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« A thunderbolt in the hammer-nail game: when hammering too hard destroys the support »

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A thunderbolt in the hammer-nail game: when hammering too hard destroys the support

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September 2024

Hammer-Nail game:

Part 1: Show your strength in the hammer-nail game: a Nim game with incomplete information Working paper Beta 2023 n°5, January 2023

Part 2: A sad lesson from the hammer-nail game: strength is better than dexterity, Working paper Beta 2023 n°31, September 2023

Part 3: A thunderbolt in the hammer-nail game: when hammering too hard destroys the support

Abstract

This paper completes two previous papers on the hammer-nail game. The hammer-nail game goes as follows: two players are in front of a nail slightly driven into a wooden support. Both have a hammer and in turn hit the nail. The winner is the first player able to fully drive the nail into the support. A player is of strength f if he is able, with one swing of the hammer, to drive the nail at most f millimeters into the support. A player is of non dexterity e if he is unable to hammer smoothly, so that, with one swing of the hammer, he drives the nail at least e millimeters into the support, with $e \geq 1$. The two players may be of different strength and dexterity. In the two previous papers we studied this Nim-game by assuming that if the remaining distance is lower than e, then lack of dexterity is not a problem because one swing of the hammer necessarily drives the nail into the support. It followed that strength was more useful than dexterity to win the game. In this paper we suppose that a player destroys the support and loses the game if the remaining distance is lower than e. This new assumption completely changes the results: we now observe that dexterity becomes more useful than strength to win this new hammer-nail game.

Keywords: Nim game, crossed cycles, Fort Boyard, subgame perfect Nash equilibrium, strength, dexterity.

JEL Classification: C72

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1.Introduction

In this short paper we show how a change in the end conditions of the Hammer-Nail Game completely changes the strategic role of strength and dexterity in this game.

We first recall the Hammer-Nail Game, a game played in the French TV show “Fort Boyard”[1]. The game goes as follows: two players, player 1 and player 2, are in front of a nail slightly driven into a wooden support (see figure 1). Both have a (same) hammer and in turn hit the nail. At the beginning of the game, the head of the nail is at a distance D from the support.

We work with both strength and dexterity. The strength is measured in numbers of millimeters. We say that a player is of strength f if he is able, with one swing of the hammer, to drive the nail at most f millimeters into the support[2]. This means that a player of strength f is able, with one swing of the hammer, to drive the nail 1 millimeter (mm), 2mm, 3mm, …up to f mm into the support. By (lack of) dexterity we just mean a possible inability to hammer smoothly, i.e. the inability to just drive the nail 1 millimeter into the support. So non dexterity -unskillfulness- is also measured in millimeters: a player of unskillfulness e, with one swing of the hammer, drives the nail at least e millimeters into the support. Hence a player of strength f but unskillfulness e is able, with one swing of the hammer, to drive the nail from e to f millimeters into the support, with $f > e \geq 1$. A player of high dexterity is characterized by $e = 1$ and we say that a player is more skilled when he is of higher dexterity than his opponent. We require that each player, at each turn of play, at least drives the nail e mm into the support, which means that he cannot simulate hammering the nail.

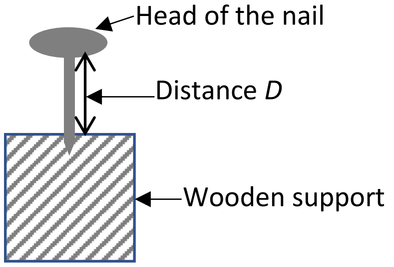

Figure 1: the nail at the beginning of the play

So the game becomes a Nim game (see Bouton, 1901, and Pacioli, 1498-1502) like other TV shows (the Fort Boyard sticks game, Umbhauer 2016, the American TV show Game of 21 Dufwenberg et al. 2010, Gneezy et al., 2010). But it is a special Nim game, given that the strategy set, $\{e, \ldots, f\}$, has a lower bound possibly larger than 1, that both players may have a different strategy set, and that they may ignore the strength or the dexterity of the opponent.

As usual in Nim games, we work with the notion of crossed cycle. A crossed cycle is the number of millimeters that at least one of both players (the player with the largest strategy set) is able to complete regardless of what is played by the opponent at the previous round. When one player has the strategy set $\{e, \ldots, f\}$ and the other has the strategy set $\{E, \ldots, F\}$, the two crossed cycles are of sizes $e + F$ and $E + f$ (see Umbhauer 2023b for more details).

[1] Adventure Live Productions/Banijay Group for France Télévision.
[2]We work with millimeters but we could work with any (smaller) measure of distance.

In two previous papers (Umbhauer 2023*a*, 2023*b*), we logically assumed that when the remaining distance *d* is lower than *e*, then lack of dexterity is not a problem because hammering like a beast for sure fully drives the nail into the support. So we assumed that in front of any distance $d \leq f$ a player always wins the game. This assumption is quite natural. Yet it has far-reaching consequences in that it implies that strength is more useful to win the game than dexterity. To show this fact we now suppose that the support is not a wooden support but a support that is destroyed when the hammer touches it. So, if a player hammers with a strength *e* larger than *d*, the support is destroyed, in which case we assume that the player loses the game. An interesting consequence of this new end condition is that it will not equilibrate the strategic weight of strength and dexterity but it will completely reverse their strategic weight, in that dexterity now becomes more powerful that strength.

We call this new game Fragile-Support Game (FSGame), and keep on calling Hammer-Nail Game (HNGame) the traditional game where a player, regardless of his dexterity, wins the game when the nail is at a distance lower than *f* from the support. In section 2 we show that if both players have the same strategy set, then the new assumption only changes the winning and losing distances but not the number of winning and losing distances in a crossed cycle. Section 3 shows that dexterity becomes more important than strength when both players have the same number of strategies but one player is stronger whereas the other player is more skilled. Section 4 confirms this evolution when both players have a different number of strategies. An example and concluding remarks are given in section 5.

2. Shift in the winning distances when both players have the same strategy set

Right through the paper we call player *A* (she) one of the players and player B the other player (he). We first study the symmetric case where both players have the same strategy set $\{e, \dots, f\}$. In this case, there is only one crossed cycle of size $e + f$.
In the Hammer-Nail Game (HNGame), we get the following Subgame Perfect Nash (SPN) Equilibria:

Proposition 1a. Hammer-Nail Game (out of Umbhauer 2023 b)
Consider two players of strength f and unskillfulness e, with f >e ≥1. Each player, at any SPN Equilibrium, at distance d from the support, with d = k(f+e)+r, k an integer and r an integer from 1 to f+e, wins the game for r from 1 to f, and loses the game for r from f+1 to f+e.

Proposition 1*a* is illustrated in figure 2*a*.

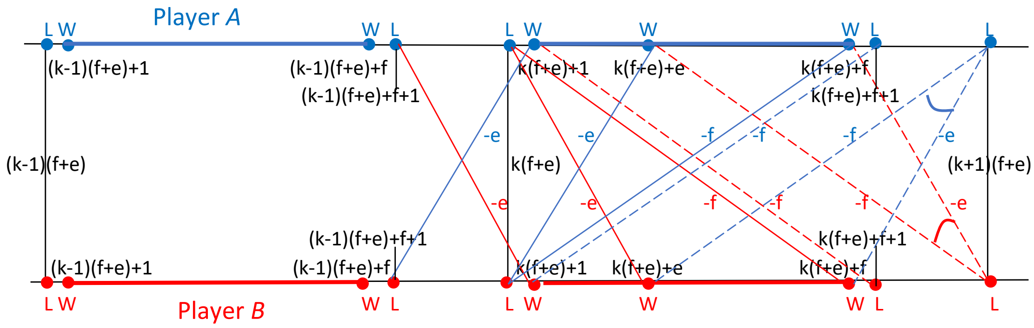

Figure 2a. Hammer-Nail Game, same strength f, same unskillfulness e, f>e>1

Legend of figure 2a: The two horizontal lines represent the distances at which player A and player B are playing. L and W mean that the player loses and wins the game at the corresponding distance. Each rectangle is a crossed cycle (f+e). The horizontal blue -red- segments represent distances where player A -player B- wins the game. The blue -red- lines going from one horizontal line to the other are possible SPN Equilibrium actions, i.e. number of millimeters played by player A -player B- compatible with the SPN Equilibria. The dashed lines are optimal ways of playing but they lead to losing the game.

In the Fragile-Support Game (FSGame), we get proposition 1*b*.

Proposition 1b. Fragile-Support Game
Consider two players of strength f and unskillfulness e, with f >e ≥1. At any SPN Equilibrium, each player, at distance d from the support, with d = k(f+e)+r, k an integer and r an integer from 0 to f+e-1, wins the game for r from e to f+e-1, and loses the game for r from 0 to e-1.

Proposition 1*b* is illustrated in figure 2*b*.

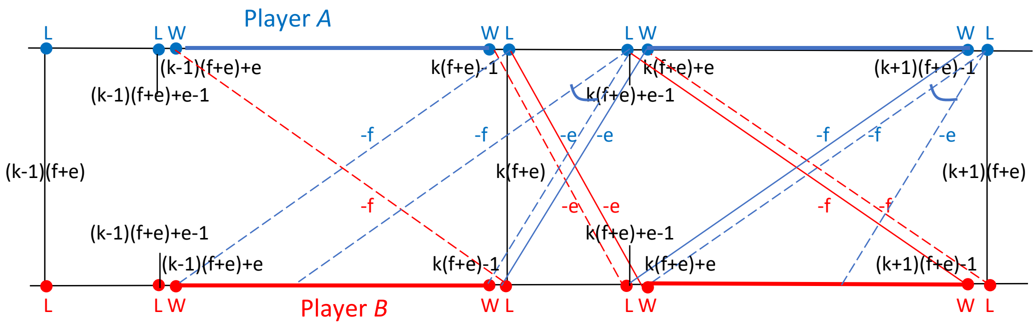

Figure 2b. Fragile-Support Game, same strength f, same unskillfulness e, f>e>1
Legend of figure 2b: similar to the legend of figure 2a

Figures 2*a* and 2*b* immediately show that proposition 1*b* shares with proposition 1*a* the fact that the players have the same number, *f*, of winning distances and the same number, *e*, of losing distances in a crossed cycle, but that the winning distances shift from the left to the right of the cycle when switching from the HNGame to the FSGame.

The main reason of this difference is linked to what happens at the distances from 1 to $f + e$.
In the HNGame (figure 2*c*), a player wins the game for *d* from 1 to *f* (obvious), and loses the game for *d* from $f + 1$ to $f + e$, because, whatever she plays (from *e* to *f*) she brings the opponent to a distance lower than or equal to f where he finishes the game.
In the FSGame (figure 2*d*), when the hammer hits the support, the support is destroyed and the player loses the game. So both players lose the game at the distances from 1 to $e - 1$ because they destroy the support. As a consequence, a player wins the game for *d* from *e* to $f + e - 1$ because, either she can finish the game, or she can bring the opponent to a distance *d* lower than *e* (namely by playing *f*), so that the opponent destroys the support by hammering. At distance $f + e$, a player loses again the game, because, whatever she plays, she brings the opponent to a distance in $[e\ ,f]$ where he wins the game. And this fact is behind the proof of proposition 1*b*.

Proof of proposition 1b
We show by recurrence that each player wins the game at the distances from $(k - 1)(f + e) + e$ to $k(f + e) - 1$ for any integer *k* larger than or equal to 1.

We already know that this fact is true for $k = 1$ So it remains to show that if the fact is true for k, then it is true for $k + 1$.

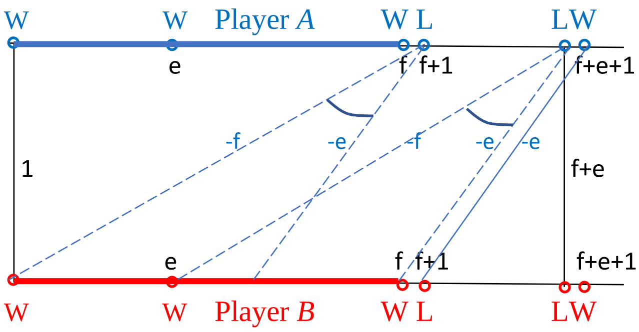

Figure 2c. HNGame, same strength f, same unskillfulness e

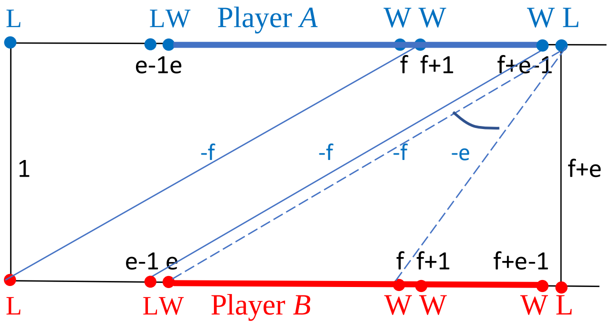

Figure 2d. FSGame, same strength f, same unskillfulness e

We assume that each player wins the game at the distances from $(k-1)(f+e)+e$ to $k(f+e)-1$. It follows that player *A* loses the game at the distances from $k(f+e)$ to $k(f+e)+e-1$, because, whatever she plays, from *e* to *f*, she brings the opponent to a distance that goes from $k(f+e)-f=(k-1)(f+e)+e$ to $k(f+e)+e-1-e=k(f+e)-1$, where player *B* wins the game by assumption. For the symmetric reasons, player *B* also loses the game at the distances from $k(f+e)$ to $k(f+e)+e-1$.
It derives from the above facts that player *A* wins the game at the distances from $k(f+e)+e$ to $(k+1)(f+e)-1$,because, by playing from *e* to *f*, she is able to bring player *B* to a distance in $[k(f+e)+e-e=k(f+e)\,,(k+1)(f+e)-1-f=\,k(f+e)+e-1]$ where he loses the game. For the symmetric reasons, player *B* wins the game at the distances from $k(f+e)+e$ to $(k+1)(f+e)-1$.■
So, if both players have the same strategy set, only the location of the winning distances is different.

3. Same number of strategies, different strategy sets: dexterity is better than strength

Things change drastically when the players have different strategy sets. In this section we suppose that both players have the same number of strategies, but that player *A*, with strategy set $\{e,\ldots,f\}$, is more skilled, and that player *B*, with strategy set $\{E,\ldots,F\}$, is stronger. So we work with $f-e=F-E$, $F>f, E>e$. In this setting there is again a unique crossed cycle of size $f+E=F+e$.

In the HNGame, the SPN Equilibria are such that the stronger player more often wins the game than the more skilled one.

Proposition 2a. Hammer-Nail Game (out of Umbhauer 2023b)
Consider two players, player A with strength f and unskillfulness e, player B with strength F and unskillfulness E, with F>f, E>e and f-e=F-E. At any SPN Equilibrium, when the more skilled player A plays at distance d from the support, she wins the game if the remainder of the division of d by f+E goes from 1 to f, and loses the game if else. When the stronger player B plays at distance d, he wins the game if the remainder of the division of d by f+E goes from 1 to F, and loses the game if else.

Proposition 2a, which is illustrated in figure 3a, amounts to saying that the losing rate (number of losing distances in a crossed cycle of $f+E$ distances) of each player is different. The losing rate of the stronger player B is $1/(1+F/e)$, whereas the losing rate of the more skilled player A is $1/(1+f/E)$, which is larger, given that $F>f$ and $E>e$. A better dexterity is always to the advantage of the opponent, given that it increases F/e; especially, when $e=1$, then the stronger player's losing rate, $1/(1+F)$, is especially low in comparison to the more skilled player's one $(E/(f+E))$. The stronger player's losing rate is unaffected by his own unskillfulness E, which only impacts the losing rate of the more skilled player.
The asymmetry between both players is especially strong when both strategy sets do not overlap. If $\{e,\ldots,f\}\cap\{E,\ldots,F\}=\emptyset$, then $1/(1+F/e)<1/2$ and $1/(1+f/E)\ >1/2$, so the more skilled player, even if she starts the game, loses the game more than half of time.

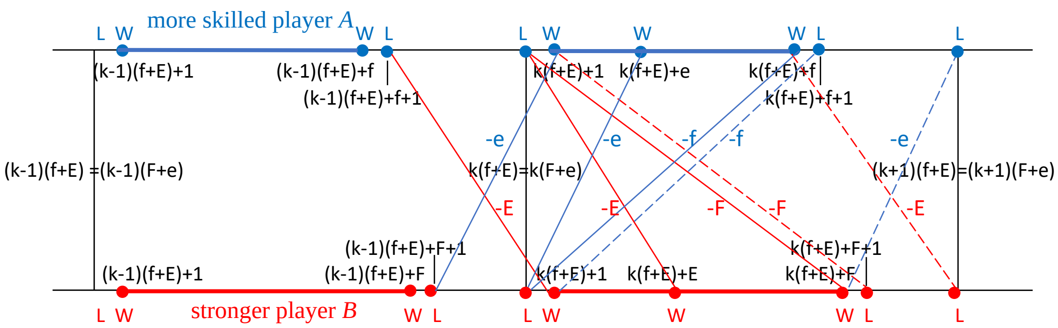

Figure 3a. Hammer-Nail game, f-e=F-E, F>f, E>e
Legend of figure 3a: similar to the legend of figure 2a

Things are completely reversed in the FSGame.

Proposition 2b. Fragile-Support Game
Consider two players, player A with strength f and unskillfulness e, player B with strength F and unskillfulness E, with F>f, E>e and f-e=F-E. At any SPN Equilibrium, when the more skilled player plays at distance d from the support, she wins the game if the remainder of the division of d by f+E goes from e to f+E-1, and loses the game if else. When the stronger player plays at distance d, he wins the game if the remainder of the division of d by f+E goes from E to f+E-1, and loses the game if else.

Proposition 2b is illustrated in figure 3b.

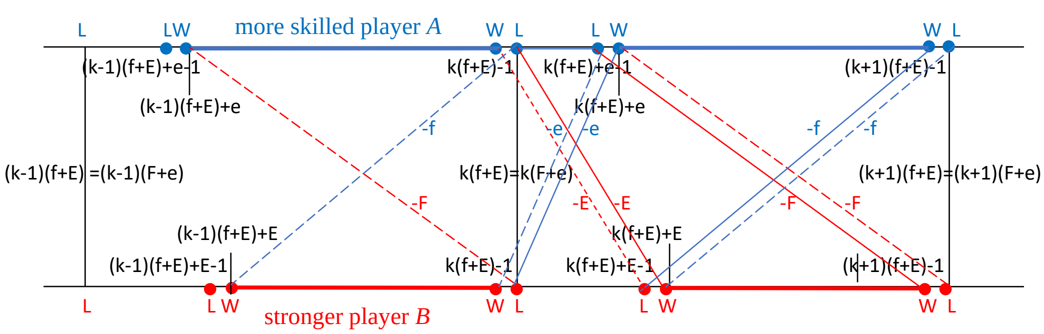

Figure 3b. Fragile-Support Game, F-E=f-e, F>f, E>e.
Legend of figure 3b: similar to the legend of figure 2a

And here is the announced thunderbolt: the more skilled player A wins the game at F distances among $f + E = F + e$, *whereas the stronger player only wins the game at f distances !* So the losing rates of both players are reversed. Now the more skilled player, with losing rate $1/(1 + F/e)$ benefits from her dexterity and from the large strength *F* of the opponent. By contrast, the stronger player, whose losing rate is $E/(E + f)$ suffers from his large unskillfulness and from the weak strength of the opponent. Now it is the stronger player who loses the game at more than half of the distances even if starting the game, when the strategy sets do not overlap ($E > f$).

This is again due to what happens at the distances from 1 to $f + E$.
In the HNGame (figure 3*c*), the more skilled player *A*, respectively the stronger player *B*, wins the game at all distances from 1 to *f*, respectively from 1 to *F*. So player *A* loses the game at all distances from $f + 1$ to $f + E$ because, whatever she plays, she brings player *B* to a distance in $[1 = f + 1 - f, F = f + E - e]$, where he wins the game. *Player B loses the game at the less numerous distances* from $F + 1$ to $F + e$ because, whatever he plays, he brings player *A* to a distance in $[F + 1 - F = 1, F + e - E = f]$, where she wins the game. So, in the first cycle $f + E$, we already observe the advantage of strength on dexterity, with more winning distances for the stronger player ($F > f$). We add that player *A*, respectively player *B*, again wins the game at distance $f + E + 1 = F + e + 1$, because player *A*, by playing *e*, brings player *B* to his losing distance $F + 1$, respectively because player *B*, by playing *E*, brings player *A* to her losing distance $f + 1$.
By contrast, in the FSGame (figure 3*d*) the more skilled player *A*, respectively the stronger player *B*, loses the game at the first $e - 1$ distances, respectively at the first $E - 1$ distances because hammering destroys the support. So player *A* wins the game at the *F* distances from e to $f + E - 1$ because, either she can finish the game or, by playing *f*, she can bring the opponent to a distance lower than or equal to $E - 1$, where he destroys the support. Symmetrically, *player B wins the game at the less numerous f distances* from *E* to $F + e - 1$, because, either he can finish the game or, by playing *F*, he can bring player *A* to a distance lower than or equal to $e - 1$ where she destroys the support. And player *A* loses again the game at distance $f + E$ because she necessarily brings player *B* to a winning distance $(f + E - e = F, f + E - f = E)$; symmetrically, player *B* loses again the game at distance $f + E$ because he necessarily brings player *A* to a winning distance $(f + E - E = f, f + E - F = e)$.

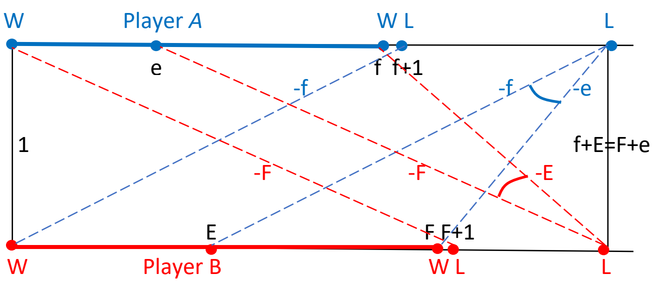

Figure3c. HNGame, F-E=f-e, F>f, E>e

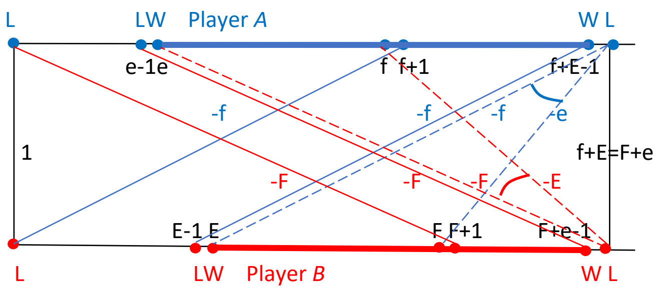

Figure3d. FSGame, F-E=f-e, F>f, E>e

These facts immediately explain the differences between proposition 2*a* and proposition 2*b*.

Proof of proposition 2b
We show by recurrence that player *A* wins the game at the distances from $(k-1)(f+E)+e$ to $k(f+E)-1$ and that player *B* wins the game at the distances from $(k-1)(f+E)+E$ to $k(f+E)-1$ for any integer *k* larger than or equal to 1.
We know that these facts are true for $k=1$. It remains to show that if these facts are true for *k*, then they are true for $k+1$.
We assume that player *A* wins the game at the distances from $(k-1)(f+E)+e$ to $k(f+E)-1$ and that player *B* wins the game at the distances from $(k-1)(f+E)+E$ to $k(f+E)-1$. It follows that player *A* loses the game at the distances from $k(f+E)$ to $k(f+E)+e-1$, because, whatever she plays, she brings player *B* to a distance that goes from $k(f+E)-f=(k-1)(f+E)+E$ to $k(f+E)+e-1-e=k(f+E)-1$, where he wins the game by assumption. And player *B* loses the game at the distances from $k(f+E)$ to $k(f+E)+E-1$, because, whatever he plays, he brings player *A* to a distance that goes from $k(f+E)-F=k(F+e)-F=(k-1)(f+E)+e$ to $k(f+E)+E-1-E=k(f+E)-1$, where she wins the game by assumption.
It derives from the above facts that player *A* wins the game at the distances from $k(f+E)+e$ to $(k+1)(f+E)-1$,because, by playing from *e* to *f*, she is able to bring player *B* to a distance in $[k(f+E)\,,k(f+E)+E-1]$ where he loses the game. And player *B* wins the game at the distances from $k(f+E)+E$ to $(k+1)(f+E)-1$ because, by playing from E to F, he is able to bring player *A* to a distance in $[k(f+E),(k+1)(F+e)-1-F=k(f+E)+e-1]$ where she loses the game.■

4. Strategy sets of different sizes: dexterity keeps an advantage on strength

In the FSGame, as in the HNGame, the player with the largest strategy set always wins the game when the initial distance is larger than a given threshold. Yet, in the FSGame, in contrast to what happens in the HNGame, dexterity keeps an advantage on strength.

4.1 Largest strategy set for the strongest player

In this first subsection, we study the case where the stronger player has the largest strategy set: $F-E>f-e$, with $F>f$ and $E>e$.
We recall the result obtained in the HNGame.

Proposition 3a. Hammer-Nail Game (out of Umbhauer 2023b)
Consider two players, player A with strength f and unskillfulness e, player B with strength F and unskillfulness E, with F>f, E>e and F-E>f-e.
If F>f+E, in any SPN Equilibrium, the stronger player B wins the game in front of any distance if he has the opportunity to play. The more skilled player A only wins the game if she starts the game at a distance lower than or equal to f.
If F<f+E, then, in any SPN Equilibrium, if the initial distance player B is confronted to is strictly larger than D=(f+E)k*, with k* the integer part of (e-1)/(F+e-(f+E)), then he always wins the game. Player A, at a distance larger than (f+E)(1+k*), always loses the game. If else, for k from 1 to k*[3], player B only loses the game at the distances from kF+(k-1)e+1 to k(f+E);*

[3] If *k**=0 player *B* always wins the game when called on to play.

this number of distances is decreasing in k . For k from 1 to 1+k, player A only wins the game at the distances from (k-1)(F+e)+1 to kf+(k-1)E; this number of distances is decreasing in k .*

Proposition 3*a* is illustrated in figure 4*a* for $F < f + E$.

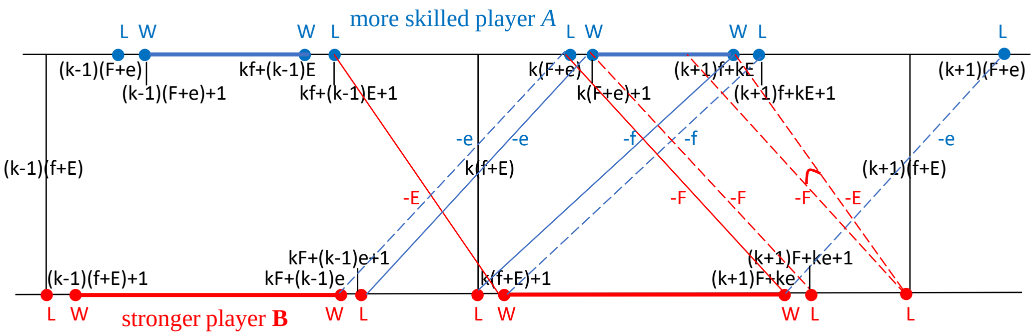

Figure 4a. Hammer-Nail Game, F>f, E>e, F-E>f-e, F<f+E
Legend of figure 4a: similar to the legend of figure 2a

In the FSGame, this proposition becomes:

Proposition 3b. Fragile-Support Game
Consider two players, player A with strength f and unskillfulness e, player B with strength F and unskillfulness E, with F>f, E>e and F-E>f-e.
In any SPN Equilibrium, if the initial distance the stronger player B is confronted to is larger than Δ=(f+E)(1+t*), with t* the integer part of (E-1)/(F+e-(f+E)), then he always wins the game. The more skilled player A, at a distance larger than (f+E)(1+t*), always loses the game. For k from 0 to t* player B only loses the game at the distances from k(F+e) to k(f+E)+E-1; this number of distances is decreasing in k. And player A only wins the game at the distances from k(F+e)+e to (k+1)(f+E)-1; this number of distances is decreasing in k .*

Proposition 3*b* is illustrated in figure 4*b*.

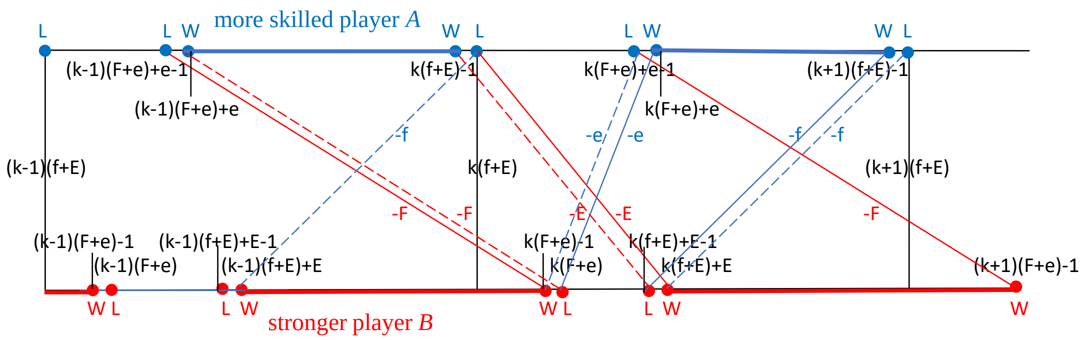

Figure 4b. Fragile-Support Game, F>f, E>e, F-E>f-e
Legend of figure 4b: similar to the legend of figure 2a

So in the FSGame we still get the result that the stronger player wins the game when the initial distance is large enough. Yet there is no longer a possibility for him to always win the game as in proposition 3*a*. Moreover, given that $E > e$, we get $t* > k*$ and so $\Delta* > D*$, so it takes more time in the FSGame than in the HNGame for the stronger player, who has the largest strategy set, to fully benefit from this advantage (i.e. the threshold distance that impedes the more skilled player from winning increases a lot).

And this fact is again due to what happens at the distances from 1 to max $(f+E, F+e) = F + e$, as can be seen in the figures 4*c* and 4*d*.

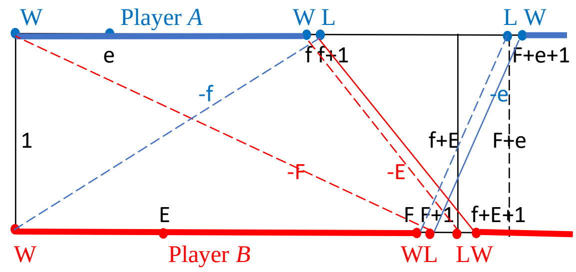

Figure 4c. HNGame, F-E>f-e, F>f, E>e

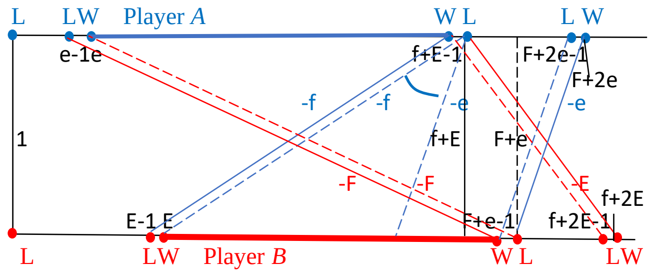

Figure 4d. FSGame, F-E, F>f, E>e

As regards the HNGame, figure 4*c* is like figure 3*c*, except that the vertical lines $f+E$ and $F+e$ are no longer superposed, so figure 4*c* needs no further explanation. The new fact, in comparison to what happens in figure 3*c*, is that the number of losing distances between the two first winning areas for player *A*, $F+e-f$, is now much larger than the number of losing distances between the two first winning areas for player *B*, i.e. $f+E-F$, because both $F-E > f-e$ and $F > f$.
In the same way, for the FSGame, figure 4*d* is like figure 3*d*, except that the vertical lines $f+E$ and $F+e$ are not superposed. So we just add that player *A* loses the game at the distances from $f+E$ to $F+2e-1$ (because she necessarily brings player *B* to a winning distance - a distance in $[E, F+e-1]$) and that player *B* loses the games at the distances from $F+e$ to $f+2E-1$ (because he necessarily brings player *A* to a distance in $[e, f+E-1]$ where she wins the game). Player *A* wins the game at distance $F+2e$ (she brings player *B* to his losing distance $F+e$), and player *B* wins the game at distance $f+2E$ (he brings player *A* to her losing distance $f+E$). This time, the number of losing distances between the two first winning areas for player *A*, $F+2e-f-E$, is not necessarily larger than the number of losing distances between the two first winning areas for player *B*, $f+2E-F-e$.
We also observe that the number of losing distances of player *A* in the FSGame between her two first winning areas is smaller than the number of losing distances between her two first winning areas in the HNGame, because $e-E<0$. In a complementary way, for the same reason, the number of losing distances of player *B* in the FSGame between his two first winning areas is larger than the number of losing distances between his two first winning areas in the HNGame.
This fact immediately induces the difference between proposition 3*a* and proposition 3*b*.

Proof of proposition 3b

We have to show that the more skilled player *A* only wins the game at the distances from $(k-1)(F+e)+e$ to $k(f+E)-1$ and that the stronger player *B* wins the game at the distances from $(k-1)(f+E)+E$ to $k(F+e)-1$ up to a given threshold value of k, 1+t*.
We know that these facts are true for $k=1$. It remains to show that if the facts are true for *k*, then they are true for $k+1$.
We assume that player *A* wins the game at the distances from $(k-1)(F+e)+e$ to $k(f+E)-1$ and that player *B* wins the game at the distances from $(k-1)(f+E)+E$ to $k(F+e)-1$. It follows that player *B* loses the game at the distances from $k\,(F+e)$ to $k(f+E)+E-1$ because he necessarily brings player *A* to a distance in $[k(F+e)-F=$

$(k-1)(F+e)+e\,, k(f+E)+E-1-E = k(f+E)-1]$ where she wins the game. Similarly, player A loses the game at the distances from $k(f+E)$ to $k(F+e)+e-1$ because she necessarily brings player B to a distance in $[k(f+E)-f = (k-1)(f+E)+E\,, k(F+e)+e-1 -e = k(F+e)-1]$ where he wins the game.
As a consequence, player B wins the game at the distances from $k(f+E)+E$ to $(k+1)(F+e)-1$ because he is able, by playing from E to F, to bring player A to a distance in $[k(f+E), k(F+e)+e-1]$ where she loses the game. Similarly, player A wins the game at the distances from $k(F+e)+e$ to $(k+1)(f+E)-1$ because she is able, by playing from e to f, to bring player B to a distance in $[k(F+e), k(f+E)+E-1]$ where he loses the game.
But this only makes sense if the number of player B's losing distances $E+k(f+E-F-e)$ is larger than or equal to1, hence $k \leq (E-1)/(f-e+F-E) = t*$ ■

4.2 Largest strategy set for the more skilled player

We now switch to the opposite configuration, where the more skilled player has the largest strategy set: $f-e > F-E, F > f, E > e$.

We recall the proposition obtained in the HNGame.

Proposition 4a. Hammer-Nail Game (out of Umbhauer 2023b)
Consider two players, player A with strength f and unskillfulness e, player B with strength F and unskillfulness E, with F>f, E>e and f-e>F-E. In any SPN Equilibrium, when the more skilled player A is confronted to an initial distance $\tilde{D}$ strictly larger than (F+e)$\tilde{k}$, with $\tilde{k}$ the integer part of (E-1)/(f+E-(F+e)), she always wins the game. The stronger player B, if confronted to a distance larger than (F+e)(1+$\tilde{k}$), always loses the game. For k from 1 to $\tilde{k}$, player A only loses the game at the distances from kf+(k-1)E+1 to k(F+e); this number of distances is decreasing in k. For k from 1 to 1+$\tilde{k}$, player B only wins the game at the distances from (k-1)(f+E)+1 to kF+(k-1)e; this number of distances is decreasing in k .

Proposition 4*a* is illustrated in figure 5*a*. It shows that player A wins the game when the initial distance is sufficiently large, but it is easy to observe that for a same value of the shortest crossed cycle and a same difference $|(f-e)-(F-E)|$, $\tilde{k}$ >k* and $\tilde{D}$ is larger than D*.

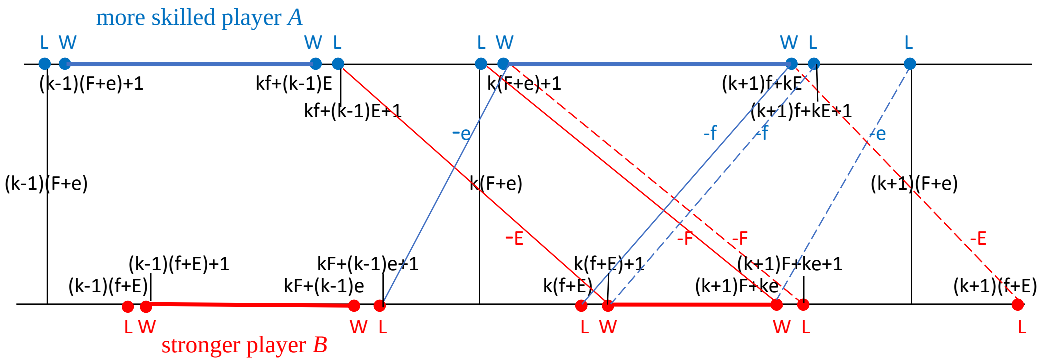

Figure 5a. Hammer-Nail Game f-e>F-E, F>f, E>e
Legend of figure 5a: similar to the legend of figure 2a

Things are opposite in the FSGame.

Proposition 4b. Fragile-Support Game
Consider two players, player A with strength f and unskillfulness e, player B with strength F and unskillfulness E, with F>f, E>e and f-e>F-E. In any SPN Equilibrium, when the more skilled player A is confronted to an initial distance $\tilde{\Delta}$ larger than (F+e)(1+$\tilde{t}$), with $\tilde{t}$ the integer part of (e-1)/(f+E-(F+e)), then she always wins the game. The stronger player B, if confronted to a distance larger than (F+e)(1+$\tilde{t}$), always loses the game. For k from 0 to $\tilde{t}$ player A only loses the game at the distances from k(f+E) to k(F+e)+e-1 : this number of distances is decreasing in k. For k from 1 to (1+$\tilde{t}$), player B only wins the game at the distances from (k-1)(f+E)+E to k(F+e)-1; this number of distances is decreasing in k .

Proposition 4*b* is illustrated in figure 5*b*.

As in the HNGame, in the FSGame we also get the result that the more skilled player wins the game when the initial distance is large enough. Yet, given that $e < E$, $\tilde{t}$ is lower than $\tilde{k}$ which helps the more skilled player *A* to benefit faster from her dexterity in the FSGame than in the HNGame.

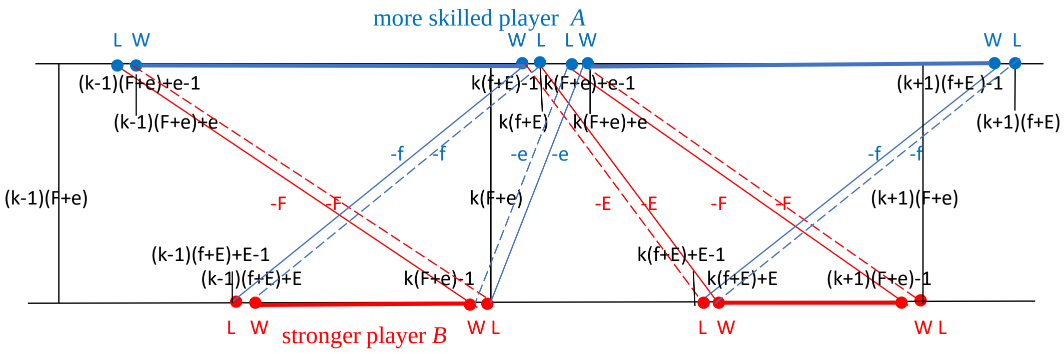

Figure 5b. Fragile-Support Game, f-e>F-E, F>f, E>e
Legend of figure 5b: similar to the legend of figure 2a

And, without surprise, this is again due to what happens at the distances from 1 to max $(f + E, F + e) = f + E$, as can be seen in the figure 5*c* and 5*d*.

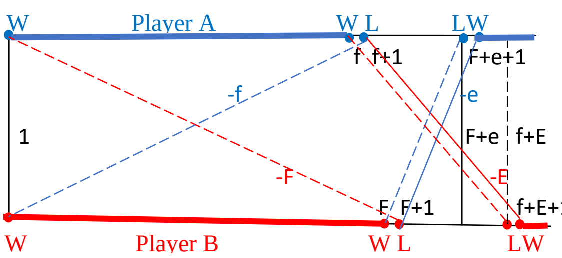

Figure 5c. HNGame, f-e>F-E, F>f, E>e

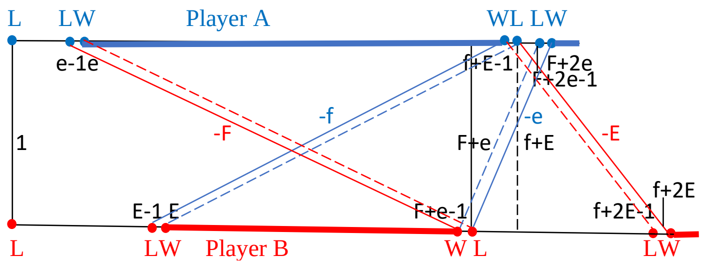

Figure 5d. FSGame, f-e>F-E, F>f, E>e

Figures 5*c* and 5*d* are like figures 4*c* and 4*d*, except that the vertical lines $F + e$ and $f + E$ are reversed, so they need no justification.

We observe that in the HNGame, player *B*'s number of losing distances between his two first winning areas, $f + E - F$, is not necessarily larger than player *A*'s number of losing distances between her two first winning areas, $F + e - f$. By contrast, in the FSGame, the number of losing distances of player B, $f + 2E - F - e$, becomes strictly larger than the number of losing distances of player A, $F + 2e - f - E$.

We also observe that the number of player A's losing distances between her two first winning areas is lower in the FSGame than in the HNGame ($F + 2e - f - E < F + e - f$), because $e < E$. By contrast, for the same reason, the number of player B's losing distances between his two first winning areas is larger in the FSGame than in the HNGame ($f + 2E - F - e > f + E - F$).

These facts explain that the more skilled player faster benefits from the advantage of having a larger strategy set in the FSGame than in the HNGame and they largely explain the difference between proposition 4b and proposition 4a.

Proof of proposition 4b: similar to that of proposition 3b.

5. Example and concluding remarks

Propositions 3a, 3b, 4a and 4b show that strength has more strategic power than dexterity in the HNGame, whereas dexterity has more strategic power than strength in the FSGame. We illustrate this fact with two examples.

In the first example, $f = 6$, $e = 4$, $F = 8$, $E = 5$ and in the second example, $f = 6$, $e = 3$, $F = 8$ and $E = 6$. In the first example, $F - E = 3 > f - e = 2$, so the stronger player has one more strategy than the more skilled one, who has 3 strategies; in the second example $f - e = 3 > F - E = 2$, so the more skilled player has one more strategy than the stronger one, who has 3 strategies. In both examples, $|(f + E) - (F + e)| = 1$, the shortest cycle is of size 11 and the longest cycle is of size 12. Both examples are therefore quite similar in structure, which allows us to illustrate the relative advantages of strength and dexterity in the HNGame and in the FSGame.

In the HNGame (figure 6a and figure 6b), both examples clearly highlight that strength keeps an advantage on dexterity. The strongest player, when he has the largest strategy set, always wins the game as soon as the initial distance D is strictly larger than 33 ($k^* = 3$), and the more skilled player can only win the game when starting at 18 distances all strictly lower than 40. By contrast, when the more skilled player has the largest strategy set, she always wins the game only if D is strictly larger than 55 ($\check{k}$=5), and the stronger player wins the game at 33 distances which go up to 63.

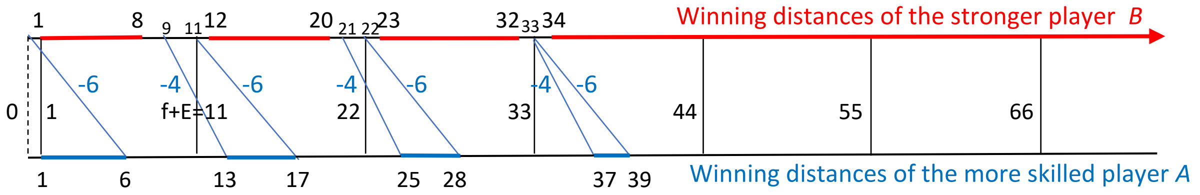

Figure 6a. Hammer-Nail Game, example 1: $f = 6$, $e = 4$, $F = 8$, $E = 5$
Legend of figure 6a: : The two horizontal lines represent the distances at which player B and player A are playing. The horizontal red -blue- segments represent distances where player B -player A- wins the game. The blue lines going from the lower horizontal line to the upper one are SPN Equilibrium actions for player A. The red arrow means that player B wins the game at each distance larger than or equal to 34.

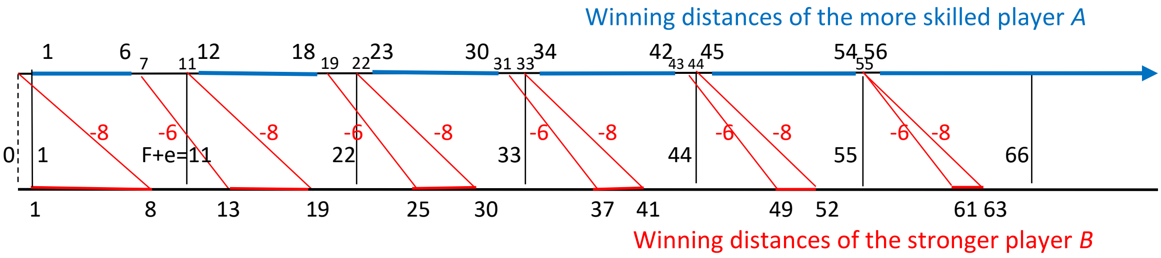

Figure 6b. Hammer-Nail Game, example 2: $f = 6, e = 3, F = 8, E = 6$
Legend of figure 6b: The two horizontal lines represent the distances at which player A and player B are playing. The horizontal blue -red- segments represent distances where player A -player B- wins the game. The red lines going from the lower horizontal line to the upper one are SPN Equilibrium actions for player B. The blue arrow means that player A wins the game at each distance larger than or equal to 56.

Things are completely reversed in the FSGame.
In the FSGame (figures 6*c* and figure 6*d*), both examples clearly highlight that dexterity keeps an advantage on strength. The strongest player, when he has the largest strategy set, always wins the game only as soon as the initial distance D is strictly larger than 48 ($t* = 4$), and the more skilled player can win the game when starting at 25 distances up to 54. By contrast, when the more skilled player has the largest strategy set, she always wins the game as soon as D is strictly larger than 24 ($\tilde{t} = 2$), and the stronger player only wins the game at 12 distances all lower than 33.

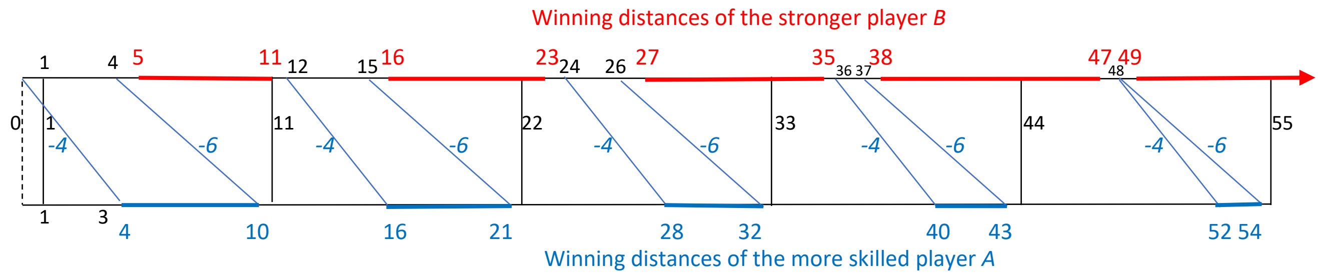

Figure 6c. Fragile-Support Game, example 1: $f = 6, e = 4, F = 8, E = 5$
Legend of figure 6c: similar to the legend of figure 6a, but the red arrow starts at 49.

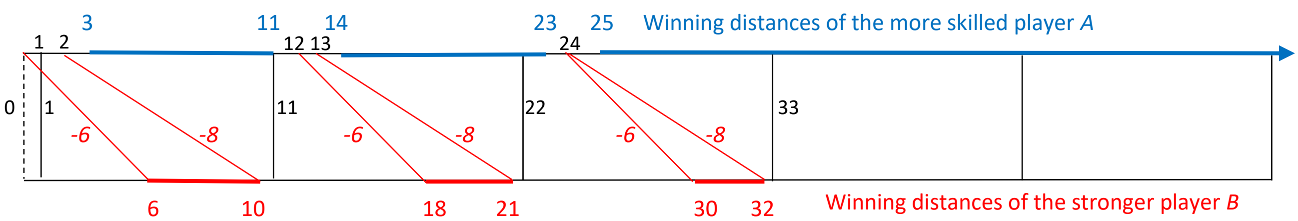

Figure 6d. Fragile-Support Game, example 2: $f = 6, e = 3, F = 8, E = 6$
Legend of figure 6d: similar to the legend of figure 6b but the blue arrow starts at 25.

Moreover, when the stronger player has the largest strategy set, the strongest player is sure to win the game as soon as the initial distance he is confronted to is larger than or equal to 34 in the HNGame, whereas it has to be larger than or equal to 49 in the FSGame.

By contrast, when the more skilled player has the largest strategy set, the more skilled player is sure to win the game as soon as the initial distance she is confronted to is larger than or equal to 25 in the FSGame, whereas it has to be larger than or equal to 56 in the HNGame.
So clearly, the relative advantages of strength and dexterity are inversed in the HNGame and in the FSGame.

To conclude, the fact that the hammer destroys or not the support when it touches the support leads to quite different results. In the Hammer-Nail Game with an indestructible wooden support, the upper bound of the strategy set is more useful to win the game than the lower bound, whereas in the Fragile-Support Game, the lower bound of the strategy set is more useful to win the game than the upper bound.

Let us add that it is quite logical to expect that a change in the game end conditions (winning conditions at small distances lower than *e*) induces a change in the winning strategies. The fact that dexterity is required at the end logically moderates the strategic importance of strength. Yet it was not expected that this change induces a reverse of the results. In other terms, it seems reasonable to conjecture that requiring dexterity at the end may equilibrate the strategic importance of strength and dexterity but it is more surprising to see that it gives more strategic weight to dexterity than to strength.

So, up to now, we conclude that the upper bound and the lower bound of the strategy set in a Nim game do not have the same strategic importance, but that each of them may be most important to win the game depending on the end conditions of the game, i.e. the winning conditions when the distance is lower than the lower bound.

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