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Household behavior and individual autonomy: A Lindahl approach

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Abstract

A comprehensive model of economic household decision is presented which incorporates both fully cooperative and fully non-cooperative variants, parameterized by the income distribution, as well as a semi-cooperative variant, parameterized in addition by a vector θ, representing the degrees of individual autonomy. In this comprehensive model, the concept of "household θ-equilibrium" is introduced through the reformulation of the Lindahl equilibrium in strategic terms. Existence is proved and some generic properties of the household θ-equilibrium derived. An example is given to illustrate. Finally a particular decomposition of the pseudo-Slutsky matrix is derived and the testability of the various models discussed.

JEL codes: D10, C72, H41

Keywords: Intra-household allocation, household financial management, degree of autonomy, Lindahl prices, local income pooling, separate spheres.

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1 Introduction

When looking at the economic behavior of households, one should be aware of the large variety of formal and informal rules and decision procedures that are used by household members. This variety is due to the different social contexts and the different social or legal norms that are in place, but also to the different types of arrangements that are freely chosen by the spouses.

This variety is reflected in the variety of models that have been proposed to analyze household behavior once discarded the so-called unitary approach, which assumes that the household acts as if it were maximizing a single utility function, possibly a well-defined social welfare function. But this variety of models can be divided into two essentially different approaches:[1] the fully cooperative, which entails Pareto-efficiency of household decisions and requires binding agreements, and the fully non-cooperative, with household decisions resulting from a Nash equilibrium of some game where each individual maximizes utility under a personal budget constraint, and where agreements should be self-enforcing.

The first approach started with models based on axiomatic bargaining theory (Manser and Brown, 1980, McElroy and Horney, 1981), which result in Pareto-efficient outcomes varying according to the specified threat point, itself possibly determined by the solution of a non-cooperative game (Lundberg and Pollak, 1993, Chen and Woolley, 2001). Subsequent papers proposed "collective" models in order to explore the restrictions on observable household behavior implied by the assumption of Pareto efficiency, without explicitly referring to a specific bargaining or other decision making process (Chiappori, 1988, 1992, Browning and Chiappori, 1998, Chiappori and Ekeland, 2006).

The second approach is based on two types of non-cooperative games, generally leading to inefficient equilibrium outcomes. In the first type each individ-

[1] A synthesis of the field is provided by Donni (2008b). See also Donni (2008a) for a general presentation of the so-called 'collective' models of household behavior. Pollak (2005) surveys both cooperative and non-cooperative bargaining models.

ual is supposed to be responsible for a “separate sphere” of joint consumption (Lundberg and Pollak, 1993). In the second type, each individual voluntarily contributes to any public good (Ulph, 1988, Chen and Woolley, 2001, Lechene and Preston, 2005, Browning, Chiappori and Lechene, 2010).

However, the inefficiency of the fully non-cooperative behavior should be considered in relative terms. The private or public (or semi-public) nature of some of the goods (and services) consumed by the household can be determined by agreement. To take an extreme example we may consider two individuals deciding to marry but to keep their consumption behavior unchanged: they both live in their own appartment (we may suppose that they are close neighbors), keep their own car, pay their own telephone bills (including rental), etc. Just as before they got married, all goods are viewed as private and the non-cooperative equilibrium is efficient. To introduce inefficiency, it should be recognized by both parties that some of these goods could be shared as public (or semi-public) goods within the household and hence generate economies of scale. But it may also be recognized that this agreement does not have to be 0-1. One objective in establishing internal rules or decision procedures within the household is to determine the nature of these goods and the degree of autonomy each spouse can keep in their consumption. This in turn will determine or constrain the decision variables on which the individual preferences of the household members are defined, and efficiency will be defined relative to these preferences. Most of such arrangements are informal ("you have your car, I have mine" or "let us share one family car"; "you pay for your telephone, I pay for mine" or "let us share the rental and each pay for our own calls"). Another example is clothing. Since spouses may be concerned by each other’s dresses, clothing can be viewed as public consumption within the household. But clothing might be partially bought together and partially bought individually.

Other arrangements are legally enforced. The decision to get married (or conversely to divorce) is already a decison of that kind. The type of marriage contract is also important. For instance, in many countries, when marrying, the spouses may choose among several types of marriage contracts leaving to each spouse more or less autonomy in property.[2] The autonomy we are referring to is also related to the way in which the household organizes its finances. An important distinction appearing in empirical sociological studies (for instance two surveys of the International Social Survey Programme of 1994 and 2002, analyzing representative samples of 38 countries) is the one between money management "systems in which couples operate more or less as single economic units" and "individualized or privatized systems in which couples operate largely as two separate, autonomous economic units" (Vogler, Brockmann and Wiggins, 2006, Pahl, 2008). The former comprehend systems in which one of the two spouses manages all the household money, except possibly a fraction left to the other spouse for his/her personal expenses, but also systems (used by more than half of the couples surveyed by the ISSP) in which all the household money is pooled in a common bank account and managed jointly by the two spouses, not necessarily on a 50-50 basis. These systems offer a good illustration of the economic household models of both the unitary and the fully cooperative approaches. In contrast with them, we find two kinds of individualized systems. The first one is the '*independent management system*' in which each spouse keeps his/her own income separate and has responsibility for different items of household expenditure. This system may be easily approached by fully non-cooperative economic household models displaying 'separate spheres', either exogenously or endogenously. The other individualized system (used by 13% of the couples in the ISSP 1994 survey, 17% in the 2002 survey) is "the *partial pool* in which couples pool some of their income to pay for collective expenditure and keep the rest separate to spend as they choose" (Vogler, Brockmann and Wiggins, 2006).[3]

[2] If they don't, a marriage contract may be imposed by law. For example, in Belgium and in France any property held before marriage (or by inheritance) remains property of the individual, and properties acquired while married are held jointly.

[3] The terminology 'partial pooling' or else 'joint pooling', applied by sociologists to specific systems of financial management within the household, should not be confused with the terminology 'income pooling' used by economists to designate situations in which households behave *as if* their income were pooled, so that it does not matter which member receives the income (see Bradbury, 2004, p.504).

Fully (or partially) non-cooperative agreements are also the result of informational constraints. When getting married, each spouse has some private information (the "type" of each spouse is only partially known by the other). Therefore, negotiation is under incomplete information and, from a mechanism design point of view, the designed agreement may imply that for some type of the spouses a "default game" is played and much of the private information revealed.[4]

Our purpose in the present paper is to propose a comprehensive equilibrium model (formulated for simplicity in terms of a two-person household) where each spouse has the right to keep some degree of autonomy in the consumption decisions about the goods that are public within the household. This model is comprehensive since assuming zero autonomy for both spouses leads to the collective model, and assuming full autonomy for both spouses is equivalent to full non-cooperation. But we get in addition a continuum of intermediate cases corresponding to partial autonomy.[5] The equilibrium concept that will be

[4] See Celik and Peters, (2011): "We argue that when the outside option of the participants is the non-cooperative play of a default game, the design problem is substantially different from the standard one where the outside option is a (possibly type contingent) exogenous allocation. In particular, we show that there are allocation rules that are implementable in this setting, only if the mechanism designer offers mechanisms which will be rejected by some types of some players. Since the participation decision is type dependent, a refusal to participate conveys information that causes the default game to be played differently than it would have been if players used only their interim beliefs. Our results provide some insight into the fact that negotiations do not always lead to successful agreements" (p.1). They illustrate the point via the example of a cartel agreement between two firms with privately known costs, the default game being Cournot duopoly.

[5] For instance, in a recent paper focusing on the household decisions concerning labor supply, Del Boca and Flinn (2010) write: "We view household time allocation decisions as either being associated with a particular utility outcome on the Pareto frontier, or to be associated with the noncooperative (static Nash) equilibrium point. In reality there are a continuum of points that dominate the noncooperative equilibrium point and that do not lie on the Pareto frontier, however developing an estimable model that allows such outcomes to enter the choice set of the household seems beyond our means" (p.2). Cf. also Lechene and Preston (2005): "neither the assumption of fully efficient cooperation nor of complete absence of collaboration is likely to be an entirely accurate description of typical household spending

introduced for all these cases will be based on a non-cooperative reformulation of the Lindahl equilibrium. This reformulation is of general interest: it is applicable to any economy with public goods.

Another important issue is empirical. It concerns the testable restrictions that are implied by some of theses models. This has been well studied under the assumption of efficiency by Browning and Chiappori (1998), where a test based on the properties of the Slutsky matrix leads to the rejection of the unitary model but not of the collective model. This test is based on the decomposition of the (pseudo-)Slutsky matrix derived from the household demand into the sum of a matrix with the Slutsky properties and a 'deviation matrix' of rank equal to 1 for a two-person household (or, more generally, equal to the number of household members minus 1). Lechene and Preston (2010) derive a similar test for the fully non-cooperative model showing that the deviation matrix will generally have a larger rank than in the collective model (more precisely, the larger the higher the number of public goods). A contribution of the present paper is to give a comprehensive derivation of the pseudo-Slutsky matrix and a particular decomposition which allows us to isolate different effects: an "aggregation effect", the only one working in the collective model, an "externality effect" appearing with non-cooperation and, finally, a "substitution effect" due to partial autonomy. Each effect increases the maximum possible rank of the deviation matrix. Hence, in principle, one can, not only test the unitary model against the collective one (as in Browning and Chiappori, 1998), the collective model against the fully non-cooperative (as in Lechene and Preston, 2010), but also the fully non-cooperative against the semi-cooperative and the semi-cooperative against the unrestricted case. The implementation of such tests becomes more and more demanding in terms of the required number of goods, more precisely in terms of the required number of private goods with respect to the number of public goods. However, as we will see, the non-cooperative and the semi-cooperative models are in principle distinguishable, at least in some

behaviour and analysis of such extreme cases can be seen as a first step towards understanding of a more adequate model" (p.19).

cases.

In Section 2, we present briefly a two-member household decision model, in both its cooperative and non-cooperative versions and, after reformulating the concept of Lindahl equilibrium, we introduce the general semi-cooperative version and define the concept of "household θ-equilibrium", with θ referring to the pair of degrees of autonomy of the two spouses. In Section 3, we prove existence and describe some generic properties of the household θ-equilibrium. In Section 4, we exploit an example already used by Browning, Chiappori and Lechene (2010), in order to illustrate the implications of varying degrees of autonomy. In particular, it will be shown that each spouse has an incentive to deviate unilaterally from full cooperation, an inecentive that is the stronger the lower the spouse's income share. The pseudo-Slutsky matrix is derived in Section 5 and the testability of the various models then discussed. We conclude in Section 6.

2 The household decision model

We study a two-adult household, consuming goods that are recognized by both spouses as being either private or public (within the household). Denote by A (the wife) and B (the husband) the two household members, and let $\left(q^A, q^B\right) \in \mathbb{R}_+^{2n}$ be the vector of consumption by the two members of the n private goods and $Q \in \mathbb{R}_+^m$ the consumption vector of the m public goods. The preferences of each spouse J ($J = A, B$) are represented by a utility function $U^J\left(q^J, Q\right)$, which is defined on $\mathbb{R}_+^{n+m}$, increasing and strongly quasi-concave. Each spouse J is supposed to receive an initial income $Y^J \geq 0$, the total household income being $Y = Y^A + Y^B > 0$. These initial incomes may be seen as the individual earnings of each spouse (for example in double-earner couples[6]) or, alternatively,

[6] In general it is supposed that only the total income of the household is observable. However some data sets give a lot of information on the labour status of each spouse and on the various income sources of families. This is the case, for example, for the 2004 German Socio-Economic Panel, a representative panel data sample of households and individuals living in

as an agreed upon income sharing of the total household income. In any case, the given income distribution can be seen as an indicator of bargaining power distribution in the household. We want to study how the household decides on its total consumption given the vector of private good prices $p \in \mathbb{R}^n_{++}$ and the vector of public good prices $P \in \mathbb{R}^m_{++}$. The first private good, assumed to be desired in any household environment, is taken as numéraire ($p_1 = 1$).

2.1 Efficient intra-household decisions

A *Pareto-optimal decision* within the household can be obtained, as well known, by solving a program:

$$\begin{aligned} &\max_{(q^A, q^B, Q) \in \mathbb{R}^{2n+m}_+} \mu U^A \left(q^A, Q\right) + (1-\mu) U^B \left(q^B, Q\right) \\ &\text{s.t. } p\left(q^A + q^B\right) + PQ \leq Y, \end{aligned} \tag{1}$$

for some Pareto weight $\mu \in [0,1]$. For $J = A, B$, let $\tau^J\left(q^J, Q\right)$ denote the marginal-willingness-to-pay vector for the public goods in terms of the numéraire:

$$\tau^J\left(q^J, Q\right) \equiv \frac{1}{\partial_{q_1} U^J\left(q^J, Q\right)} \partial_Q U^J\left(q^J, Q\right). \tag{2}$$

Under usual regularity conditions, the Pareto-optimal decisions (corresponding to all values of μ in $[0,1]$) are characterized by the Bowen-Lindahl-Samuelson conditions:

$$\tau^A\left(q^A, Q\right) + \tau^B\left(q^B, Q\right) = P, \tag{3}$$

together with the usual first order conditions for private consumption and the budget equation $p\left(q^A + q^B\right) + PQ = Y$.

As well discussed in Browning, Chiappori and Lechene (2006), if the Pareto weight is independent of the *environment* (p, P, Y), while possibly depending on distributional factors, then the efficient intra-household decision model reduces to the unitary model, in the sense that the household decides as a single decision

Germany. Beninger (2010) uses this panel to explore the influence of the perceived tax system on household behavior.

unit, maximizing under the common budget constraint $pq + PQ \leq Y$ the utility function

$$\widetilde{U}(q, Q) \equiv \max_{\left\{(q^A, q^B) \in \mathbb{R}_+^{2n} | q^A + q^B = q\right\}} \mu U^A\left(q^A, Q\right) + (1 - \mu) U^B\left(q^B, Q\right). \qquad (4)$$

However, as soon as the Pareto weight does depend on the environment (p, P, Y), the function $\widetilde{U}$ becomes a 'generalized' utility function, depending through μ on prices and household income, so that the *collective model* must indeed be distinguished from the *unitary model*.

2.2 Fully non-cooperative decisions

An alternative non-unitary model of household decisions is non-cooperative,[7] with each spouse having full autonomy in allocating income to private and public consumption. Referring to the way in which the household finances are organized, as described in the Introduction, this would correspond to the 'independent management system', in which each spouse keeps a separate account to be autonomously used. However this is only an external manifestation since the reasons why a couple acts non-cooperatively can be of many sorts (social, cultural, due to personal history, informational etc.). Also, it should be understood that this "full non-cooperation" may be much reduced as compared to a non-marital situation before marriage (or after a divorce) where the possibility of shared consumption for the m public goods is non-existent (or suppressed). In the bargaining theory framework, one can interpret the fully non-cooperative solution as the (realized) threat point or, in a mechanism design approach, as the solution to a fully-revealing "default game".

Accordingly, one may define a *game with voluntary contributions to public goods* where each spouse J chooses a strategy $\left(q^J, g^J\right) \in \mathbb{R}_+^{n+m}$ (q^J denoting J's private consumptions and g^J his/her contributions to public goods) in order to

[7] See Ulph (1988), Chen and Woolley (2001), Lechene and Preston (2010), Browning, Chiappori and Lechene (2010).

solve the programme:

$$\max_{(q^J,g^J)\in\mathbb{R}_+^{n+m}} U^J\left(q^J, g^J + g^{-J}\right) \tag{5}$$

$$\text{s.t. } pq^J + Pg^J \leq Y^J. \tag{6}$$

A Nash equilibrium of this game can be characterized by the first order conditions (for $J = A, B$):

$$\begin{aligned} \frac{1}{\partial_{q_1} U^J\left(q^J, g^J + g^{-J}\right)} \partial_q U^J\left(q^J, g^A + g^B\right) &\leq p \\ \tau^J\left(q^J, g^A + g^B\right) &\leq P \\ pq^J + Pg^J &= Y^J, \end{aligned} \tag{7}$$

with an equality for any private good i s.t. $q_i^J > 0$ or any public good k s.t. $g_k^J > 0$.

2.3 The Lindahl approach to cooperative collective decisions

Our purpose now is to propose a more general strategic approach, which will include as sub-cases the two extreme models, the collective and the non-cooperative, but will also include a continuum of intermediate cases. For that purpose we start from the concept of Lindahl equilibrium, which is the best-known "decentralized" procedure[8] to allocate efficiently the cost of public goods within a group. However, the version we give of the concept will be strategic. In the context of household decision[9] and referring again to the way in which the household finances are organized, let us suppose that, rather than keeping only separate accounts, the two spouses pool into a common account the fractions of their incomes to be devoted to public consumption and that these fractions are determined according to the Lindahl decentralized procedure. This corresponds to a variant of the "partial pool management system" in which the spouses pool

[8] Introduced by Lindahl (1919) and popularized by Samuelson (1954).

[9] Cherchye, De Rock and Vermeulen (2007) also use Lindahl prices to analyze household decisions.

the part of their income needed to pay for collective expenditure and keep the rest separate to spend as they choose on private goods.

The Lindahl approach consists in supposing that there exists a pair of personalized (Lindahl) price vectors $\left(P^A, P^B\right) \in \mathbb{R}_+^{2m}$, satisfying $P^A + P^B = P$, which are posted within the household. We assume that each spouse chooses strategically, for each public good k, a quantity to be bought by the household under the following cost allocation scheme. Each spouse, say the wife A, anticipating for each public good k a contribution $g_k^B \in \mathbb{R}_+$ from her husband, suggests an additional quantity $g_k^A \in \mathbb{R}_+$ knowing that she will have to transfer to the common account a corresponding amount $P_k^A \left(g_k^A + g_k^B\right)$ from her own account. For private goods, she chooses the quantity vector $q^A \in \mathbb{R}_+^n$ to be bought in the market at prices $p \in \mathbb{R}_{++}^n$, and paid from her own account. We can then define a Lindahl equilibrium for the household.

Definition 1 *A vector* $\left(q^A, g^A, q^B, g^B, P^A, P^B\right) \in \mathbb{R}_+^{2n+4m}$*, with* $P^A + P^B = P$*, is a* Lindahl household equilibrium *if it satisfies the "budget consistency condition"*

$$P_k^J \left(g_k^A + g_k^B\right) = P_k g_k^J, \text{ for } J = A, B \text{ and any public good } k, \tag{8}$$

and if the pair $\left(q^J, g^J\right)$ *solves the program*

$$\max_{(q^J, g^J) \in \mathbb{R}_+^{n+m}} U^J \left(q^J, g^J + g^{-J}\right) \tag{9}$$

$$s.t.\ pq^J + P^J \left(g^J + g^{-J}\right) \leq Y^J, \tag{10}$$

for $J = A, B$.

The *budget consistency condition* (8) can be interpreted as a kind of "participation constraint": what each spouse, say the wife A, transfers for public good k into the common account corresponds to the amount she would like to spend in the market (given the quantity contributed by the other). It is also this condition which ensures the equivalence of this household equilibrium to the standard definition of a Lindahl equilibrium where individualized contributions g_k^J are not introduced, but instead each individual chooses a desired total consumption of public good k. At equilibrium, condition (8) eliminates the case where spouse J would like to diminish the consumption of public good k but cannot since the non-negative constraint on g_k^J is binding. Indeed, by condition (8), $g_k^J = 0$ and $g_k^{-J} > 0$ imply $P_k^J = 0$, and hence a contradiction since, with $P_k^J = 0$, $g_k^J = 0$ could not be optimal ($U^J\left(q^J, Q\right)$ is increasing in Q_k). Hence, at a Lindahl household equilibrium, g_k^A and g_k^B are either both positive or both nil for any public good k.

For the sake of later comparisons, recall the first order conditions for a Lindahl household equilibrium (for $J = A, B$):

$$\begin{aligned} \frac{1}{\partial_{q_1} U^J\left(q^J, g^J + g^{-J}\right)} \partial_q U^J\left(q^J, g^J + g^{-J}\right) &\leq p \\ \tau^J\left(q^J, g^J + g^{-J}\right) &\leq P^J \\ pq^J + P^J\left(g^J + g^{-J}\right) &= Y^J, \end{aligned} \tag{11}$$

with an equality for any private good i s.t. $q_i^J > 0$ or any public good k s.t. $g_k^J > 0$. They entail the Bowen-Lindahl-Samuelson conditions for any interior solution.

2.4 Extending the Lindahl approach to semi-cooperative decisions

In order to introduce a more comprehensive model, allowing for semi-cooperation, we assume that there are arrangements within the household which are variants of the one we have described to define the Lindahl household equilibrium. Each spouse J, for reasons that may be of many kinds, may want (or, sometimes, may be obliged) to keep some *degree of autonomy* $\theta^J \in [0, 1]$ in spending for the public goods.

In the household finance management context, this would still correspond to the "partial pool" management system in which the spouses pool some part of their income needed to pay for collective expenditure while keeping a separate account to be autonomously used. In the bargaining theory approach the semi-cooperative solution could be a less extreme threat point, and in the mechanism design approach the result of another kind of fully revealing default game.

The difference with the pure Lindahl case is that the total public good expenses Pg^J of spouse J do not all transit through Lindahl taxation (say via the household common account): a portion $\theta^J Pg^J$ is autonomously spent by J directly in the market and only the remaining portion $\overline{\theta}^J Pg^J$, with $\overline{\theta}^J = 1-\theta^J$, is subject to Lindahl taxation. This consists again in posting within the household a pair of *contributive shares* P_k^A and P_k^B for each public good k, such that $P_k^A + P_k^B = P_k$. Then each spouse, say the wife A, anticipating for each public good k a contribution $g_k^B \in \mathbb{R}_+$ from her husband, chooses her own contribution $g_k^A \in \mathbb{R}_+$ which will determine the monetary amount $P_k^A \left(\overline{\theta}^A g_k^A + \overline{\theta}^B g_k^B\right)$ that she will have to transfer from her own account to the common account (observe that if $\overline{\theta}^A = \overline{\theta}^B = 1$, we are back to the pure Lindahl case). She will buy the basket $\theta^A g^A$ of public goods directly in the market at prices P, together with the basket $q^A \in \mathbb{R}_+^n$ of private goods that she wants to consume, at prices p. This leads to the following comprehensive equilibrium concept:

Definition 2 *A vector* $\left(q^A, g^A, q^B, g^B, P^A, P^B\right) \in \mathbb{R}_+^{2n+4m}$*, with* $P^A + P^B = P$*, is a* household θ-equilibrium *with degrees of autonomy* $\left(\theta^A, \theta^B\right) \in [0,1]^2$ *if it satisfies the "budget consistency condition"*

$$P_k^J \left(\overline{\theta}^A g_k^A + \overline{\theta}^B g_k^B\right) = P_k \overline{\theta}^J g_k^J, \text{ for } J = A, B \text{ and any public good } k, \tag{12}$$

and if the pair $\left(q^J, g^J\right)$ *solves the following program:*

$$\max_{(q^J, g^J) \in \mathbb{R}_+^{n+m}} U^J \left(q^J, g^J + g^{-J}\right) \tag{13}$$

$$\text{s.t. } pq^J + P\theta^J g^J + P^J \left(\overline{\theta}^J g^J + \overline{\theta}^{-J} g^{-J}\right) \leq Y^J, \tag{14}$$

for $J = A, B$.

Notice that, for the extreme case $\theta^A = \theta^B = 0$, spouse J is exclusively confronted to the personalized price vector P^J for public goods, so that we obtain the definition of a Lindahl household equilibrium, and hence a Pareto efficient outcome. In the other extreme case $\theta^A = \theta^B = 1$, the contributive shares cease to play a role, so that J's program reduces to the corresponding program in the fully non-cooperative game with voluntary contributions to public goods, with the budget consistency condition (12) vanishing.

Similarly to what has been shown in the preceding subsection, this condition may be reformulated for, say, the wife A and public good k, as $P_k g_k^A = P_k \theta^A g_k^A + P_k^A \left(\overline{\theta}^A g_k^A + \overline{\theta}^B g_k^B \right)$ meaning that the market value $P_k g_k^A$ of the wife's voluntary contribution to public good k exactly decomposes into the market value of the autonomous portion $P_k \theta^A g_k^A$ and the remaining portion subject to Lindahl taxation. Moreover, we get $P_k^A \left(\overline{\theta}^B g_k^B \right) = P_k^B \left(\overline{\theta}^A g_k^A \right)$, so that $P_k^J = 0$ whenever $\overline{\theta}^J g_k^J = 0$ while $\overline{\theta}^{-J} g_k^{-J} > 0$. Knowing that her husband is not fully non-cooperative (*i.e.* $\theta^B < 1$) and that he is willing to contribute to public good k (*i.e.* $g_k^B > 0$), the wife A should not be taxed for public good k, either if she is fully non-cooperative (*i.e.* $\theta^A = 1$) or if she would rather like to decrease the household consumption of good k. Accordingly, the *budget consistency condition* (12) confers a *voluntariness* property to the Lindahl taxation imposed on each spouse.

A consequence of this voluntariness property in the semi-cooperative case ($0 < \theta^J < 1$, $J = A, B$) is that, whenever it exists, a *separate spheres* equilibrium, namely an equilibrium where $g_k^A g_k^B = 0$ for all k, coincides with an equilibrium of the game with voluntary contributions to public goods, played when the spouses are fully non-cooperative.

Proposition 3 *Let $0 < \theta^J < 1$ for $J = A, B$. Suppose $\left(q^A, g^A, q^B, g^B, P^A, P^B\right) \in \mathbb{R}_+^{2n+4m}$, with $P^A + P^B = P$, is a household θ-equilibrium such that $g_k^A g_k^B = 0$ for all k (*separate spheres*). Then $\left(q^A, g^A, q^B, g^B\right)$ is a Nash equilibrium of the game with voluntary contributions to public goods.*

Proof. Let us take the wife's viewpoint and denote g_A^A the vector of public goods to which she contributes and P_A their corresponding market prices. Her budget constraint (14) becomes $pq^A + P_A g_A^A \leq Y^A$ (since the contributive share P_k^A is zero if she does not contribute to public good k, and P_k if she does). This constraint is then equivalent to the budget constraint (6) of the game with voluntary contributions to public goods at the given separate spheres equilibrium. So, the two programs for the wife coincide as far as the private goods that she purchases and the public goods to which she contributes are concerned. Also, for any other public good k, if she deviated starting to contribute to it (by choosing $\widetilde{g}_k^A > g_k^A = 0$), she would have to pay $\left(P_k\theta^A + P_k^A\overline{\theta}^A\right)\widetilde{g}_k^A = \theta^A P_k \widetilde{g}_k^A < P_k\widetilde{g}_k^A$, less than under full non-cooperation. Hence, if she prefers not to deviate in the semi-cooperative case, so would she in the fully non-cooperative one. ■

Observe finally that the budget consistency condition (12) implies that the budget constraint (14) takes the form $pq^J + Pg^J = Y^J$ at equilibrium, and also that, whenever the two individual budget constraints are satisfied, we obtain, by addition, collective feasibility in the sense that

$$p\left(q^A + q^B\right) + P\left(g^A + g^B\right) \leq Y^A + Y^B = Y. \tag{15}$$

3 Generic properties of household θ-equilibria

Existence can be obtained by using an argument which is standard for competitive equilibrium.[10]

Proposition 4 *For every* $\left(\theta^A, \theta^B\right) \in [0,1]^2$*, there exists a household* θ*-equilibrium.*

Proof. Consider the household θ-equilibrium as an equilibrium of a generalized game (where strategy spaces are non constant correspondences). If $\left(\theta^A, \theta^B\right) \neq (1,1)$ we introduce, in addition to spouses A and B, a fictitious player with strategy space $S^0 = \left\{\left(P^A, P^B\right) \in \mathbb{R}_+^{2m} : P^A + P^B = P\right\}$ and payoff function $-\sum_{k=1}^m \left|P_k^A\left(\overline{\theta}^A g_k^A + \overline{\theta}^B g_k^B\right) - P_k\left(\overline{\theta}^A g_k^A\right)\right|$. The strategy spaces of the two spouses can be compactified by defining for $J = A, B$:

$$S^J = \left\{\begin{array}{c}\left(q^J, g^J\right) \in \mathbb{R}_+^{n+m} : q_i^J \leq Y^J/p_i, g_k^J \leq Y^J/P_k, \text{ all } i, \text{ all } k, \text{ and} \\ pq^J + P\theta^J g^J + P^J\left(\overline{\theta}^J g^J + \overline{\theta}^{-J} g^{-J}\right) \leq Y^J\end{array}\right\}.$$

Since all relations are linear in the relevant strategy variables and the payoff functions are continuous and quasi-concave, the best reply correspondences of

[10] See, for example, Mas-Colell, Whinston and Green, 1995, ch.17, app. B.

the two spouses as well as the one of the fictitious player are upper hemicontinuous and convex-valued. Hence, there exists a "social equilibrium" by Debreu (1952) theorem. Clearly, at this equilibrium, both spouses' programs (conditionally on P^A and P^B) are solved, and $P_k^J\left(\overline{\theta}^A g_k^A + \overline{\theta}^B g_k^B\right) = P_k\left(\overline{\theta}^J g_k^J\right)$ for any J and any k, verifying the budget consistency condition (12). ■

As to *efficiency*, it is naturally violated outside the fully cooperative case $\theta^A = \theta^B = 0$. Take the first order conditions relative to the public good k for both spouses' programs (13):

$$\tau_k^J\left(q^J, g^J + g^{-J}\right) \equiv \frac{\partial_{Q_k} U^J\left(q^J, g^J + g^{-J}\right)}{\partial_{q_1} U^J\left(q^J, g^J + g^{-J}\right)} \leq \theta^J P_k + \overline{\theta}^J P_k^J,\ J = A, B, \quad (16)$$

with equality if $g_k^J > 0$. For efficiency, the Bowen-Lindahl-Samuelson condition requires for each public good k that the sum $\tau_k^A + \tau_k^B$ of the two marginal willingnesses to pay be equal to the corresponding market price $P_k = P_k^A + P_k^B$. This condition is generally violated as soon as cooperation is less than full. Indeed, the sum of the two marginal willingnesses to pay is equal, if both spouses contribute to public good k, to $P_k + \theta^A P_k^B + \theta^B P_k^A$, larger than P_k outside the case $\theta^A = \theta^B = 0$, and the more so the higher the degrees of autonomy of the two spouses. Also, if a spouse, say the wife, contributes alone to public good k, $\tau_k^A = P_k$, so that $P_k < \tau_k^A + \tau_k^B$, leading to a similar conclusion.

Finally, let us address the question of *local determinacy* of household θ-equilibria. For an environment (p, P, Y) and an income distribution $\left(Y^A, Y^B\right)$, take a less than fully non-cooperative equilibrium $\left(q^A, g^A, q^B, g^B, P^A, P^B\right) \in \mathbb{R}_+^{2n+4m}$ with degrees of autonomy $\left(\theta^A, \theta^B\right) \neq (1, 1)$. Further, consider a partition $\left\{M^A, M^B, M^{AB}, M^0\right\}$ of the set M of public goods, where M^A and M^B are the subsets of goods exclusively contributed by spouses A and B, respectively, M^{AB} is the subset of goods to which both spouses contribute and M^0 is the subset of goods that are not at all consumed by the household at this equilibrium. Denote by m^A, m^B, m^{AB} and m^0 the cardinals of the respective subsets in this partition.

Clearly, $m^A+m^B+2m^0$ unknowns characterizing the equilibrium are trivially determined, namely $g_k^J = 0$ for $k \in M^{-J} \cup M^0$, $J = A, B$. Besides, $2m^0$

Lindahl prices corresponding to the public goods which are not consumed by the household can be ignored. In order to determine the remaining $2n + 4m - \left(m^A + m^B + 4m^0\right)$ unknowns, we have 2 budget equations, $2\left(n-1\right)$ equations expressing the first order conditions for the private goods,[11] $m^A + m^B + 2m^{AB}$ equations expressing the first order conditions for public goods

$$\tau_k\left(q^J, g^A + g^B\right) = \theta^J P_k + \overline{\theta}^J P_k^J, \quad k \in M^J \cup M^{AB},\ J = A, B, \qquad (17)$$

$m - m^0$ equations $P_k^A + P_k^B = P_k$ and the $m - m^0$ corresponding budget consistency conditions (12). To sum up, we have $2n + m^A + m^B + 2m^{AB} + 2\left(m - m^0\right)$ equations in $2n + 4m - \left(m^A + m^B + 4m^0\right)$ unknowns, implying an excess $2\left(m - \left(m^A + m^B + m^{AB} + m^0\right)\right) = 0$ of the number of unknowns over the number of equations. Hence, a Lindahl household equilibrium is (generically) locally determinate outside the fully non-cooperative case.

In the fully non-cooperative case $\theta^A = \theta^B = 1$, we further eliminate the $2\left(m - m^0\right)$ unknowns P_k^A and P_k^B for $k \in M^A \cup M^B \cup M^{AB}$ and the corresponding $m - m^0$ equations $P_k^A + P_k^B = P_k$ together with the $m - m^0$ corresponding budget consistency conditions (12), that is, the same number of unknowns and equations. However, because of the elimination of the contributive shares P_k^A and P_k^B, the equation system contains now a subsystem with $2\left(n-1\right) + m^A + m^B + 2m^{AB}$ first order condition equations in only $2n + m^A + m^B + m^{AB}$ unknowns, namely q_i^J (for $J = A, B$ and $i = 1, ..., n$), g_k^J (for $J = A, B$ and $k \in M^J$) and $g_k^A + g_k^B$ (for $k \in M^{AB}$). Hence, there is in this subsystem an excess of the number of equations over the number of unknowns equal to $m^{AB} - 2$. As a consequence, there is generically *over*determinacy if $m^{AB} > 2$, or even if $m^{AB} = 2$, since the household consumption is entirely determined in this case by the sole first order conditions, independently of any budget constraint. If $m^{AB} = 0$ (*separate spheres*), the two individual budget equations make the whole system determinate. If $m^{AB} = 1$ (*separate spheres up to one public good*), in order to obtain determinacy of the whole system

[11] We are assuming for simplicity that all the n private goods are consumed by both spouses at the equilibrium we refer to.

we replace by aggregation the two individual budget constraints by the single household budget equation. In this case, the splitting of Y into Y^A and Y^B does not influence the equilibrium outcome: the property of *local income pooling* is an essential characteristic of the regime of joint contribution to a single public good by the two spouses.

We thus reach the same conclusion as Browning, Chiappori and Lechene (2010), that there are generically only two possible regimes under full non-cooperation: pure separate spheres and separate spheres up to one public good, the latter regime being characterized by local income pooling. We must however emphasize that the non-genericity of the joint contribution to more than one public good which holds in the fully non-cooperative case does not generalize to the semi-cooperative and cooperative cases, because spouse J's first order condition for each contributed public good k involves then the contributive share P_k^J as a further unknown.

4 An example

In order to study the consequences for household decisions of varying the income shares as well as the degrees of autonomy of the two spouses, and to make comparisons with previous results on the game with voluntary contributions to public goods obtained by Browning, Chiappori & Lechene (2010), we use the same example, with Cobb-Douglas preferences over one private good and two public goods. We denote by x and z the private consumptions of spouses A and B, respectively, and by X and Z the quantities of the two public goods. The utility functions are given by:

$$U^A(x, X, Z) = xX^aZ^\alpha \text{ and } U^B(z, X, Z) = zX^bZ^\beta, \tag{18}$$

with positive parameter values a, α, b and β. The wife A is supposed to care more about the first public good, and the husband B about the second, so that

$$\frac{\alpha/a}{\beta/b} < 1, \tag{19}$$

where the term on the LHS can be taken as the *degree of symmetry* of the spouses' preferences for the two public goods. We use the normalization

$$p_x = p_z = P_X = P_Z = Y = 1, \tag{20}$$

with an income distribution given by $Y^A = \rho$ and $Y^B = 1 - \rho$.

4.1 The different regimes

Browning, Chiappori & Lechene (2010) show the existence of three kinds of regimes. Two kinds correspond to separate spheres, prevailing both for extremely unequal income distributions, where the spouse with the higher income contributes alone to both public goods, and for relatively equal income distributions, where each spouse contributes to her/his preferred public good. The third kind appears in intermediate cases of income distribution and is characterized by separate spheres up to one public good: the spouse with the higher income contributes to both public goods, while the other spouse contributes solely to her/his preferred public good. According to our previous analysis, we expect the regime where both spouses contribute to both public goods to be generically possible too.

Consider the regime where A contributes to her preferred public good (X) and B to his (Z). By the budget consistency condition, $P_Z^A = P_X^B = 0$, and by the first order conditions for public goods,

$$\begin{aligned} ax/X &= \theta^A + \overline{\theta}^A P_X^A = 1 \text{ and } \alpha x/Z < \theta^A + \overline{\theta}^A P_Z^A = \theta^A, \\ \beta z/Z &= \theta^B + \overline{\theta}^B P_Z^B = 1 \text{ and } bz/X < \theta^B + \overline{\theta}^B P_X^B = \theta^B. \end{aligned} \tag{21}$$

Using the equilibrium budget equations

$$x + X = \rho \text{ and } z + Z = 1 - \rho, \tag{22}$$

we easily obtain the solution

$$x = \frac{\rho}{1+a},\ X = \frac{a\rho}{1+a},\ z = \frac{1-\rho}{1+\beta},\ Z = \frac{\beta(1-\rho)}{1+\beta}. \tag{23}$$

This solution is constrained by the two first order conditions on the non-contributed goods, expressed as inequalities:

$$\frac{\alpha}{\beta}\frac{1+\beta}{1+a}\frac{\rho}{1-\rho} < \theta^A \text{ and } \frac{b}{a}\frac{1+a}{1+\beta}\frac{1-\rho}{\rho} < \theta^B. \tag{24}$$

Clearly, one of these two conditions will be violated for small enough or large enough values of ρ, so that separate spheres with both spouses contributing to public consumption can indeed prevail only if the income distribution between the two spouses is not too unequal. Also, by multiplying both sides of the first inequality by the corresponding sides of the second, we obtain

$$0 < \frac{\alpha/a}{\beta/b} < \theta^A\theta^B \leq 1. \tag{25}$$

Hence, existence of the regime of separate spheres with both spouses contributing to public consumption requires a relatively high average degree of autonomy of the two spouses, the higher the larger the degree of symmetry of their preferences for the two public goods. The fully non-cooperative case, where $\theta^A\theta^B = 1$, always satisfies this condition, provided there is no full symmetry in the spouses' preferences.

Now consider the regime where both spouses contribute to both public goods, which is generically excluded under full autonomy of the two spouses. By first order conditions, the marginal willingnesses-to-pay for the public goods are:

$$\begin{aligned} ax/X &= \theta^A + \overline{\theta}^A P_X^A \text{ and } \alpha x/Z = \theta^A + \overline{\theta}^A P_Z^A, \\ \beta z/Z &= \theta^B + \overline{\theta}^B P_Z^B \text{ and } bz/X = \theta^B + \overline{\theta}^B P_X^B. \end{aligned} \tag{26}$$

Division of both sides of the second and third equations by the corresponding sides of the first and fourth, respectively, leads to

$$\frac{a}{\alpha}\theta^A \leq \frac{a}{\alpha}\frac{\theta^A + \overline{\theta}^A P_Z^A}{\theta^A + \overline{\theta}^A P_X^A} = \frac{X}{Z} = \frac{b}{\beta}\frac{\theta^B + \overline{\theta}^B P_Z^B}{\theta^B + \overline{\theta}^B P_X^B} \leq \frac{b}{\beta}\frac{1}{\theta^B}, \tag{27}$$

the two inequalities being easily checked to be true (by taking the extreme values $P_X^A = 1$ and $P_Z^A = 0$). We thus obtain

$$0 \leq \theta^A\theta^B \leq \frac{\alpha/a}{\beta/b} < 1, \tag{28}$$

an existence condition just opposite to the one we found for separate spheres. For both spouses to contribute to both public goods their average degree of autonomy must be small enough, the smaller the more asymmetric their preferences for the public goods.

Thus, for not too unequal income distributions, separate spheres appear as a characteristic of high individual autonomy in household decisions. As the spouses become less and less autonomous, the regime prevailing when their incomes are not too different is rather the one where both contribute to both public goods, which is the rule under full cooperation.

More generally, in order to represent the parameter configurations leading to the different regimes, we take the same values as in Browning, Chiappori and Lechene (2010), namely $a = 5/3$, $\alpha = 8/9$, $b = 15/32$ and $\beta = 1/2$, leading to a degree of symmetry equal to 0.5, and we stick now to equal degrees of autonomy $\theta^A = \theta^B = \theta$. This is given in Figure 1, with θ and ρ varying from 0 to 1 along the horizontal and the vertical axes, respectively. Six different regimes are possible: (I) where B is the only spouse to contribute to (both) public goods, (II) where A contributes to her preferred public good and B still contributes to both, (III) where each spouse specializes on his/her preferred public good (separate spheres with both spouses contributing), (IV) and (V) symmetric to (II) and (I) respectively (with inverted roles of A and B), and (VI) where both spouses contribute to both public goods.

As already mentioned, we see that the regime (III) of separate spheres exists only for a sufficiently high degree of autonomy (higher than the square root of the degree of symmetry $\sqrt{1/2} = 0.707$), and for relatively equal income shares. The corresponding regime (VI) for a lower degree of autonomy is the one where both spouses contribute to both public goods, allowing for more and more income disparities as we approach full cooperation. By contrast, the regimes (I) and (V) of exclusive contribution to public spending by the richer spouse are compatible with a lower and lower amplitude in income distribution as we approach full non-cooperation.

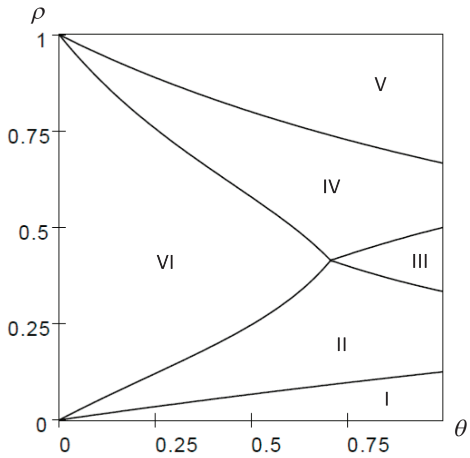

Figure 1: Regime switching values of ρ as θ varies

4.2 Income distribution, household public consumption and spouse welfare

The different regimes will naturally influence the way in which public consumption evolves as the income distribution varies. We illustrate this point in Figure 2 by plotting the household consumption of the wife's preferred public good (X) as her income share ρ increases from 0 to 1. The upper thin line corresponds to full cooperation, the lower thin broken line to full non-cooperation. The two thick curves correspond to intermediate equal degrees of cooperation, $\theta = 1/3$ for the upper one and $\theta = 4/5$ for the lower.

Except for the line corresponding to full cooperation ($\theta = 0$), all the curves are broken lines, reflecting regime switches as ρ increases (I-II-III-IV-V for $\theta = 4/5$ and $\theta = 1$, I-II-VI-IV-V for $\theta = 1/3$). The line corresponding to full non-cooperation ($\theta = 1$) exhibits two horizontal segments (relative to regimes II and IV). These segments illustrate local income pooling, a phenomenon which is peculiar to this case. Finally observe that, for both curves corresponding to intermediate degrees of cooperation, the segments relative to regimes of separate

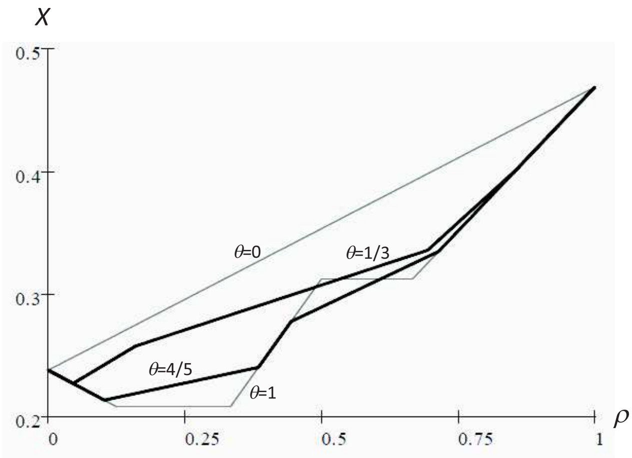

Figure 2: Household public consumption X as ρ varies

spheres (I, III and V), coincide with the fully non-cooperative curve.

As far as, say, the wife's utility is concerned, one can expect that it will be increasing in her income share, while the ultimate effect on her utility of her degree of autonomy combined with the one of her husband is ambiguous. Indeed, there is an efficiency loss as autonomy increases, but this loss may be compensated by the ensuing decrease in Lindahl taxation. We illustrate this ambiguity in Figure 3, where A's utility is represented as an increasing function of her income share ρ for three different configurations of the degrees of autonomy. The smooth curve corresponds to full cooperation ($\theta^A = \theta^B = 0$), the upper (thick) broken line to a configuration where the wife is more autonomous than the husband ($\theta^A = 3/4 > 1/4 = \theta^B$), and the lower broken line to the opposite case ($\theta^A = 1/4 < 3/4 = \theta^B$).

Clearly, more autonomy while keeping constant the average degree of autonomy is beneficial: the upper broken line completely dominates the lower broken one. Moreover, both broken lines dominate the smooth curve (corresponding to full cooperation) for a low income share: in spite of the efficiency loss, autonomy is beneficial for the poor spouse even when the other spouse is more

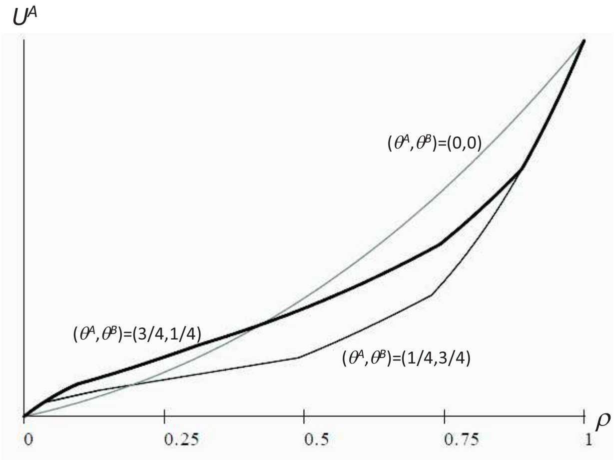

Figure 3: A's utility as ρ, θ^A and θ^B vary

autonomous. As better illustrated in Figure 4, this effect is a strong incentive for, say, the wife to deviate alone from full cooperation when her income is relatively low, thus cumulating the benefits of an absolute and a relative increase in her degree of autonomy.

The three curves correspond now to a zero degree of autonomy of the husband ($\theta^B = 0$), and different degrees of autonomy of the wife ($\theta^A = 0$ for the smooth curve, $\theta^A = 1/5$ for the lower, thick, broken curve, and $\theta^A = 4/5$ for the higher, thin, broken curve). The wife's utility is seen to respond positively, and significantly for a low income share, to an increase in her degree of autonomy, while the husband remains fully cooperative. As the income share becomes higher and higher, this effect eventually vanishes though.

5 Local properties of household demand

Local properties of the household demand can be used for empirical testing, by allowing in particular to discriminate between different models of household behavior. In this section, we first establish the foundations of the spouses'

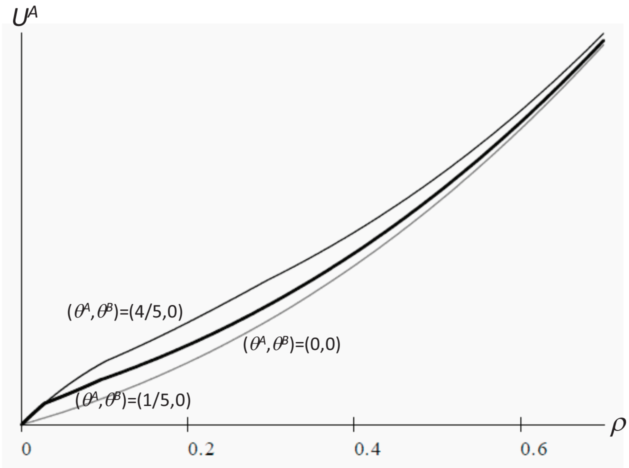

Figure 4: A's utility as ρ and θ^A vary, θ^B remaining nil

demand functions, to be aggregated into the household demand function. We then proceed with the analysis of its local properties in our comprehensive model of household behavior, after a preliminary examination of the two extreme cases of full cooperation ($\theta^A = \theta^B = 0$) and full autonomy ($\theta^A = \theta^B = 1$).

5.1 Foundations of the spouses' demand functions

The Marshallian demand function of spouse $J \in \{A, B\}$, conditional to a given choice $g^{-J} \in \mathbb{R}_+^m$ of the other spouse, can be straightforwardly derived from his/her utility maximization program:

$$x^J\left(p, \mathcal{P}^J, \mathcal{Y}^J, g^{-J}\right) \equiv \arg \max_{\substack{(q^J, g^J) \in \mathbb{R}_+^{n+m} \\ pq^J + \mathcal{P}^J g^J \leq \mathcal{Y}^J}} U^J\left(q^J, g^J + g^{-J}\right) \tag{29}$$

with $\mathcal{P}^J \equiv \theta^J P + \overline{\theta}^J P^J = P - \overline{\theta}^J P^{-J}$ and $\mathcal{Y}^J \equiv Y^J - P^J \overline{\theta}^{-J} g^{-J}$.

We fix both the degrees of autonomy $\left(\theta^A, \theta^B\right)$ of the two spouses and the income distribution ρ, defined by $\left(Y^A, Y^B\right) \equiv \left(\rho^A, \rho^B\right) Y \equiv (\rho, 1-\rho) Y$. They will in general be omitted, for simplicity of notation, as arguments of the functions to be introduced in the following. By contrast, we shall consider perturbations of the environment $\omega \equiv (p, P, Y) \in \mathbb{R}_+^{n+m+1}$. More precisely, we consider an open set $\Omega \subset \mathbb{R}_+^{n+m+1}$ of environment values, assuming equilibrium *uniqueness* for any element of Ω, so that we can refer to the functions $G^J : \Omega \to \mathbb{R}_+^m$ and $P^J : \Omega \to \mathbb{R}_+^m$ $(J = A, B)$, associating with each environment the individual contributions to public consumption and the contributive shares at equilibrium.[12] Of course, these functions are related by the budget consistency condition (12).

We further suppose that the private goods purchased by each spouse and the public goods to which she/he *actually* contributes (corresponding to the non-zero elements of equilibrium vectors $\left(q^A, g^A\right)$ and $\left(q^B, g^B\right)$) are the same for each element of Ω (*no regime switching* over this set). We keep the notation of the preceding subsection: M^J, M^{AB} and M^0 for the sets of public goods contributed by spouse J, both spouses and no spouse, respectively, and m^J, m^{AB}, m^0 for the corresponding cardinals. Finally, we assume differentiability of the functions G^J and P^J, so that we can obtain differentiable Marshallian conditional demand functions x^J $(J = A, B)$.

5.2 Full cooperation

In the *unitary model*, where the Pareto weights $\mu^A = \mu$ and $\mu^B = 1 - \mu$ of the two spouses are fixed and the household is maximizing the utility function (4) under the common budget constraint $pq + PQ \leq Y$, the expenditure shares $\rho^A = \rho$ and $\rho^B = 1 - \rho$ have to be adjusted to changes in the environment by lump sum transfers within the household, so that they are in fact functions of the environment ω. The contributions to public goods G^A and G^B, as well as the contributive shares P^A and P^B, are also functions of the environment, as previously assumed, although also partially through ρ in this context. We can

[12] Lechene and Preston (2010), studying the fully non-cooperative case, also rely on the uniqueness assumption, although with a slightly different game where each spouse J chooses, rather than his/her own contribution g^J, his/her preferred household consumption Q^J (which should not be less than g^{-J}), with $Q^A = Q^B = Q$ at equilibrium.

accordingly express the individual demand functions as follows:

$$\xi^J\left(\omega,\rho^J(\omega)\right)\equiv x^J\left(p,P^J(\omega),\rho^J(\omega)Y-P^J(\omega)G^{-J}(\omega),G^{-J}(\omega)\right), \quad (30)$$

and the corresponding household demand function as

$$\widetilde{\xi}(\omega)\equiv\xi(\omega,\rho(\omega))\equiv\xi^A\left(\omega,\rho^A(\omega)\right)+\xi^B\left(\omega,\rho^B(\omega)\right). \quad (31)$$

The household demand $\widetilde{\xi}$ has the usual properties of Marshallian demand functions, in particular a symmetric and negative semi-definite Slutsky matrix[13]

$$\Sigma=\overbrace{\left[\partial_{(p,P)}\xi\right]+\left[\partial_Y\xi\right]\left[{}^T\xi\right]}^{\Psi}+\overbrace{\left[\partial_\rho\xi\right]\left(\left[\partial_{(p,P)}\rho\right]+\left(\partial_Y\rho\right)\left[{}^T\xi\right]\right)}^{\Delta}. \quad (32)$$

The matrix $\Delta=\Sigma-\Psi$ expresses a deviation of the equivalent Ψ of a Slutsky matrix computed for $\xi(\cdot,\rho)$ (a pseudo-Slutsky matrix which does not take income share adjustments into account and which does not have the same properties) from the genuine Slutsky matrix Σ of the function $\xi(\cdot,\rho(\cdot))$. We make the expression of this matrix more explicit in the following lemma.

Lemma 5 *The Slutsky matrix Σ of the household demand $\widetilde{\xi}(\cdot)\equiv\xi(\cdot,\rho(\cdot))$ of the unitary model is the sum of a* pseudo-Slutsky matrix $\Psi=\left[\partial_{(p,P)}\xi\right]+\left[\partial_Y\xi\right]\left[{}^T\xi\right]$ *and a* deviation matrix Δ *which can be expressed in terms of individual demand functions x^A and x^B as*

$$\Delta=\left(\left[\partial_{\mathcal{Y}}x^A\right]-\left[\partial_{\mathcal{Y}}x^B\right]\right)\left[{}^T\left(\rho^Bx^A-\rho^Ax^B\right)\right]. \quad (33)$$

Proof. By (30), (31) and (32),

$$\Delta=\left(\left[\partial_{\mathcal{Y}}x^A\right]-\left[\partial_{\mathcal{Y}}x^B\right]\right)Y\left(\left[\partial_{(p,P)}\rho\right]+\left(\partial_Y\rho\right)\left[{}^T\xi\right]\right). \quad (34)$$

Using $\rho(\omega)=(1/Y)\left[(p,P)\right]\left[\widetilde{\xi}^A\right]$, with $\widetilde{\xi}^A(\omega)\equiv\xi^A(\omega,\rho(\omega))$, we have:

$$\begin{aligned}&Y\left(\left[\partial_{(p,P)}\rho\right]+\left(\partial_Y\rho\right)\left[{}^T\xi\right]\right) && (35)\\ =\ &\left[{}^T\widetilde{\xi}^A\right]+\left[(p,P)\right]\left[\partial_{(p,P)}\widetilde{\xi}^A\right]+\left(-\rho+\left[(p,P)\right]\left[\partial_Y\widetilde{\xi}^A\right]\right)\left[{}^T\widetilde{\xi}\right].\end{aligned}$$

[13]Non-scalar matrices are denoted by an expression inside square brackets. As concerns quantities (resp. prices), each line (resp. column) corresponds to a different good. For instance, $[\xi]$ is a column matrix of order $(n+m,1)$, and $[(p,P)]$ is a line matrix of order $(1,n+m)$.

By symmetry of the Slutsky matrix $\left[\partial_{(p,P)}\widetilde{\xi}^A\right] + \left[\partial_Y \widetilde{\xi}^A\right]\left[{}^T\widetilde{\xi}\right]$ and by Euler's identity applied to ξ^A, a homogeneous function of degree 0, we can establish:

$$\begin{aligned} & [(p,P)]\left(\left[\partial_{(p,P)}\widetilde{\xi}^A\right] + \left[\partial_Y \widetilde{\xi}^A\right]\left[{}^T\widetilde{\xi}\right]\right) \qquad (36) \\ = \; & [(p,P)]\left({}^T\left[\partial_{(p,P)}\widetilde{\xi}^A\right] + \left[\widetilde{\xi}\right]\left({}^T\left[\partial_Y \widetilde{\xi}^A\right]\right)\right) \\ = \; & [(p,P)]\left({}^T\left[\partial_{(p,P)}\widetilde{\xi}^A\right]\right) + \overbrace{[(p,P)]\left[\widetilde{\xi}\right]}^{Y}\left({}^T\left[\partial_Y \widetilde{\xi}^A\right]\right) = 0. \end{aligned}$$

Hence, by (30) and (31), we obtain

$$\begin{aligned} Y\left(\left[\partial_{(p,P)}\rho\right] + \left(\partial_Y \rho\right)\left[{}^T\xi\right]\right) &= \left[{}^T\xi^A\right] - \rho\left(\left[{}^T\xi^A\right] + \left[{}^T\xi^B\right]\right) \qquad (37) \\ &= \rho^B\left[{}^T\xi^A\right] - \rho^A\left[{}^T\xi^B\right] = \left[{}^T\left(\rho^B x^A - \rho^A x^B\right)\right], \end{aligned}$$

so that $\Delta = \left(\left[\partial_{\mathcal{Y}} x^A\right] - \left[\partial_{\mathcal{Y}} x^B\right]\right)\left[{}^T\left(\rho^B x^A - \rho^A x^B\right)\right]$. ■

In the *collective model*, if we take as fixed the income distribution given by the parameter ρ, and implicitly consider a Pareto weight μ which varies with the environment, the effects of the adjustment in the income distribution required to keep μ fixed are absent ($\Delta = 0$). As a consequence, the corresponding household demand $\xi(\omega,\rho)$ (with fixed ρ) has only the pseudo-Slutsky matrix Ψ which differs from the genuine Slutsky matrix Σ of $\widetilde{\xi}(\omega)$ by the deviation matrix Δ, an outer product, hence with rank at most equal to 1. This observation is formally stated in the following proposition, which reproduces the main result in Browning and Chiappori (1998, Proposition 2).

Proposition 6 *Under* full cooperation *($\theta^A = \theta^B = 0$), the household demand function $\xi(\omega,\rho) = \xi^A(\omega,\rho) + \xi^B(\omega,\rho)$ has a pseudo-Slutsky matrix Ψ which deviates from a Slutsky matrix Σ by an outer product, which can be expressed in terms of the individual demands x^A and x^B as*

$$\Delta = \left(\left[\partial_{\mathcal{Y}} x^A\right] - \left[\partial_{\mathcal{Y}} x^B\right]\right)\left[{}^T\left(\rho^B x^A - \rho^A x^B\right)\right]. \qquad (38)$$

Using again (30) and (31), we may be more precise about the expression of matrices Ψ^A and Ψ^B (with sum equal to Ψ) in terms of individual demands x^A

and x^B:

$$\Psi^J = \left[\partial_p x^J\right]\left[I_n \vdots 0_{n\times m}\right] + \left(\left[\partial_{\mathcal{P}} x^J\right] - \left[\partial_{\mathcal{Y}} x^J\right]\left[{}^T G^{-J}\right]\right)\Pi^J \tag{39}$$
$$+ \left(\left[\partial_g x^J\right] - \left[\partial_{\mathcal{Y}} x^J\right]\left[P^J\right]\right)\Gamma^{-J} + \left[\partial_{\mathcal{Y}} x^J\right]\rho^J\left[{}^T\left(x^A + x^B\right)\right],$$

where I_n is the identity matrix of order n, $0_{n\times m}$ the zero matrix of order $n\times m$, $\Pi^J \equiv \left[\partial_{(p,P)} P^J\right] + \left[\partial_Y P^J\right]\left[{}^T\left(x^A + x^B\right)\right]$ and $\Gamma^J \equiv \left[\partial_{(p,P)} G^J\right] + \left[\partial_Y G^J\right]\left[{}^T\left(x^A + x^B\right)\right]$. Notice that, in the absence of any public good, $\Pi^J = \Gamma^{-J} = 0$, implying:

$$\Psi^J = \left[\partial_p x^J\right] + \left[\partial_{\mathcal{Y}} x^J\right]\rho^J\left[{}^T\left(x^J + x^{-J}\right)\right] \tag{40}$$
$$= \overbrace{\left[\partial_p x^J\right] + \left[\partial_{\mathcal{Y}} x^J\right]\left[{}^T x^J\right]}^{\Sigma^J} - \left[\partial_{\mathcal{Y}} x^J\right]\left[{}^T\left(\rho^{-J} x^J - \rho^J x^{-J}\right)\right], \tag{41}$$

where Σ^J is a Slutsky matrix. Thus,

$$\overbrace{\Psi^A + \Psi^B}^{\Psi} = \overbrace{\Sigma^A + \Sigma^B}^{\Sigma} - \overbrace{\left(\left[\partial_{\mathcal{Y}} x^A\right] - \left[\partial_{\mathcal{Y}} x^B\right]\right)\left[{}^T\left(\rho^B x^A - \rho^A x^B\right)\right]}^{\Delta}, \tag{42}$$

making it clear that the deviation matrix Δ expresses an *aggregation effect*, working in the general case where there is no "representative consumer", independently of the existence of public goods.

5.3 Full autonomy

From now on, we shall always take the income distribution as fixed and, for simplicity of notation, omit the parameter ρ as an argument of the functions ξ^J and ξ. The pseudo-Slutsky matrix $\Psi = \left[\partial_{(p,P)}\xi\right] + \left[\partial_Y \xi\right]\left[{}^T\xi\right]$ of the household demand function $\xi \equiv \xi^A + \xi^B$ can be easily decomposed by detailing the different effects of the environment through the arguments of the sum $x^A + x^B$:

$$\Psi = \overbrace{\left[\partial_{(p,\mathcal{P})} x^A\right] + \left[\partial_{\mathcal{Y}} x^A\right]\left[{}^T x^A\right]}^{\Sigma^A} + \overbrace{\left[\partial_{(p,\mathcal{P})} x^B\right] + \left[\partial_{\mathcal{Y}} x^B\right]\left[{}^T x^B\right]}^{\Sigma^B} \tag{43}$$
$$-\overbrace{\left(\left[\partial_{\mathcal{Y}} x^A\right] - \left[\partial_{\mathcal{Y}} x^B\right]\right)\left[{}^T\left(\rho^B x^A - \rho^A x^B\right)\right]}^{\Delta} + \overbrace{\left[\partial_g x^A\right]\Gamma^B + \left[\partial_g x^B\right]\Gamma^A}^{\Xi},$$

where $\Gamma^J \equiv \left[\partial_{(p,P)} G^J\right] + \left[\partial_Y G^J\right]\left[{}^T\left(x^A + x^B\right)\right]$, as above. The Slutsky matrices Σ^A and Σ^B of the individual demand functions x^A and x^B express the direct effects on individual optimizing decisions of a change in the environment. Their sum Σ has also the properties of a Slutsky matrix. The matrix Δ is an outer product, with a rank at most equal to $r_\Delta = 1$. It was already present in the fully cooperative case, resulting from an *aggregation effect*.

The matrix Ξ is new. It expresses an *externality effect*, when this effect ceases to be fully compensated by the response of Lindahl taxation to changes in the environment, as it was the case under full cooperation. Because of the assumption of no regime switching over Ω, if $g_k^J = 0$ for some k, then $\partial_g x_{n+k}^J = 0$ and $\partial_\omega G_k^J = 0$, so that the matrix $\left[\partial_g x^J\right]$ (resp. Γ^J) has at most $n+m^J+m^{AB}$ (resp. $m^J + m^{AB}$) non-zero rows (we recall that m^J is the number of public goods exclusively contributed by spouse J and m^{AB} the number of goods contributed by both spouses). In the absence of public consumption or, more generally, under preference separability, when the utility derived from each spouse's private and public consumption is unaffected by the other spouse's exclusive contributions to public goods, the matrix $\left[\partial_g x^J\right]$ vanishes (at least in the regime of separate spheres, where $m^{AB} = 0$), so that $\Xi = 0$, bringing us back to the result of the fully cooperative case: the deviation matrix $\Sigma - \Psi$ has a rank at most equal to $r_\Delta = 1$. In the other generic regime of separate spheres up to one public good k ($m^{AB} = 1$), under the same separability assumption, the matrices $\left[\partial_g x^A\right]$ and $\left[\partial_g x^B\right]$ will have only one non-zero column, the k-th, so that the maximum rank of Ξ will be $r_\Xi = 1$, leading to a maximum rank of the deviation matrix $r_\Delta + r_\Xi = 2$. But these results (Lechene and Preston, 2010, Theorems 4 and 2), due to inoperative externality effects, are of course lost as soon as we abandon separability. The generic result requires the rank of the deviation matrix to be at most equal to some upper bound which is introduced in the following proposition (Lechene and Preston, 2010, Theorems 3 and 1).

Proposition 7 *Under* full autonomy *($\theta^A = \theta^B = 1$), the household demand function $\xi(\omega) = x^A\left(p, P, \rho Y, G^B(\omega)\right) + x^B\left(p, P, (1-\rho)Y, G^A(\omega)\right)$ has a pseudo-Slutsky matrix Ψ which deviates from a Slutsky matrix $\Sigma = \Sigma^A + \Sigma^B$ by a matrix*

$\Delta - \Xi$ *of rank at most equal to*

$$r_{\Delta-\Xi} = m - m^0 + \min\left\{n - \max\left\{m^A - m^B, 1\right\}, 1\right\},$$

where $m^A - m^B$ *is assumed non-negative WLOG. The upper bound* $r_{\Delta-\Xi}$ *can be neither higher than* $m - m^0 + 1$ *nor lower than* 1 *(for* $n = 1$*,* $m^B = 0$ *and* $m^A + m^{AB} = 1$*).*

Proof. (**Separate spheres**) This is the simpler case. We first determine the maximum possible rank of Ξ. The matrix $\left[\partial_g x^J\right]$ has at most $n + m^J$ non-zero rows, which however cannot be linearly independent since $[(p, P)]\left[\partial_g x^J\right] = 0$ (consumption changes induced by the sole externality effect should not modify the expenditure $(p, P) \cdot x^J$, which has to be kept equal to J's income). Hence, the rank of $\left[\partial_g x^J\right]$ is at most equal to $n + m^J - 1$. The matrix Γ^J has at most m^J non-zero rows so that the rank of the matrix $\Xi = \left[\partial_g x^A\right]\Gamma^B + \left[\partial_g x^B\right]\Gamma^A$ cannot be higher than

$$r_\Xi = m^B + \min\left\{n + m^B - 1, m^A\right\} = m^A + m^B + \min\left\{n - 1 - \left(m^A - m^B\right), 0\right\}.$$

Now, by applying Euler's identity to the functions ξ and x^J, which are homogeneous of degree 0, we see that $[(p, P)]\left[{}^T\Psi\right] = [(p, P)]\left[{}^T\Sigma\right] = 0$, implying $[(p, P)]\left[{}^T(\Delta - \Xi)\right] = 0$, so that the columns of the matrix $\Delta - \Xi$ are not linearly independent. Hence, the rank of this matrix is at most equal to $n + m^A + m^B - 1$, since it has only $n + m^A + m^B$ non-zero columns (variations in the prices of the m^0 public goods which are not consumed by the household cannot induce changes in the spouses' contributions). Taking into account this upper bound, recalling that $m^A + m^B = m - m^0$ (since $m^{AB} = 0$), and simply adding r_Δ and r_Ξ completes the proof:

$$\begin{aligned} r_{\Delta-\Xi} &= \min\left\{n + m - m^0 - 1, 1 + m^A + m^B + \min\left\{n - 1 - \left(m^A - m^B\right), 0\right\}\right\} \\ &= m - m^0 + \min\left\{n - \max\left\{m^A - m^B, 1\right\}, 1\right\}. \end{aligned}$$

(**Joint contribution to public consumption**) Now suppose that both spouses contribute to the k-th public good, for any $k \in M^{AB}$ ($m^{AB} > 1$ is non-generic, as we have seen in section 3). Notice that it is then possible to realize a transfer of the individual contributions of the two spouses (so that $g_k^A + \varepsilon_k$ and $g_k^B - \varepsilon_k$) without violating the non-negativity constraint and without changing the individual utilities. A new equilibrium prevails provided there is a compensating income transfer ($P_k\varepsilon_k$, leading to incomes $Y^A + P_k\varepsilon_k$ and $Y^B - P_k\varepsilon_k$). Because of this *local income pooling* property, the equilibrium outcome (except as concerns the way the household consumption Q_k is decomposed into the voluntary contributions g_k^A and g_k^B) will be the same at given prices and household income if we let, say, the wife make the whole purchase of public good k, while compensating her by a transfer from her husband equal to $P_k G_k^B$. This transfer triggers the appearance of a new component of the pseudo-Slutsky matrix of household demand, namely

$$\left(\left[\partial_{\mathcal{Y}} x^A\right] - \left[\partial_{\mathcal{Y}} x^B\right]\right) \sum_{k \in M^{AB}} \left(P_k \Gamma_k^B + e_{n+k} G_k^B\right),$$

where $e_{n+k} = \left[\partial_{(p,P)} P_k\right] + \left(\partial_Y P_k\right) \left[^T \left(x^A + x^B\right)\right]$ is the $n+k$-th row of the identity matrix I_{n+m}. Clearly, this component does not increase the rank of the deviation matrix, since it can be added to $-\Delta$ without changing its nature of outer product. Otherwise, the income transfer brings us back to a regime of separate spheres with $m^A + m^{AB}$ and m^B public goods contributed by spouses A and B, respectively. Hence, the maximum rank of Γ^B is now m^B. However, the relevant upper bound for the rank of $\left[\partial_g x^B\right]$ remains $n + m^B + m^{AB} - 1$, since we cannot apply in this context the implication $x_k^B = 0 \Longrightarrow \partial_g x_k^B = 0$ imposed by the assumption of no regime switching over Ω. Indeed, B's marginal willingness to pay for the k-th public good remains equal to P_k (whereas it is generically smaller than its price for any non contributed public good), making it eligible for a contribution by B in response to any perturbation of his environment. By simply reproducing the argument developed for the case of separate spheres, we thus obtain for the maximum rank of the deviation matrix:

$$\begin{aligned} r_{\Delta-\Xi} &= \min\left\{n+m-m^0-1, 1+m^B+\min\left\{n+m^B+m^{AB}-1, m^A+m^{AB}\right\}\right\} \\ &= m-m^0+\min\left\{n-1, \min\left\{n-\left(m^A-m^B\right), 1\right\}\right\} \\ &= m-m^0+\min\left\{n-\max\left\{m^A-m^B, 1\right\}, 1\right\}. \end{aligned}$$

■

5.4 Semi-cooperation

The analysis of the intermediate cases where both spouses have some degree of autonomy, but also cooperate through Lindahl taxation, generalizes the two previous cases. The pseudo-Slutsky matrix $\Psi = \left[\partial_{(p,P)}\xi\right] + \left[\partial_Y \xi\right]\left[{}^T\left(x^A+x^B\right)\right]$ of the household demand function at some equilibrium

$$\xi(\omega) \equiv \sum_{J=A,B} x^J\left(p, \theta^J P + \overline{\theta}^J P^J(\omega), \rho^J Y - P^J(\omega)\overline{\theta}^{-J} G^{-J}(\omega), G^{-J}(\omega)\right), \tag{44}$$

with $\omega = (p, P, Y)$, can be expressed, by using again the notations $\Gamma^J \equiv \left[\partial_{(p,P)}G^J\right] + \left[\partial_Y G^J\right]\left[{}^T\left(x^A+x^B\right)\right]$ and $\Pi^J \equiv \left[\partial_{(p,P)}P^J\right] + \left[\partial_Y P^J\right]\left[{}^T\left(x^A+x^B\right)\right]$, as the sum of the two matrices Ψ^J for $J = A, B$:

$$\begin{aligned} \Psi^J &= \left[\partial_p x^J\right]\left[I_n \vdots 0_{n\times m}\right] + \theta^J\left[\partial_{\mathcal{P}} x^J\right]\left[0_{m\times n} \vdots I_m\right] \\ &\quad + \left(\overline{\theta}^J\left[\partial_{\mathcal{P}} x^J\right] - \overline{\theta}^{-J}\left[\partial_{\mathcal{Y}} x^J\right]\left[{}^T G^{-J}\right]\right)\Pi^J \\ &\quad + \overline{\theta}^{-J}\left(\left[\partial_g x^J\right] - \left[\partial_{\mathcal{Y}} x^J\right]\left[P^J\right]\right)\Gamma^{-J} + \theta^{-J}\left[\partial_g x^J\right]\Gamma^{-J} \\ &\quad + \left[\partial_{\mathcal{Y}} x^J\right]\rho^J\left[{}^T\left(x^A+x^B\right)\right], \end{aligned} \tag{45}$$

of which the expressions for full cooperation and full autonomy are readily seen to be particular cases, for $\theta^J = \theta^{-J} = 0$ and $\theta^J = \theta^{-J} = 1$ respectively.

By Proposition 1, the non-cooperative and the semi-cooperative models are observationally equivalent under separate spheres, so that we then expect the expression of the pseudo-Slutsky matrix to coincide with the one given in the preceding subsection. Indeed, $P^J g^{-J} = 0$ in this regime, so that $\left[P^J\right]\Gamma^{-J} = \left[{}^T G^{-J}\right]\Pi^J = 0$. Also, as $P_k^J = P_k$ if $k \in M^J$, and zero otherwise, the only non-zero elements of the matrix Π^J are $\Pi^J_{k,n+k} = 1$ for any $k \in M^J$. Finally, the k-th column of $\left[\partial_{\mathcal{P}} x^J\right]$ is zero if $k \notin M^J$. We thus obtain under separate spheres:

$$\Psi^J = \overbrace{\left[\partial_{(p,\mathcal{P})} x^J\right] + \left[\partial_{\mathcal{Y}} x^J\right]\left[{}^T x^J\right]}^{\Sigma^J} - \left[\partial_{\mathcal{Y}} x^J\right]\left[{}^T\left(\rho^{-J} x^J - \rho^J x^{-J}\right)\right] + \overbrace{\left[\partial_g x^J\right]\Gamma^{-J}}^{\Xi^J}, \tag{46}$$

so that the sum $\Psi = \Psi^A + \Psi^B$ has clearly the same expression as in (43).

Outside the regime of separate spheres, the observable household behavior under semi-cooperation differs from the one under non-cooperation. As already stated, joint contribution by the two spouses to m^{AB} public goods, with $m^{AB} > 1$, is not anymore a singular property. Also, for $m^{AB} = 1$, the property of local income pooling does not extend from non-cooperation to semi-cooperation. Besides, the pseudo-Slutsky matrix of the household demand function exhibits now a further component. By using the condition $P^A + P^B = P$, hence $\Pi^A + \Pi^B = \partial_{(p,P)} P = \left[0_{m\times n} \vdots I_m\right]$, we obtain the expression

$$\begin{aligned}
\Psi \quad = \quad & \overbrace{\left[\partial_{(p,\mathcal{P})} x^A\right] + \left[\partial_{\mathcal{Y}} x^A\right]\left[{}^T x^A\right]}^{\Sigma^A} + \overbrace{\left[\partial_{(p,\mathcal{P})} x^B\right] + \left[\partial_{\mathcal{Y}} x^B\right]\left[{}^T x^B\right]}^{\Sigma^B} \\
& - \overbrace{\left(\left[\partial_{\mathcal{Y}} x^A\right] - \left[\partial_{\mathcal{Y}} x^B\right]\right)\left[{}^T\left(\rho^B x^A - \rho^A x^B\right)\right]}^{\Delta} \\
& + \overbrace{\left(\left[\partial_g x^A\right] - \overline{\theta}^B\left[\partial_{\mathcal{Y}} x^A\right]\left[P^A\right]\right)\Gamma^B + \left(\left[\partial_g x^B\right] - \overline{\theta}^A\left[\partial_{\mathcal{Y}} x^B\right]\left[P^B\right]\right)\Gamma^A}^{\Xi} \\
& - \overbrace{\begin{pmatrix} \overline{\theta}^A\left(\left[\partial_{\mathcal{P}} x^A\right] + \left[\partial_{\mathcal{Y}} x^B\right]\left[{}^T G^A\right]\right)\Pi^B \\ +\overline{\theta}^B\left(\left[\partial_{\mathcal{P}} x^B\right] + \left[\partial_{\mathcal{Y}} x^A\right]\left[{}^T G^B\right]\right)\Pi^A \end{pmatrix}}^{\Theta},
\end{aligned} \tag{47}$$

with a deviation matrix $\Sigma - \Psi = \Delta - \Xi + \Theta$. In addition to the aggregation and externality effects described by the matrices Δ and Ξ, respectively, we now have *substitution effects* of price changes through the contributive shares, expressed by the matrix Θ. It is the rank of this matrix that may increase the maximum possible rank of the deviation matrix, as we make precise in the following proposition (covering both regimes, of separate spheres and of joint contribution to some or all public goods).

Proposition 8 *Under* semi-cooperation *($\left(\theta^A, \theta^B\right) \in (0,1)^2$), the household demand function*

$$\xi(\omega) = \sum_{J=A,B} x^J \left(p, \theta^J P + \overline{\theta}^J P^J(\omega), \rho^J Y - P^J(\omega) \overline{\theta}^{-J} G^{-J}(\omega), G^{-J}(\omega)\right)$$

has a pseudo-Slutsky matrix Ψ which deviates from a Slutsky matrix $\Sigma = \Sigma^A + \Sigma^B$ by a matrix $\Delta - \Xi + \Theta$ of rank at most equal to

$$r_{\Delta - \Xi + \Theta} = m - m^0 + \min\left\{n-1, \min\left\{n - \left(m^A - m^B\right), 1\right\} + 2m^{AB}\right\},$$

where $m^A - m^B$ is assumed non-negative WLOG. The upper bound $r_{\Delta-\Xi+\Theta}$ can neither be higher than $1 + 2\left(m - m^0\right)$ nor lower than 1 (for $n = 1$, $m^B = 0$ and $m^A + m^{AB} = 1$).

Proof. The matrix Δ is an outer product, with rank at most equal to $r_\Delta = 1$. Concerning matrix Ξ, even if local income pooling does not prevail anymore, the argument used in the second part of the proof of Proposition 7 still holds: in the case of joint contribution to m^{AB} public goods, we may assume a reallocation of individual contributions accompanied by a compensating income transfer between the two spouses, so as to recover separate spheres without changing their aggregate demand, provided we keep the contributive shares fixed. If both spouses contribute to some k-th public good, it is possible to reallocate the whole contribution of, say, the husband to the wife (whose contributions thus become $g_k^A + g_k^B$ and 0, respectively), while keeping inalterate all the individual first order conditions. The consequence of this reallocation on the individual budgets is a deficit $\left(\theta^A P_k + \overline{\theta}^A P_k^A - \overline{\theta}^B P_k^A\right) g_k^B = \left(\theta^A P_k^B + \theta^B P_k^A\right) g_k^B$ for the wife and a surplus $-\left(\theta^B P_k + \overline{\theta}^B P_k^B - \overline{\theta}^A P_k^B\right) g_k^B = -\left(\theta^A P_k^B + \theta^B P_k^A\right) g_k^B$ for the husband, which can be eliminated by a compensating income transfer from the latter to the former. This transfer introduces a new component of the pseudo-Slutsky matrix of household demand, namely

$$\Delta' = \left(\left[\partial_Y x^A\right] - \left[\partial_Y x^B\right]\right) \sum_{k \in M^{AB}} \left(\left(\theta^A P_k^B + \theta^B P_k^A\right) \Gamma_k^B + \left(\theta^A \Pi_k^B + \theta^B \Pi_k^A\right) G_k^B\right),$$

which however does not increase the rank of the deviation matrix, since the sum $-\Delta + \Delta'$ remains an outer product. As to the new matrix Ξ after the reallocation, it is straightforward to establish, along the lines of the proof of Proposition 7 for separate spheres, that its rank cannot be larger than

$$\begin{aligned} r_{\Xi} &= m^B + \min\left\{n + m^B + m^{AB} - 1, m^A + m^{AB}\right\} \\ &= m - m^0 + \min\left\{n - \left(m^A - m^B\right) - 1, 0\right\}. \end{aligned}$$

Concerning matrix Θ, first observe that the k-th column of matrix $\left[\partial_{\mathcal{P}} x^J\right]$ is zero for any $k \in M^{-J} \cup M^0$, since a variation in the price of a public good to which spouse J does not contribute cannot induce changes in J's demand for any good. So is obviously the k-th column of matrix $\left[{}^T G^J\right]$, hence of matrix $\overline{\theta}^J \left(\left[\partial_{\mathcal{P}} x^J\right] + \left[\partial_{\mathcal{Y}} x^{-J}\right]\left[{}^T G^J\right]\right)$. Thus, in the product of this matrix with matrix Π^{-J} the corresponding k-th line of the latter might as well be zero. But Π^{-J} has m^J further zero lines, namely any j-th line such that $j \in M^J$, because of the budget consistency condition. Hence, the product of these matrices is upper bounded by m^{AB} and the rank of Θ (the sum of two such products) by $r_{\Theta} = 2m^{AB}$.

By adding $r_{\Delta} + r_{\Xi} + r_{\Theta}$, and taking into account the upper bound of the rank of the deviation matrix, which has at most $n + m - m^0 - 1$ linearly independent non-zero columns (since $[(p,P)]\left[{}^T\Psi\right] = [(p,P)]\left[{}^T\Sigma\right] = 0$, so that $[(p,P)]\left[{}^T(\Delta - \Xi + \Theta)\right] = 0$), we finally obtain:

$$\begin{aligned} & r_{\Delta-\Xi+\Theta} = \\ & \min\left\{n + m - m^0 - 1, 1 + m - m^0 + \min\left\{n - \left(m^A - m^B\right) - 1, 0\right\} + 2m^{AB}\right\} = \\ & m - m^0 + \min\left\{n - 1, \min\left\{n - \left(m^A - m^B\right), 1\right\} + 2m^{AB}\right\}. \end{aligned}$$

■

The upper bound imposed upon the rank of the deviation matrix can be used to test the different models of household behavior. Browning and Chiappori (1998) have used this upper bound to discriminate between the unitary model (which predicts that the matrix $\Psi - \left({}^T\Psi\right)$ has rank 0 because of the symmetry of $\Psi = \Sigma$) and the collective model (which predicts that $\Psi - \left({}^T\Psi\right)$ has rank at most 2, since $\Psi = \Sigma - \Delta$, with a rank of Δ at most equal to 1 for a couple).

They have also shown that this test requires at least 5 goods, a requirement that stems from the fact that the rank of $\Psi - \left({}^{T}\Psi\right)$ cannot be higher than 2 if the number $n+m$ of goods is not larger than 4 (given the linear dependence of the columns of Ψ introduced by the homogeneity of degree zero of the demand functions).

Lechene and Preston (2008) have shown that, in order to reject the non-cooperative model, one must have $n \geq m+5$. Their Lemma A.1 shows indeed that, if $\Psi - \left({}^{T}\Psi\right)$ has rank at most $n+m-1$, then Ψ can always be expressed as the sum of a symmetric matrix and a matrix of rank not higher than r such that $2r+1 \geq n+m-1$. For $r = m+1$, as in the non-cooperative case, the test works only if $2(m+1)+1 < n+m-1$, that is, if $n \geq m+5$.

As previously emphasized, there is no possibility of discriminating between full and partial autonomy under separate spheres, since the non-cooperative and the semi-cooperative models are then observationally equivalent. However, the discrimination between the two models is possible under joint contribution to at least one public good. In the semi-cooperative case, if we apply this lemma to our Proposition 8, we see that $n \geq m + 4m^{AB} + 5$ is needed to discriminate between full and partial autonomy. If, for instance, there is only one public good to which both spouses contribute, at least 10 private goods are required. The maximum possible rank of $\Psi - \left({}^{T}\Psi\right)$, given homogeneity of degree 0 of the demand functions, is then 10. As the observed rank increases from 0 to 10, the test successively rejects the unitary model (at 2), full cooperation (at 4), full autonomy (at 6) and the semi-cooperative model as a whole (at 10).

Finally, the discrimination between the non-cooperative model and the semi-cooperative one is less demanding as soon as we know that there are more than one public good to which both spouses contribute. In a recent empirical application of the noncooperative consumption model to data drawn from the Russia Longitudinal Monitoring Survey, Cherchye, Demuynck and De Rock (2011) could not reject joint contributions to two or three public goods by some households.[14] In that case full autonomy is (generically) excluded and, if the observed

[14] This fact should however be taken with care, since the number of observations per house-

rank is more than 4, full cooperation is also excluded, leaving (for rank less than 10) partial autonomy as an appropriate model.

6 Conclusion

In recent years, the 'terra incognita' of economic household behavior has been progressively reduced, firstly by extending the unitary model to the collective model, hence exploring further the territory of efficient household decisions, and secondly by entering the area of full non-cooperation. The present paper has covered yet another territory, the one in-between, where household members keep some degree of autonomy in their public good contributions. For that purpose we have introduced the general concept of θ-equilibrium based on a non-cooperative reformulation of the Lindahl equilibrium. Combining the generic local properties of equilibria in the various models with a comprehensive derivation of the pseudo-Slutsky matrix we have shown that the possibility of testing the different models exists. For that we use the result already known about the non-cooperative case, namely that only two regimes (pure separate spheres and separate spheres up to one public good) are generic, and a particular decomposition of the pseudo-Slutsky matrix into the different effects that are specific to each model, each additional effect adding to the maximum possible rank of the deviation matrix. The required number of goods and, more specifically, the required number of private goods with respect to the number of public goods, are increasing with the rank of the deviation matrix, making testability more and more difficult to implement. However, in the case of at least two jointly contributed public goods, the three models (with full, semi- and nil cooperation) can be more easily separated since nil cooperation is generically excluded.

An important issue that is raised by these results, either with full or partial autonomy, is the identification of a good consumed by the household as being

hold in the data set is very small.

private or public (or semi-public).[15] The public or private nature of a good is of course linked to some objective characterictics, like the possibility of being non-exclusively consumed or the presence of external effects, but also linked to the recognition of such characteristics by the spouses themselves and their agreement to share the good. Another related issue is the fixing of the autonomy parameters. All these issues are here supposed to be preliminarily resolved. But such issues should be treated both theoretically and empirically. For example, one could introduce a preliminary stage in the household game where the autonomy parameters are set in some (enforceable) contractual agreement and study the equilibrium of the two-stage game. For testability, also, other approaches and techniques can be used such as the revealed-preference approach[16] or the techniques used by the New Empirical Industrial Organization when estimating conduct parameters.[17] Further work is obviously required.

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[15] As well emphasized by Browning and Chiappori (1998), such identification is not needed in the collective model. Chiappori and Ekeland (2006) show that the private or public nature of consumption within the household is not testable from aggregate data alone. However, using a global approach through revealed preferences, Cherchye, De Rock and Platino (2010) draw the reverse conclusion.

[16] See Cherchye, De Rock and Vermeulen (2007) and Cherchye, Demuynck and De Rock (2011).

[17] These so-called conduct parameters measure the relative weight of competitive toughness and play in the analysis of firm behavior a role similar to the degrees of autonomy in the analysis of household behavior.

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