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A Learning Model with Memory in the Financial Markets

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Abstract: Learning is central to a financial agent's aspiration to gain persistent strategic advantage in asset value maximization. The implicit mechanism that transforms this aspiration into value gain is the speed of error corrections (equivalently, an agent's speed of learning) whilst facing increased uncertainty. In such a setting, perpetual learning can lead to a type of persistent memory that financial agents can exploit as information-set to plan for their next strategic decisions. The existing literature focuses predominantly on a learning model with a long-lag characterization of asset values. However, recent methodological advances show that long memory can be generated from a learning model with just one lag. In this paper, we exploit this strand of literature and integrate the new construct into our proposed duration-compliant “social distance” mechanic as an alternative robust model of learning in a financial market. We assess the efficacy of our approach with a numerical example and a calibration exercise.

Keywords: Learning, Memory, Distance, Short-Lag, Financial Markets.

JEL Codes: C1, C73, D83, E3, N2.

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1 Introduction

“*May you never forget what is worth remembering, nor ever remember what is best forgotten*”. –

Irish Blessing

The above quote from Irish Blessing aligns with a reflection of the hidden architecture of model building by economic theorists and econometricians based on some quantitative convolution of the idea: ‘*never forget what is worth remembering*’. That is, the persistence of a positive signal for the agent to predict a pattern that is most likely to occur in interaction with complex socio-economic systems. Further, it becomes increasingly difficult, although ideally plausible, to least remember negative shocks because a long-term memory of such a shock can trigger a system instability (Mishra et al. 2023). Clearly, this is what neither a central planner nor an individual agent would ever want. However, in a real-life scenario, both events occur in tandem at either the instant of time, t , or at different time points such as $t+1, t+2, \dots$. This means that the agent needs a learning rule to negotiate both negative and positive shocks at the same time or within a short temporal difference: i.e., trying to forget (ideally very rapidly because this brings stability within the shortest time duration) and remember the positive one (because this can create a diffusion path of positive expectations).

Memory (irrespective of its lengths, short and long) is an ‘identity of an agent because this ‘identity helps the agent design his next best strategic action maximizing value. Researchers offer several theoretical possibilities as potential sources of *memory* (particularly, long-term memory), such as *aggregation, heterogeneity* of actions, and the *individual effects of learning*. This paper emphasizes the distinct effects of learning (particularly the speed of learning) and embeds the same in a ‘distance-theoretic’ framework to propose an alternative approach to asset value optimisation under uncertainty.

Our identification mechanism consists of two important edifices. First, we consider a setting where persistence of *information asymmetry* and their varying magnitudes produce a realistic duration-dependent type of shock survival process. A variant of this mechanism has been investigated in Parke (1999). Abstracting from conventional infinite-order autoregressive (AR) and moving average (MA) driven framework, the error duration model assumes that observed errors follows a stochastic *duration* instead of dissipating uniformly over time. The duration aspects allow a financial agent to evaluate persistence profiles of shocks for the (asymmetric) duration. In other words, the length of duration of shocks or their survival probabilities vary around the intervention (short or long duration before and after the intervention). Indeed, in a financial setting, one can possibly identify asset positions as one would model elements that have stochastic durations, where those durations reflect ‘persistence or memory’ of certain degrees. Instead of a *uniform* duration or interval (a feature which does not comply often with real life data), we allow an asymmetric duration. For

instance, shocks display greater persistence after the intervention of a negative policy shock (such as an increase in interest rate) and lower persistence prior to the intervention. Such heterogeneous persistence profiles can produce varied magnitudes of memory so much so that asset price volatility can be fractionally integrated (with long memory) if ‘a few asset positions last longer than would be predicted by the lifetimes of typical positions (Parke, 1999).

The basic mechanism for an error-duration model is the overlay of a sequence of shocks which are of both stochastic magnitude and stochastic duration. The *stochastic* element accommodates a type of characteristic that may be persistent but may have a random assignment/occurrence. In other words, an observed time series such as stock prices (returns) or volumes at a given point of time, is actually a mathematical representation of *sum of those shocks that survive to that point*. Eventually, the nature of the *distribution of the duration of these shocks* actually determines the order of integration (or fractional integration) of the series. Technically (as we shall explore in Section 3), the fractional integration would require that only a small percentage of the shocks have *long durations* (or more persistent), which is typically the case in financial markets. We use this mechanic as our baseline source of long memory and combine it with an agent’s learning behavior considering his relative economic positioning.

Recently, in an intuitively appealing and theoretically robust framework, Chevillon and Mavroeidis (2017) demonstrated that *learning can generate memory*. Bauwens et al. (2023) expanded the methodological underpinning of Chevillon and Mavroeidis to introduce a model that could generate a long-memory with substantial lag-size reduction (in their case, with just *one lag*). This dimensionality reduction is important because in real life financial or economic worlds, agents ‘best remember a lag 1 shock’, assuming that long-lag shocks are somewhat incorporated within the immediate lag (lag 1). It is within this robust lag-assumption, agents often engage in the unavoidable dynamics of comparison across social or economic positions. As an example, consider a case where a financial agent wishes to predict an asset position for himself. He would take into account not only his own position at $t - 1$, but also, of the position of another agent (j) at $t - 1$. A predictive conformity of his position at $t + 1$ (in general, an equilibrium decision) is based on his *social position* in relation to others. This position drives the core of learning at an instant of time. We recognise the centrality of such a process, that is, an agent’s ‘social positioning’ with asymmetric error duration, one which is likely to originate from persistent information asymmetry. This eventually helps us introduce a long-memory social distance-driven learning model (an acronym that we denote as DSDLM).

The rest of the paper is planned as follows. Section 2 provides a synoptic overview of the learning and long-memory literature eliciting how learning and memory are intertwined. Section 3 presents our model design. Section 4 provides a numerical example. Finally, Section 5 presents a discussion of our main arguments and

summarizes the results.

2 Literature

The literature on learning and long memory has developed ostensibly over the years. In our view, both streams of thoughts have grown in parallel and remarkably well in their own respective fields of development. Lately, researchers have been seeking to explore the source of long-memory in the form of endogenous learning of agents. There are still substantive ambiguities on how learning (discrete and discontinuous) determines long memory when agents invariably make comparisons of their own as well as others' social status. In other words, an agent may be wary of the financial gains from an asset position of another agent (making her socially and financially more secured and identifiable). The question is how one can combine those central features of learning (about social positions) and exploit duration nature of error dependence with persistence asymmetric information. Often, then the question focuses on rationality of choice of agents. Can people learn enough to make useful decisions by observing the choices and experiences of others? If there are multiple equilibria in a strategic model, perhaps reflecting low and high levels of financial economic activity, can the process by which agents approach equilibrium be used to predict the outcome? In this section, we present a summary of the existing literature on learning and memory, which guide us in designing our research question.

Learning is central to agents' identity and their ability to grow in any environment. In addition, in a financial market setting, agents who demonstrate the quicker ability to learn tend to earn better financial rewards. The speed of learning is often seen to be proportional to the inverse of time loss to correct or innovate a strategy. Learning and memory are fundamental processes that play a crucial role in various aspects of human life, including financial decision-making. In recent years, there has been growing interest in developing novel approaches to learning and memory in finance, with the aim of improving financial literacy and decision-making abilities among individuals and organizations.

In this section, we provide a comprehensive overview of existing work on learning and memory in finance, highlighting the current state of knowledge, and identifying areas for future research. Several theories have been proposed to explain how people learn and remember information related to finance. One of the most widely accepted frameworks is the Atkinson-Shiffrin (A-S) model, which posits that information first enters sensory memory before being transferred to short-term memory and eventually long-term memory (Atkinson & Shiffrin, 1968). However, this model has been criticized for oversimplifying the complexity of human memory and neglecting factors such as motivation, attention, and prior knowledge (Tulving & Thomson, 1973; Eichenbaum et al., 1999).

Another influential theory is the Information Processing Theory (IPT), which suggests that learning and memory are functions of the amount and quality of information processed (Sweller, 1988). According to IPT, people process information through a series of stages, including attention, perception, encoding, storage, and retrieval. However, this theory assumes that all information is equally important, failing to account for the fact that people typically focus on specific aspects of financial information while ignoring others (Hoffrage, 2015).

A more nuanced perspective on learning and memory comes from the Levels of Processing Framework (LOPF), which proposes that information can be processed at multiple levels, including shallow or surface level, deep or semantic level, and finally, the level of relational or contextual connections (Craik & Lockhart, 1972). Although LOPF acknowledges the importance of deeper processing, it does not fully address the issue of motivation and goals in learning and memory (Bjork & Bjork, 1992). Recent Advances in Neuroscience and Financial Decision Making. Wang et al. (2018) explore the application of deep learning techniques to financial markets prediction. The authors begin by discussing the challenges associated with forecasting financial markets, such as non-stationarity, volatility, and high dimensionality, and argue that deep learning models are well positioned to address these challenges due to their ability to learn complex patterns and relationships in large datasets. They then provide a comprehensive overview of various deep learning architectures that have been applied to financial market prediction, including feedforward neural networks, recurrent neural networks, and convolutional neural networks.

In another stream of work, Khalil and Pipa (2021) use news analytics to assess the potential of deep learning and natural language processing (NLP) in financial forecasting. They examine the strengths and limitations of these techniques and discuss how they can be used to improve financial prediction. The authors provide a detailed analysis of various studies that have applied deep learning and NLP to financial forecasting tasks such as stock price prediction, sentiment analysis, and text classification. In addition, they explore the challenges associated with implementing these methods in real-world scenarios and highlight future research directions in this field. Overall, Khalil and Pipa's work offers valuable insights into the application of advanced computational methods in finance and contributes significantly to our understanding of their capabilities and limitations.

Various other related research covers a wide range of models including linear regression, decision trees, random forest, support vector machines, neural networks, convolutional neural networks, long-short-term memory (LSTM) networks, gated recurrent units (GRU) and attention-based LSTM. The authors noted that while these models have shown promising results in predicting stock prices, there are still challenges such as dealing with high-dimensional data, handling non-linear relationships, and addressing the problem of

overfitting. In general, the paper provided valuable insights into the use of machine learning and deep learning models for stock market price forecasting. It served as a useful resource for researchers and practitioners interested in this area.

Hsua and Lessmanna (2022) present a comprehensive overview on the current state of financial market forecasting, comparing the strengths and weaknesses of machine learning and financial economic approaches. The authors analyze the advantages and disadvantages of various machine learning models, such as linear regression, decision trees, and artificial neural networks, in terms of their precision, interpretability, and ability to handle complex financial data. They also examine the limitations of traditional financial economic models, including their reliance on simplified assumptions and the difficulty in capturing market dynamics. Furthermore, the authors discuss recent advances in hybrid models that combine elements from both machine learning and financial economics, highlighting their potential to improve forecast accuracy and provide more robust results. Through their literature review, Hsua and Lessmann demonstrate the importance of bridging the gap between these two fields to develop more effective and practical financial market forecasting tools.

Many authors exploit deep learning models where their use can help predict future price patterns. The researchers show that their method outperforms traditional statistical models in some cases, demonstrating the potential of deep learning techniques for financial forecasting. In general, the substantive literature contributes to the growing body of research applying deep learning to financial markets and suggests that universal features of price formation may exist across different assets and time scales. However, the authors acknowledge that further study is needed to fully understand the origins and implications of these characteristics and to improve the accuracy and robustness of deep learning models for financial forecasting.

In a recent work, Anh, Inoue and Kasahara (2022) present a novel approach to modeling financial markets that takes into account the idea of innovation processes and expected utility maximization. The authors propose a framework that incorporates memory effects and allows the consideration of multiple factors that influence investment decisions. Their model captures the notion of "innovation" as a way to describe how new information is incorporated into market prices, leading to a more realistic representation of market behavior. Using this approach, the authors are able to demonstrate how their model can reproduce fat-tailed distributions and volatility clustering, two phenomena that are observed in real-world financial markets but difficult to capture using traditional models (see Bekiros et al. 2021 for clustering implications of correlated shocks at the fat-tailed, leading to different learning dynamics). Additionally, they show how their model can be used to analyze the impact of policy interventions and changes in market structure on market behavior. Overall, the paper represents a significant contribution to the field of financial econophysics, offering a fresh perspective on the complex nature of financial markets and their behavior.

An agent-based mechanism in memory generation has been presented by Zheng et al. (2022), where the authors provide a comprehensive examination of long memory in financial markets through the lens of a heterogeneous agent model. The authors integrate concepts from cognitive psychology, behavioral finance, and agent-based modeling to create a rich and nuanced portrait of market behavior. They convincingly demonstrate that the presence of long memory in investors' decision-making processes can lead to the emergence of complex patterns in market activity, including volatility clustering and fat-tailed returns. The model developed by the authors captures the essential characteristics of financial markets, including the coexistence of rational and irrational behaviors and the role of social influence in shaping market opinion. Furthermore, they show that their model can replicate many stylized facts of financial markets, such as the anomalous behavior of stock prices and trading volume. This paper makes a valuable contribution to our understanding of financial markets and offers deep insights into the development of more accurate and realistic models for market simulation and prediction.

3 Model

In this section, we present and characterize a distance-based and learning driven long-memory model for an asset in a financial market. Our strategy requires robust identification of potential sources of long-memory based on which a learning rule can be designed. Several influential works preside over our choice. Firstly, we follow the running and the most popular strand of literature that recognizes the role of heterogeneity and aggregation (see, e.g., Granger, 1980; Abadir and Talmain, 2002; Zaffaroni, 2004; and Schennach, 2018). Secondly, we are driven by recent contributions in the field focusing on the assumption of a representative agent framework with constant parameters (Chevillon and Mavroeidis, 2017). The latter assumption avoids confounding inferences with various models of long-range dependence (discussed, for instance, in Granger, 1980, and similar scores of research following him). To fully exploit the distinguishing features of our model, we present below required preliminaries.

3.1 Long-memory and Learning: Some preliminaries

We begin with a conventional expression of an integrated process of order d for a time series, y_t (for $t = 1, \dots, T$).

$$(1 - L)^d y_t = \psi(L) \varepsilon_t \tag{1}$$

where $(1 - L)^d$ is the (fractional) difference operator of order d with d is fractional and lies between 0 and

1. In an extreme case where $d = 1$, $(1 - L)^1 y_t = y_t - y_{t-1} = \Delta y_t$. It is a first order difference equation, the root of which becomes explosive as time grows, indicating the instability and irreversibility of the system to the steady-state value. $\psi(L^j)$ is the coefficient of the error term (ε) at each specific time period $t - j$ with $\sum_{j=0}^{\infty} |\psi(L^j)| < \infty$, $j = 0, 1, 2, \dots$, and the error term (ε_t) is a white noise process with zero mean and constant variance, viz. $\varepsilon_t \sim iid(0, \sigma^2)$. As explained in Hamilton (1994), if $d > 1/2$, y_t will no longer be stationary as the inverse of $(1 - L)^d$ approaches infinity.

$$y_t = (1 - L)^{-d} \psi(L) \varepsilon_t \quad (2)$$

Based on the power series expansion technique, the operator $(1 - L)^{-d}$ can be demonstrated as

$$(1 - L)^{-d} = \sum_{j=0}^{\infty} \gamma_j L^j \quad (3)$$

where $\gamma_0 \equiv 1$ and

$$\gamma_j = \frac{(d + j - 1)(d + j - 2) \cdots (d + 2)(d + 1)(d)}{j!} \quad (4)$$

where $\gamma_j \cong (j + 1)^{d-1}$ given that $d < 1$ and j is large.

In summary, the series y_t can be a mean-reverting process when the superscript $d - 1$ in $\gamma_j \cong (j + 1)^{d-1}$ is less than 0, that is, $d < 1$. This indicates that the impacts of the shocks from previous periods on y_t will gradually decrease over time. The conventional mechanics of shock propagation, as described above and summarized below in Table 1, assume that the memory of a dynamic system can be described primordially as a function of long lags. Here, each lag should contain the best set of information at that point of time about both the present and the past values of the system. However, because the available information is imperfect at any point of time, there is always a remnant of the past that flows through the passage of time to the present moment. Such a flow can simply be modeled in a linear setting or to approximate real life dynamics, can be modelled non-linearly.

Insert Table 1 about here

In this setting, learning occurs at each time lag and progressively accumulates until the system stabilises. An ideal condition would be fast-paced learning through successive lags, possibly with just one lag! If all information from the past can be stored in the immediate lag in a financial time series, then this immediate lag is technically the best predictor of the time series. It not only reduces the dimension by improving the degrees of freedom, but also offers a powerful opportunity to project all past values within the immediate lag

of a series. Whilst such an approach has remarkable technical reliability in an atomistic world of ‘no social interaction’. we, however, live in a highly interactive expectation-driven world. The one-lag assumption as an optimum driver of the current value of a time series, then appears at odds with highly probable dynamic interaction with other time series within a system.

It is not always possible to generalise that all time series could generate long memory with just one lag, as the governing features of those series may accrue to a different data generating process. Therefore, the high-dimensional interaction that takes place in a financial market can reveal that such a step-by-step learning and adjustment process faces inescapable impacts of greater speed of arrival of both endogeneous and exogeneous shocks. They make learning very complex. Thus, the long-memory process that germinates from such a learning dynamics can be hard to characterize with long-lag dependence structure because the relationship between each remote lag with the succeeding one, and the latter with the current value of the series can be highly non-linear and very likely stochastic. The model of one-lag theory of Bauwens et al. (2023) is a clever approach in this regard, but its efficacy as a reliable and powerful predictor of the current value of the series is problematic given complex ‘social interactions of various lags of the series with other series in the growing system’.

What we have presented so far is the context of memory defined from an individual agent’s perspective. But because we live in the dimension of a system, then, what one would be seeking is the transformation of individual memory into collective memory. The latter leads to an emergent and a recursive system defining a new narrative of learning-based long-memory characterisation. What matters here is the pace at which negative shocks taper-off over time. Denoting the memory of the individual or the system as S , and by allowing S to be a function of some known factors (X) and some unknown factors (ϵ) at time, t , then temporal change in ‘memory’ - keeping other things constant - would converge in probability, to (\bar{S}) .

$$S_t = f(X_t, \epsilon_t) \tag{5}$$

$$\lim_{t \rightarrow T} \frac{d(S_t)}{dt} \rightarrow \bar{S} \tag{6}$$

This implies that depending on the convergence speed and length of time (very short or long), the system settles to a stable system. This notion of asymptotic convergence to stability of the system encapsulates the role of a type of learning that is inherently drawn from a weakly inefficient system. In case, one assembles - as in a typical financial market - both private and public information and exploit the historical nature of the data reasonably well, the system gives rise to a strong efficiency. The learning in this context is not supposed to be driven solely by the asymptotic character. Rather, as recent authors (we will discuss about them shortly)

would argue a rather short-span dependent autoregressive phenomenon to characterise a long-memory. Note that the convergence speed of shock duration can be asymmetric (greater speed of convergence before the intervention of a shock and smaller speed in the aftermath of intervention) as well as tightly interlinked with spillover learning dynamics that arise within an integrated system, such as the highly immersive financial world. The question one pose is whether persistence profiles depict ‘smooth’ trajectory reflecting continuous improvement through various lags.

In real life financial markets or economic conditions we often come across situations where (positive and negative) shocks can exhibit *duration-specific* correlation, one which can demonstrate a mean-reverting but long-memory process. In other words, we have ignored so far the fact that there can a duration within which the errors can survive up to a point of time with certain probability. As noted earlier, we regularly come across circumstances in financial worlds where errors do not display a continuous persistence without being disrupted somewhere with certain probability of duration. Indeed, *duration* enables persistence of errors or shocks with the power of asymmetry and the latter, we know from theory is a major source of non-linearity. Whilst a linear dependence of errors over time is a theoretically plausible and computationally less demanding feature, asymmetry and non-linearity are more reality driven. Agents in any (complex interactive) system demonstrate a natural inclination to learning, but this learning is not continuous due to the way errors survive certain period of being persistent and being non-persistent in other duration. An *error-duration dependent learning process* is closer to real life financial dynamics where the memory that originates from such mechanisms of learning presents a robust mechanism to reflect on agents’ complex behaviour in a far more complex system. Accordingly, we present below a duration model of the error drawing on Parke (1999). We then extend this framework by accommodating a ‘distance’ metric among interacting variables in a ‘social or economic’ space.

3.2 Duration Model of Error and Long-memory

Parke (1999) introduced an error-duration dependent long-memory process. We will exploit the rich properties of this model to build our proposed construct. To avoid notational ambiguities, denote as before ϵ_t ($t = 1, 2, \dots, T$) as a series of *i.i.d.* shocks with mean zero (no bias) and finite variance σ^2 (i.e., a stable system). Let us assume that the error ϵ_t has a *stochastic duration* $d_s \geq 0$, indicating that *it has survived* from period s to period $s + d_s$. Survivability here is proximus to the broad idea of persistence; a shock that survives for a certain period can be alternatively explained by a shock that is persistent for the duration of the period of survival. The duration-driven persistence is both technically and intuitively different from the conventional persistence properties of a time series; whilst the former can estimated varied persistence patterns specific to

durations or survival of shocks, the latter assumes a smoothing and linearisation of shocks over the course of time. Duration models are popular in economics and finance, especially when modeling the effects of some policy intervention, for instance, in the financial markets.

Denote $I_{s,t}$ as an indicator function with respect to a situation where error ϵ_s survives to period t . In other words, if the shock survives the entire duration $s + d_s$, we describe this by $I_{s,t} = 1$ for $t = s + d_s$. Likewise, if $t > s + d_s$, that is, time t is greater than the duration of survival ($s + d_s$), then we denote this by the indicator function $I_{s,t} = 0$. We can assign probabilities to survival of shocks. Denote p_k as the probability that ϵ_s survives to the period $s + k$: that is, $P_k = P(I_{s,s+k} = 1)$. If $P_k, k = 0, 1, 2, \dots$ is monotone non-increasing, then following Parke (1999), we can show that the realization of values in y_t (such as stock returns or stock prices at time t) is the sum of all errors $\epsilon_{t-i}, i = 0, 1, 2, \dots$ that survive until period t :

$$y_t = \sum_{i=-\infty}^t I_{s,t} \epsilon_s \quad (7)$$

Here, survival probabilities specified by (p_0, p_1, p_2, \dots) are for a specific duration and are crucial parameters of the concept of error-duration (ED) representation of y_t . For this process, the autocovariance function (γ_k) can establish a link between the model that describes the duration of the error and other representations. Parke (1999) showed that $\gamma_k = \sigma^2 \sum_{j=k}^{\infty} p_j$ and $p_k = (\gamma_k - \gamma_{k+1}) / \sigma^2$. The most important mechanic in Parke's idea of error-duration driven long memory feature is how certain shocks survive until a specific period. This makes no assumption that the survivability has to be at the full duration of the sample or the entire time path of the history of the series. Shocks can 'survive' only for a sub-period of time within the observed longer time horizon, but this survival characteristic is long-enough to generate a persistence-type feature.¹

Exploiting these results and following Parke (1999), we can now establish that the process y_t has a *long memory* if $\lim_{n \rightarrow \infty} \sum_{k=1}^n k p_k$ or $\lim_{n \rightarrow \infty} \sum_{k=1}^n k (\gamma_k - \gamma_{k+1}) / \sigma^2$ is non-finite. In the following, we describe a process that has an implicit character of error duration but with a short time lag. Basically, the idea, as we will see shortly, concerns how a short-duration lag (particularly, lag 1) can generate a long-memory feature. However, before reconciling the models of Parke (1999) (error-duration) with Chevillon and Mavroeidis (2010) (on the role of learning in long-memory), and Bauwens et al. (2023) (the importance of one time lag in generating memory), we need to understand the general mechanic of how learning can generate memory. We briefly outline the model of Chevillon and Mavroeidis (2010) below and introduce an extension later.

¹For this specification, the autocovariances of $(1 - L)y_t$ can be represented in the form of $\sigma^2(p_k - p_{k-1})$ as the probability of survival between two time periods drive the growth of variance.

3.3 Learning and long memory (persistence)

In a simple *expectation* driven framework, a typical learning model can be presented by expressing a linear correspondence between an expected value of the time series, y_t at time $t+1$ (denoted by y_{t+1}^e and its current value (y_t).

$$y_t = \gamma y_{t+1}^e + x_t \tag{8}$$

Here y_{t+1}^e represents the expectation of y_{t+1} conditional on information up to time t . Furthermore, γ (which should be typically > 0 and < 1) can describe the learning parameter: a positive effect of learning occurs when $\gamma > 0$ and the learning deliver a stable outcome when $\gamma < 1$. As γ is close to 1, the system takes longer time to stabilise in contrast to when γ is close to 0. Importantly, note that when γ is close to 0, the speed of learning is faster and transformation of the system from instability to stability happens in the quickest time. However, when γ is close to 1 (typically, greater than 0.5), the speed of error correction in the system is slow and the system takes longer time to stabilise. Within this long duration, the arrival of some other exogenous/endogenous shocks are quite likely and if this happens, the 'survival' of shocks or error term to the next period can mask a non-converging pattern. In other words, shocks may appear to last forever - a feature that is often unreliably accepted/rejected with the conventional test of a unit root in a time series. Evans and Honkapohja (2001) and latter, many authors have extensively studied this model.

Equation 8 can be generalised to an infinite-horizon expectation framework (thus introducing an autoregressive feature) by assuming that $y_t = f(x_t, x_{t+1}^e)$:

$$y_t = x_t + \sum_{j=1}^{\infty} \gamma_j x_{t+1}^e \tag{9}$$

If agents in a financial market are rational, then we can set $y_{t+1}^e = E_t(y_{t+1})$ and $x_{t+j}^e = E_t(x_{t+1})$ assuming rational expectations (where, E_t captures expectations). In other words, agents' expectations about the future should be technically equivalent to the average of future values of the variable in question. Further, with the condition that $|\gamma| < 1$ and there are no bubbles ($\lim_{T \rightarrow \infty} |E_t(y_T)| < \infty$ for all $t < \infty$), the specifications (8) and (9) are equivalent (see, e.g., Blanchard and Fischer, 1989). The same is not true under adaptive learning mechanism (see Preston, 2005, 2006). Although equations 8 and 9 represent linear learning rules, in practice, one may use non-linear mechanics (such as a quadratic specification) to ensure stability in the effects of learning. In that case, how expected value of financial gains from an asset investment, for instance, would determine the current value can be reasonably mapped from a linear to a series of admissible

values in a non-linear space.

In a setting describing adaptive learning (Sargent, 1993; Evans and Honkapohja, 2001), the limitations that financial agents would face are the same as empirical economists because the latter are known to assume models, but they are not aware of their true parameter values. Hence, empirical economists estimate the parameters econometrically, throwing thus an inherent challenge of reconciling true learning rules with the empirically compliant one. In particular, agents would like to build expectations driven by a law of motion of the processes y_t or x_t . The parameters of these processes are estimated recursively given the information that is accessible to them. In this setting, the agents' forecasts can be expressed as weighted averages of past data (a typical feature of a conventional learning process). However, the weights may vary over time to accurately reflect the information along with the increase in the sample. This is an important feature of learning. Researchers often use linear learning algorithms, so that they can envisage that a long range dependence can arise without the need for nonlinearities (on which recent econometric literature on the source of long-memory has advanced) (see Diebold and Inoue, 2001; Davidson and Sibbertsen, 2005; Miller and Park, 2010, and the surveys by Granger and Ding, 1996; Davidson and Teräsvirta, 2002).

In the case of linear learning mechanic, one can conveniently express this by mean plus a random noise model. The 'mean' represents average learning - a feature of group characteristics. Basically, it means that our typical learning is a group phenomenon as long as we live in a society full of competitive and boundedly rational agents (Sargent, 1993). The noise has a random feature so that there is no bias and the unit variance dictates stability of learning. Another assumption concerns that random errors are not correlated over time so that the serial correlation of unknowns does not impact the learning feature (an average). We thus represent the idea in the form of the assumption (below).

The process x_t is determined by $x_t = \mu + \epsilon_t$, where μ is a constant and ϵ_t is i.i.d., with $E(\epsilon_t) = 0$ and $E(\epsilon_t^2) < \infty$. Such an assumption ensures that $|\gamma| < 1$ in equations 8 and 9 are stable. To envisage a long-lag feature of learning, we can introduce a recursive system: $y_{t+1}^e = \sum_{i=1}^t y_i$ so that the expectation of y_t at $t+1$ is the sum of all y over the time horizon t . What happens if there is a mean-shift in the learning model. The mean can shift in situations whenever there are, for instance, structural re-positioning of the system following significant 'innovation' (for instance, policy). This mechanism allows us to model the mean in a time-varying structure, specifically an autoregressive (AR) process of μ_t . Limiting the process to AR(1) (as we know from theory that an AR(1) series can give rise an an MA(∞) process): $\mu_t = \mu_{t-1} + \zeta_t, t \geq 1$. Here $\alpha_0 = \alpha$ and ζ_t are i.i.d. with zero mean and finite variance.

The signal-to-noise ratio of the above process is given by $\Gamma_t = var(v_t)/var(\epsilon_t)$, which indicates the proportion of signals weighed by noise. Eventually, this proportion can drive the agent's learning speed. It

is easy to see that y_{t+1}^e is a function of the current and past values of y_t and μ_t . In summary, the learning algorithm can be represented by linear functions of the past values of y_t . This can accommodate time-varying coefficients to capture the dynamic learning and adjustment process.

$$y_{t+1}^e = \sum_{j=0}^{t-1} \kappa_{t,j} y_{t-j} + \psi_t \quad (10)$$

The faster an agent discount effects from past observations, the greater becomes the speed of adjustment the efficacy of learning. We can estimate this using the mean delay of κ_t , denoted as $m(\kappa_t) = \frac{1}{\kappa_t(1)} \sum_{j=1}^{t-1} j \kappa_{t,j}$. An econometrician can use the magnitude of $m(\kappa_t)$ relative to the sample size as an indicator of the ‘*length of the learning window*’. Chevillon and Mavoredis (2018) show that this drives the memory of the process that is induced by learning dynamics.

To summarise, if an agent starts with a sensible model, if the environment is stationary, if it is costless to obtain and process information, then eventually the agent learns enough about the environment to make optimal decisions. Following our representation of learning algorithms in static and dynamic settings, our next task is to show when an agent learns, how quickly she converges to a ‘group’ outcome so that the ‘conformist’ representation helps her position own financial status relative to the competitors - which is what investors in a financial market do.² By introducing such a construct we aim to solve two problems that are not discussed in the extant literature. First, we show that how an agent discounts past information in order to build her current model depends on the relative ‘positioning’ of the agent in the ‘social space.’ Eventually, the magnitude of learning can be driven by the error-duration dependence in such a way that an agent’s relative social positioning within that period can allow her to either drift away or converge to a social optimum. The memory that lingers from such a strategy would ideally drive the speed at which past observations should exert dissipating effect on the current value of the system. We describe such a mechanism in the following section.

3.4 The instrumentality of ‘space’ in learning and memory

Agents’ choices are sparsely atomistic. Rather, the relative nature of choices, irrespective of whether observing another agent at the same point of time or comparing her choices with herself and others at different points of time, makes learning a complex phenomenon. One can still apply a linear learning rule to individual choices (e.g., whilst forecasting a pattern of stock price movements based on her own social/financial position as well as the same position of others within a system). In other words, whether she really wants it or not, she is

²See Gupta, Mallick and Mishra (2018) for a related discussion in the context of household consumption.

naturally exposed to the tangency of relativism.³ Following on our representation of learning and memory in the preceding section, we now expand the model behavior by introducing a status and a conformist model to understand learning behaviour of agents in the financial markets.

To formalize the above idea, let us assume that there are two time periods for an individual's choice: influenced by 'intrinsic' value of location and expected benefits from social exchange, i.e. the benefits gained due to proximity with one's neighbors or observations at two different points of time (0 and 1). These two components are captured in a utility function U , for each agent, which is maximized to determine the direction of their movement along the real line. We build two models (status and conformist) drawing on the seminal ideas of Akerlof (1997).

3.4.1 Status Model

In this model an agent chooses the status-producing variable x (such as a financial asset, a crypto or a green asset as an example) to maximize the indirect utility function

$$U = -\alpha(\bar{x} - x) - ax^2 + bx + c \quad (11)$$

As long as the agent falls behind everyone else in her choice of x , the agent loses utility in amount $\alpha(\bar{x} - x)$, where \bar{x} is the choice of everyone else. For instance, choosing a green asset is like a status seeking investment. If everyone is investing in this asset (\bar{x}) but the agent concerned chooses to refrain from investing in this asset, then this creates a status problem. Not only that, she will be likely to be neglected in social space (because green asset investment conforms sustainability objective of an economy). It is not therefore, difficult to assume that x has an intrinsic value to her of $-ax^2 + bx + c$ because this value may decline over time in case the expected social return is less over time (the quadratic component). The last term C is a constant that drives the innate value of the asset. Therefore, in equilibrium, the agent chooses

$$x = (b + d)/2a \quad (12)$$

3.4.2 Conformist Model

The agent cannot ever remain alienated from the group. There will be a point of time, she would like to 'conform' to the group's identify. If everyone around is investing in green assets, the agent herself - upon delaying the idea of the investment for sometime - can eventually give up and converge to everyone's idea.

³See Tamvada, Shrivastava and Mishra (2021) for a discussion with respect to a developing country.

Whether the agent really gains from 'conforming' to everyone's decision is an issue (which is beyond the scope of this paper), however, it satisfies her to assume that if everybody gains, she gains. If everybody loses, she loses too. In other words, being in the 'herd' and identifying herself in that group, the agent may not feel like a sole 'loser' in the competitive social race. In a crypto investment scenario, for instance, the volatility of cryptos can be attributed primarily to 'herd-mentality' driven investing aptitude (to either take risks like others, where the latter may be irrationally driven). The above examples help us build a conformist model, the agent would like to minimize the social distance between herself and others. In this case she does not seek to be better than other people, but instead wants to be as much like them as possible. The utility function is

$$U = -\alpha|x - \bar{x}| - ax^2 + bx + c \quad (13)$$

In the above, the agent can experience a loss of utility of magnitude $\alpha|x - \bar{x}|$ because he does not comply with others (such as our example of buying green assets). Because everyone is similar in equilibrium, $x = \bar{x}$. This setting produces multiple equilibria as long as the distance $\alpha > 0$. The distinct clusters of agents who have various values of α represent a dynamic learning environment where they would move quickly - in the social space - towards the point where everyone has achieved a social optimum (i.e., everyone's choice). The problem facing each individual i is how to choose x_{1i} conditional on her initial social position, x_{0i} . If we compare this with Parke's (1999) error duration framework, we can envisage the distance that shocks survive from an initial position to a terminal position, a length that determines the extent of memory or persistence.

However, the agent has to make expectations about the position of her potential trading partners in social/financial exchange. One can expect multiple outcomes depending on how these expectations are formed. The simplest assumption concerns how the acquired social position 'of all the other individuals will coincide with their initial position' (Akerlof, 1997). With such static expectations about social position, $\alpha_{1,ij}^e$, i 's expected acquired distance between herself and j will be $x_{1i} - x_{0j}$. In sum, each respective agent i chooses the respective value of x_{1i} to maximize:

$$U_i(x) = \sum_{j \neq i} \frac{\psi}{(f_1 + |x_{0,i} - x_{0,j}|)(f_2 + |x_{1,i} - x_{0,j}|)} + [-ax_{1,i}^2 + bx_{1,i} + z + w_i] \quad (14)$$

- ψ : Constant of proportionality; a , b are arbitrary constants.
- $f_1 + (x_{0,i} - x_{0,j})$ - *Inherited social position*: For instance. Expected benefit from social interactions and is the key term in emphasising the motive for movement towards a particular neighbour - as the distance between individuals i and j decreases the benefits increase significantly.

- $f_2 + (x_{1,i} - x_{0,j})$ - *Attenuator of social distance*: For instance, 'learning status'. These are exogenous contextual factors
- $-ax_{1,i}^2 + bx_{1,i} + z + w_i$ - *Intrinsic value*: here z represents exogenous factors, w_i represents factors, such as demographic variables (age, gender, social and economic classes, etc. The quadratic component in the function captures the fact that as the intrinsic value rises, the utility will increase at a decreasing rate.

3.5 A Learning-driven memory model with social positioning of agents

So far, we have presented fundamental characteristics of learning rules leading to long-memory, and duration-dependent models that can drive long-memory persistence. Furthermore, we have also presented a social distance model of choice of risk positions of financial investments triggering a persistence type of behaviour depending on the financial agent's relative social positioning. In this section, we combine the elements of the distance approach (status and conformist) into the learning based and duration driven long-memory characterisation of of a financial asset. The starting point of our formalization is to allow survival of shocks with specific durations (realistically observed). Assuming two time points (t_0 and t_1) where shocks survive and then do not persist at all, one can generate learning rules separately for the duration of errors. In fact, the relative social position of an agent and the loss function as well as the intrinsic values (see Equations 14 and 15, respectively) can differ significantly depending on the duration of shocks' survival (eventually being driven by the survival probability, p). In equation 8 we have presented the realization of y_t with respect to durations. Embedding it into Equation 9, we now obtain:

$$y_t = \beta \sum_{s=-\infty}^t [I_{s,t+1} \epsilon_s]^e + x_t \quad (15)$$

and

$$y_{t+1}^e = \sum_{j=0}^{t-1} \kappa_{t,j} [I_{s,t-1}] + \psi_t \quad (16)$$

Using representations in equations 15 and 16 and using some algebraic manipulations, we arrive at the following:

$$U(y_{t+1}^e) = \frac{\psi}{[f_1 + (I(s,0)_i - I(s,0)_j)] [f_2 + (I(s,1)_i - I(s,0)_j)]} + [-ax_{1,i}^2 + bx_{1,i} + z + w_i] \quad (17)$$

The expected utility gain for a financial agent at $t + 1$ ($U(y_{t+1}^e)$) is modelled here as a function of first,

the agent's inherited social position in the sense that as the agent observes other agents investing in a risky asset, he learns by thinking that the other agent might have a reliable information for investment. The agent then keeps investing in such an asset until the point he realises that the investment is no more profitable and move to a new social positioning. In other words, the investor's actions survive the duration of the time he believes in other agent's decisions. Hence, one may expect a certain degree of persistence in investment decisions (a memory process) - a feature reflective of duration model of Parke (1999). Furthermore, the investor compares his social position of today (that is, his total utility from a prior investment) with another investor's position yesterday. This status-seeking social positioning also leads to a persistence like feature, one that can be identified as a long-memory process. Thus, in our proposed formalisation, we identify the source of a long-memory by social positioning of the investor in asset-value optimisation between time periods and between other investors. We are now able to present the following proposition.

Proposition: *If the distance (d) decreases between i and j both at time t and between time t and $t - 1$, then greater is the duration (i.e., with stronger asymmetry) of errors (shocks), it is essential for i to learn faster to minimise a negative effect of memory.*

Proof: It is straightforward to see from equation 17 that as the distance between two agents (i, j) declines (the second function in the denominator), the agent learns quickly to arrive at a social optimum. Eventually, the last term (the quadratic component) helps speed up the declining weight of the distance - or error spacing or duration between two time periods, leading to smaller magnitude of memory. In other words, the greater is the distance (the second term in the numerator ($f_1 + (I(s, 1)_i - I(s, 0)_j$) as well as the first term ($f_2 + (I(s, 0)_i - I(s, 0)_j$), the greater will be the magnitude of memory. This can drive a further alienation from the desired equilibrium.

4 Numerical example

We characterise equation 17 by exploiting the basic learning rule as in equation 8. Initial parameterisations of equation 8 along with equation 11 (the status model) help us produce Table 2 where we have undertaken Monte Carlo simulation for over 10,000 replications. In the Table 2, α is the indicator of distance (viz., the distance between two successive values in the duration function or the distance between two agents' estimates at two different points in time). We have used three cases: $\alpha = 0.1$ and $\alpha = 0.9$ are, respectively, lowest and highest distance in the sense that whenever, for instance, $\alpha = 0.9$ there is a long gap between

observations in the duration function or the expected returns from an asset between two time periods is the highest between two agents. Under this ‘distance’ numeric we generate different estimates of memory (or d) using both Robinson (1995) and local Whittle estimator of Shimotsu and Phillips (2005). The empirical rejection frequencies of one-sided 5% level tests of our null hypotheses are tested against the alternatives: $H_0: d = 0$ against $H_1: d > 1$ and $H_0: d = 1$ against $H_1: d < 1$.

Insert Table 2 about here

In Table 2, we have recorded the mean estimates of d . Technically, we expect a greater strength of memory as the distance increases (α becomes large). Basically, we find that $E(\hat{d})$ increases in α . Because T is fixed, a higher α corresponds to a shorter learning window, so much so that the actual memory of the process becomes a decreasing function of the length of the learning window (a result similar to Chevillon and Mavroeidis, 2017).

5 Discussion and Conclusions

This paper exploits this advancement in modeling and proposes a novel short-lag linear autoregressive model in a dynamic learning environment influenced by an investor’s social positioning. Our proposed mechanic is supposedly holistic in nature, aimed at delivering greater predictive power than traditional models used for financial data. In fact, our main identifying mechanism is a process of learning with fast-paced error correction, where an immediate lag and its projections in social positioning setting can help capture longer period of temporal dependence in financial data. With real-life examples in the financial market, we compare the efficacy of our estimates with the conventional model and show that our mechanism can be computationally less burdensome and have better predictive power. As we reflect on the memorable quote by Irish Blessing, financial markets - like any other dynamic system - should remember positive shocks (so that the propagation of relatively less risky return becomes more realistic). At the same time, these markets should forget (or quickly smooth out) the effects of bad shocks so that the system becomes asymptotically stable. The extension to test for correct specification of a DSDLM model and its predictive power, in a setting of less liquid markets, such as commodities, remains an avenue for future research.

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Table 1: Memory properties of y_t with different d values

d Value	Memory	Stationarity	Mean/Variance	Learning	Shock Duration
$d < 0$	Long	Stationary	Mean-reversion/Finite	Fast-paced	Long-lived
$d = 0$	Short	Stationary	Mean-reversion/Finite	Fast-paced	Short-lived
$0 < d < 0.5$	Long	Stationary	Mean-reversion/Finite	Fast-paced	Long-lived
$0.5 \leq d < 1$	Long	Non-stationary	Mean-reversion/Infinite	Extremely slow	Long-lived
$d = 1$	Permanent	Non-stationary, unit root process	No Mean-Reversion/Infinite	No learning	Permanent
$d > 1$	Permanent	Non-stationary	No Mean-Reversion/Infinite	No learning	Permanent, the effects increase over time

Table 2: **Simulated estimates**

α	τ	Average of \hat{d}		Pr(Reject $d = 0$)		Pr(Reject $d = 1$)	
		Robinson	Local Whittle	Robinson	Local Whittle	Robinson	Local Whittle
0.1	0.1	0.005	-0.015	0.129	0.105	0.801	0.933
	0.5	0.056	0.044	0.178	0.199	0.723	0.897
	0.8	0.223	0.208	0.401	0.451	0.587	0.756
	0.9	0.326	0.297	0.499	0.598	0.501	0.674
	0.99	0.467	0.388	0.702	0.774	0.433	0.621
0.5	0.1	0.017	-0.019	0.131	0.13	0.834	0.938
	0.5	0.115	0.071	0.222	0.238	0.747	0.879
	0.8	0.321	0.252	0.498	0.534	0.584	0.747
	0.9	0.419	0.358	0.654	0.649	0.526	0.687
	0.99	0.566	0.477	0.789	0.786	0.456	0.655
0.9	0.1	0.033	-0.002	0.145	0.152	0.822	0.946
	0.5	0.1198	0.132	0.333	0.372	0.712	0.85
	0.8	0.472	0.451	0.684	0.741	0.549	0.778
	0.9	0.649	0.587	0.847	0.862	0.555	0.689
	0.99	0.794	0.739	0.957	0.941	0.52	0.67