each bid from 0 to $V-1$ with probability $1/V$ (see for example Umbhauer 2016) and that the NE payoff is 0.5.

So each bid from 0 to $V-1$ leads to the payoff 0.5. It follows that, in the GBRM equilibrium:

bid 0 leads to the net payoff $0.5V/(V+1)$;

bid 1 leads to the net payoff $0.5V/(V+1) - 1/(V+1)$ (because we have to add the payoff obtained by bid 1 when confronted to bid V);

bid i leads to the net payoff $0.5V/(V+1) - i/(V+1)$ i from 2 to $V-1$;

bid V leads to the net payoff $0 - 0.5V/(V+1)$.

So the GBRM equilibrium net payoff is: $0.5V^2/(V+1)^2 - (1+2+\dots+V-1)/(V+1)^2 - 0.5V/(V+1)^2 = [0.5V^2 - 0.5V(V-1) - 0.5V]/(V+1)^2 = 0$.

We now focus on the second price all-pay auction.

We start by calculating the GBRM net payoff.

Bid 0 leads to the net payoff $0.5V/(V+1)$.

Bid 1 leads to the net payoff $[V+V/2-V]/(V+1)$.

Bid i leads to the net payoff $[\sum_{j=0}^{i-1}(V-j) + V/2 - i(V-i+1)]/(V+1)$ i from 2 to V .

We get $[\sum_{j=0}^{i-1}(V-j) + V/2 - i(V-i+1)]/(V+1) = [V/2 + i(i-1)/2]/(V+1)$ (for i from 1 to V).

We show it by recurrence. It is true for $i = 1$

We suppose it is true for i and we calculate: $[\sum_{j=0}^i(V-j) + V/2 - (i+1)(V-i)]/(V+1)$

We get $[\sum_{j=0}^i(V-j) + V/2 - (i+1)(V-i)]/(V+1) =$

$[\sum_{j=0}^{i-1}(V-j) + (V-i) + V/2 - i(V-i+1) - (V-i) + i]/(V+1) =$

$[\sum_{j=0}^{i-1}(V-j) + V/2 - i(V-i+1) + i]/(V+1) = [V/2 + i(i-1)/2 + i]/(V+1) = [V/2 + i(i+1)/2]/(V+1)$.

So the GBRM equilibrium net payoff becomes:

$$[0.5V(V+1)+0.5\sum_1^V i^2-0.5\sum_1^V i]/(V+1)^2 = 0.5[V(V+1)+V(V+1)(2V+1)/6-0.5V(V+1)]/(V+1)^2 \\ = 0.5V[0.5+(2V+1)/6]/(V+1) = (V^2+2V)/(6(V+1)).$$

We now show that, regardless of the played bid, the player gets a positive net payoff. This immediately follows from the fact that bid 0 leads to the net payoff $0.5V/(V+1)$ and that bid i leads to the net payoff $[V/2+i(i-1)/2]/(V+1)$ for i from 1 to V . This payoff is always positive.

Appendix 5 Proof of proposition 3

- Each bid i , i from 1 to V is a best reply to all bids j , j from 0 to $i-1$.
- Each bid i , i from $V+1$ to M is a best reply to all bids j , j from 0 to V .
- Bid 0 is a best reply to V and is the only best reply to bid j , j from $V+1$ to M .

So, given that we look for a symmetric BRM equilibrium, we get:

$$q_0 = q_V/(M-V+1) + \sum_{i=1}^{M-V} q_{V+i}.$$

It immediately follows that:

$$\sum_{i=V+1}^M q_i < q_0 < \sum_{i=V}^M q_i.$$

We also have:

$$q_1 = q_0/M,$$

$$q_2 = q_0/M + q_1/(M-1),$$

$$q_i = \sum_{j=0}^{i-1} q_j / (M - j) \quad i \text{ from } 1 \text{ to } V,$$

$$q_i = \sum_{j=0}^{V-1} q_j / (M - j) + q_V / (M - V + 1) \quad i \text{ from } V+1 \text{ to } M.$$

It follows that $q_1 = q_0/M$,

$$q_2 = q_0/M + q_0/((M-1)M) = q_0/(M-1),$$

$$q_3 = q_0/M + q_1/(M-1) + q_2/(M-2) = q_0/(M-1) + q_0/((M-1)(M-2)) = q_0/(M-2).$$

By recurrence, if $q_i = q_0/(M-i+1)$,

$$q_{i+1} = q_i + q_i/(M-i) = q_0/(M-i+1) + q_0/((M-i+1)(M-i)) = q_0/(M-i) \quad \text{for } i \text{ from } 1 \text{ to } V-1.$$

$$q_i = q_0/(M-V+1) + q_V/(M-V+1) = q_0/(M-V+1) + q_0/(M-V+1)^2 = q_0(M-V+2)/(M-V+1)^2 \quad \text{for } i \text{ from } V+1 \text{ to } M$$

$$\text{And } q_0 + q_0/M + q_0/(M-1) + \dots + q_0/(M-V+1) + q_0 - q_0/(M-V+1)^2 = 1$$

$$\text{So } q_0(2 + \sum_{i=0}^{V-1} 1/(M-i) - 1/(M-V+1)^2) = 1$$

Appendix 6 Proof of proposition 4

We have $Eg(0) = M + q_0V/2$,

$$Eg(1) = M + q_0V + q_1(V/2 - 1) - 1(1 - q_0 - q_1) = M + q_0V + q_1V/2 - 1(1 - q_0).$$

More generally $Eg(i) = M + \sum_{j=0}^{i-1} (V - j)q_j + q_iV/2 - i(1 - \sum_{j=0}^{i-1} q_j)$ i from 1 to M .

It follows: $Eg(i+1) - Eg(i) = (V-i)q_i + q_{i+1}V/2 - q_iV/2 + iq_i - (1 - \sum_{j=0}^i q_j) = Vq_i/2 + Vq_{i+1}/2 - \sum_{j=i+1}^M q_j$.

And $Eg(i+2) - Eg(i+1) = Vq_{i+1}/2 + Vq_{i+2}/2 - \sum_{j=i+2}^M q_j$.

Given that $q_{i+2} \geq q_i$ for i from 1 to $M-2$, and given that $-\sum_{j=i+2}^M q_j > -\sum_{j=i+1}^M q_j$ for i from 0 to $M-2$, it immediately follows that $Eg(i+2) - Eg(i+1) > Eg(i+1) - Eg(i)$ for any i from 1 to $M-2$. So $Eg(i+1) - Eg(i)$ is increasing in i , for i from 1 to $M-1$.

We get, for i from 1 to $M-V-1$, $Eg(V+i+1) - Eg(V+i) = Vq_{V+i}/2 + Vq_{V+i+1}/2 - \sum_{j=i+1}^{M-V} q_{V+j} =$

$q_0(M-V+2)(2V+i-M)/(M-V+1)^2$ given that $q_{V+j} = q_0(M-V+2)/(M-V+1)^2$ for any i from 1 to $M-V$. It follows that if $M > 2V$, $Eg(V+i+1)-Eg(V+i)$ becomes positive only for $i > M-2V$.

Hence $Eg(j+1)-Eg(j)$ becomes positive only for $j > M-V$.

Putting all the results together, it derives that, for $M > 2V$, $Eg(b)$ decreases for b from 1 to $M-V$ and increases from $M-V+1$ to M .

Appendix 7 Proof of proposition 5

For $M > 2V$, we know that $q_0 > 1/3$, so q_0 is higher than the continuous NE probability $f(0)=1/V$ when $V \geq 3$.

The result also holds for $V \leq M < 2V$:

$q_0 = 1/[2+\sum_{i=0}^{V-1} 1/(M-i)-1/(M-V+1)^2]$ so we get $q_0 = 1/[2+1/M+1/(M-1)+\dots+1/(M-V+1)-1/(M-V+1)^2]$ which we approximate by $1/[2+\ln(M/(M-V))+1/(2M)-1/(2(M-V))-1/(M-V+1)^2]$ if $M > V$, by $1/(1+\ln(V)+\gamma+1/(2V))$ if $M = V$ (where γ is Euler's constant 0.577)

If $M = V$, $1/(1+\ln(V)+\gamma+1/(2V)) > 1/V$ as soon as $V \geq 3$

If $M > V$, we set $M = V+a$ with a an integer higher or equal to 1. We show that

$$1/[2+\ln(M/(M-V))+1/(2M)-1/(2(M-V))-1/(M-V+1)^2] > 1/V$$

$$\text{i.e. } 2+\ln((V+a)/a)+0.5/(V+a)-0.5/a-1/(a+1)^2 < V$$

$\ln((V+a)/a)$ is decreasing in a . So we get for any V and a :

$$2+\ln((V+a)/a)+0.5/(V+a)-0.5/a-1/(a+1)^2 < 2+\ln((V+a)/a)+0.5/(V+a)$$

$$< 2+\ln((V+1))+0.25$$

$$< V \text{ as soon as } V \geq 4.$$

Appendix 8 proof of proposition 6 and proposition 7

Proof of proposition 6

We get $\sum_{i=1}^{V-1} q_i = q_0(1/(V+2))+1/(V+3)+\dots+1/(2V)$ which can be approximated by $q_0 \ln(2V/(V+1))+1/4V-1/(2V+2)$, close to $\ln(2)q_0$ if V is large .

We also get $\sum_{i=V}^M q_i = q_0+q_0(M-V)/(M-V+1)^2 = q_0(1+V/(V+1)^2)$ close to q_0 for large values of V .

So we get $q_0 \approx 1/(2+\ln(2)) = 0.371$, $\sum_{i=V}^M q_i \approx 0,371$ and $\sum_{i=1}^{V-1} q_i \approx 0.258$.

Proof of proposition 7

For $M = V$, $q_0 = 1/(2+1/V+1/(V-1)+1/(V-2)+\dots+1/1) \approx 1/(1+\ln(V)+\gamma+1/(2V))$ where γ is Euler's constant 0.577, given that we approximate $1+1/2+\dots+1/V$ by $\ln(V)+\gamma+1/(2V)$.

Appendix 9 proof of proposition 8

First part of the proposition.

We calculate the mean payoff for each player when M is large. We already know that, when M is large (say $M \rightarrow \infty$) and V is a constant, then q_0 goes to $1/2$, $\sum_{i=1}^V q_i$ goes to 0, $\sum_{i=V+1}^M q_i$ goes to $1/2$, which means, given that each action is played with the same probability, that $q_i = 1/(2(M-V))$ for each i from $V+1$ to M .

We look for the payoff obtained with each played bid. We forget the bids from 1 to V, given that they lead to a payoff which is a constant that will be multiplied by a probability (to play the bid) that is so small that even the sum of the payoffs obtained with these bids goes to 0 (because each q_i goes to 0 and $\sum_{i=1}^V q_i=0$.) For similar reasons we forget the payoff a player gets when he meets a player who plays a bid from 1 to V, because the sum of the payoffs is again close to 0. So we start by looking for the payoff obtained with bid 0, then the payoff obtained with bid $V+i$, i from 1 to $M-V$, and then we calculate the mean payoff.

Net payoff obtained with bid 0 = $q_0V/2 = V/4$.

Net payoff obtained with bid $V+1 =$

$$q_0V+a(V/2-V-1)-(V+1)(M-V-1)a = q_0V+a(V/2)-a(V+1)(M-V).$$

Net payoff obtained with bid $V+2 = q_0V-a+a(V/2-V-2)-(V+2)(M-V-2)a =$

$$q_0V-a+a(V/2)-a(V+2)(M-V-1).$$

Net payoff obtained with bid $V+i = q_0V- a \dots -(i-1)a+a(V/2)-a(V+i)(M-V-i+1)$ i from 2 to $M-V$.

So, for i from 1 to $M-V$, the net payoff with bid $V+i$ is equal to:

$$V/2-a(i-1)i/2-aV(M-V+1/2)-ia(M-2V+1)+ai^2 = V/2+ai^2/2-aV(M-V+1/2)-ia(M-2V+1/2).$$

Now we calculate the mean net payoff, by multiplying $V/4$ by q_0 , each net payoff with bid $V+i$ by a , $i=1$ to $M-V$, and by summing these payoffs. So we get:

$$V/8+a(M-V)V/2-a^2(M-V)V(M-V+1/2)+a \sum_{i=1}^{M-V} \left(\frac{ai^2}{2} - ia \left(M - 2V + \frac{1}{2} \right) \right).$$

$$a \sum_{i=1}^{M-V} \left(\frac{ai^2}{2} - ia \left(M - 2V + \frac{1}{2} \right) \right) =$$

$$a^2(M-V)(M-V+1)(2M-2V+1)/12 - a^2(M-V)(M-V+1)(M-2V+1/2)/2 \text{ which goes to}$$

$2M/48-M/8 = -M/12$ because $a = 1/(2(M-V))$ and because V and the other constants are small in comparison to M .

$V/8+a(M-V)V/2- a^2(M-V)V(M-V+1/2)$ goes to $V/8+V/4-V/4$ because $a = 1/(2(M-V))$.

So the BRM equilibrium net payoff goes to $-M/12$ (given that we can forget the terms in V)

This amounts to saying that, for M very large (in comparison to V), the player loses $1/12^{\text{th}}$ of the budget M .

Second part of the proposition,

Given that $M \gg 2V$, we know from proposition 4 that the lowest payoff is obtained for the bid $M-V$, i.e. the bid $V+(M-2V)$. This net payoff is equal to:

$$V/2+ai^2/2-aV(M-V+1/2)-ia(M-2V+1/2) \text{ with } i = M-2V \text{ and } a = 1/(2(M-V)).$$

So it is equal to : $V/2+a(M-2V)^2/2-aV(M-V+1/2)-(M-2V)a(M-2V+1/2)$ which goes to $-M/4$ (we can forget V and the others constants given that only M goes to infinity).