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«A sad lesson from the hammer-nail game: strength is better than dexterity»

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A sad lesson from the hammer-nail game: strength is better than dexterity

Hammer-Nail game:

Part 1: Show your strength in the hammer-nail game: a Nim game with incomplete information" Working paper Beta 2023 n°5, January 2023

Part 2: A sad lesson from the hammer-nail game: strength is better than dexterity

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Abstract

In this second paper on the hammer-nail game, we confront strength with dexterity. The hammer-nail game, a game played in the French TV show "Fort Boyard", goes as follows: two players are in front of a nail slightly driven into a wooden support. Both have a hammer and in turn hit the nail. The winner is the first player able to fully drive the nail into the support. A player is of strength f if he is able, with one swing of the hammer, to drive the nail at most f millimeters into the support. A player is of non dexterity e if he is unable to hammer smoothly, so that, with one swing of the hammer, he drives the nail at least e millimeters into the support, with $e > 1$. In a previous paper, we mainly studied the impact of strength, both players being of high dexterity ($e = 1$), and we transformed the hammer-nail game into a Nim game with incomplete information on strength. In this paper we study the impact of both strength and dexterity. We confront two players of different strength and dexterity and namely show a sad result: strength is more useful than dexterity to win the game. We also study the behavior in front of incomplete information, either on strength or on dexterity.

Keywords: Nim game, crossed cycles, Fort Boyard, subgame perfect Nash equilibrium, strength, dexterity, incomplete information, heuristics of behavior.

JEL Classification: C72

1. Introduction

As in an earlier paper (Umbhauer, 2023), we study a game played in the French TV show "Fort Boyard"[1], the hammer-nail game. The game goes as follows: two players, player 1 and player 2, are in front of a nail slightly driven into a wooden support (see figure 1). Both have a (same) hammer and in turn hit the nail. At the beginning of the game, the head of the nail is at a distance

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[1] Adventure Live Productions/Banijay Group for France Télévision.

D from the support. In the French TV show Fort Boyard, player 1 is a candidate who wants to win the game, in order to get more time to catch the "Boyards" (coins) and player 2 is a "Maître du temps", a person who belongs to the Fort Boyard TV team, who wants to impede the candidate from winning.

In this paper we work with both strength and dexterity. The strength is measured in numbers of millimeters. We say that a player is of strength 5 if he is able, with one swing of the hammer, to drive the nail at most 5 millimeters into the support[2]. This means that a player of strength 5 is able, with one swing of the hammer, to drive the nail 1 millimeter (mm), 2mm, 3mm, 4mm or 5mm into the support. The notion of dexterity we introduce in this paper is somewhat limited in that a lack of dexterity just means an inability to hammer smoothly, i.e. the inability to just drive the nail 1 millimeter into the support. So non dexterity -unskillfulness- is also measured in millimeters: a player of unskillfulness e, with one swing of the hammer, drives the nail at least e millimeters into the support. Hence a player of strength f but unskillfulness e is able, with one swing of the hammer, to drive the nail from e to f millimeters into the support, with $f > e \geq 1$. A player of high dexterity is characterized by $e = 1$ and we say that a player is more skilled when he is of higher dexterity than his opponent.

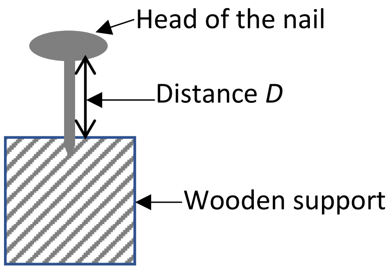


Figure 1 : the nail at the beginning of the play

We require that each player, at each turn of play, at least drives the nail e mm into the support, which means that he cannot simulate hammering the nail. As a matter of fact, once the head of the nail is close to the support, but too far from it to be fully driven into with one swing of the hammer, a player may be incited to not drive the nail further into the support, in order to impede the opponent from winning the game at his next turn of play[3].

So the game becomes a special Nim game (see Bouton, 1901). TV shows sometimes work with Nim games (see for example the sticks game also played in the Fort Boyard TV show (Umbhauer, 2016) or the Game of 21, played in an American TV show (Dufwenberg et al. 2010, Gneezy et al., 2010)). In this special Nim game, the (pure) strategy set of a player of strength f and unskillfulness e contains $f - e + 1$ (pure) strategies[4]. Throughout the paper, we will note f and e, respectively F and E, the strength and the unskillfulness of one player, respectively of the other player.

In Umbhauer (2023) we mainly worked with two players of high dexterity $(e = 1)$ but of different strengths and we introduced incomplete information on strength. Each player knew

[2]We work with millimeters but we could work with any (smaller) measure of distance.

[3] This requirement does not exist in the TV Fort Boyard game, but it is mathematically necessary to avoid that the game never stops.

[4] We further only say strategies instead of "pure" strategies, given that we only work with pure strategies.

his strength but had incomplete information on the strength of the opponent. We built a perfect Bayesian equilibrium without making use of the prior probability distributions on the unknown strengths, which is not usual when dealing with games with incomplete information. In this paper we confront strength with dexterity. We show that strength is more useful than dexterity to win the game. To put it more precisely, when both players have the same number of strategies but are of different strength and dexterity, the stronger player wins more often than the more skilled one. When the two players have not the same number of strategies, then the player with the larger strategy set wins the game when D is sufficiently large, but strength keeps an advantage on dexterity.

In this paper we also draw attention to crossed cycles, to the link between incomplete information and crossed cycles and to heuristics of behavior, namely in contexts with incomplete information, either on strength or on dexterity.

In section 2, we briefly recall some results out of Umbhauer (2023) we use in this paper and we introduce the notion of crossed cycle. Section 3 shows the impact of dexterity on the optimal strategies when both players are of same strength and same dexterity. Section 4 studies the optimal behavior of two players of same strength but different dexterity and provides a first result about the impact of incomplete information on dexterity. Section 5 is the core section of the paper. We show that at the subgame perfect Nash equilibrium, if two players have the same number of strategies $(f-e=F-E)$, but one player is more skilled and the opponent is stronger ($e<E$ but $F>f$), then starting the game being stronger leads to more often winning the game than starting the game being more skilled. Section 6 gives the optimal behavior when the more skilled player has either a smaller or a larger strategy set than the stronger player. We show that the player with the largest strategy set always wins the game when D is sufficiently large, but that strength keeps an advantage on dexterity. We conclude the paper in section 7 by giving heuristics of behavior in a context of partial incomplete information, either on strength or on dexterity.

2. Some results on strength

We here recall some results on strength (out of Umbhauer, 2023) when both players are of high dexterity ($e=1$).

Throughout the paper, we simply say that a player plays s to say that he drives the nail s millimeters into the support.

Proposition 1 (out of Umbhauer, 2023)
Consider two players of same strength f and of same high dexterity e=1. We call r the remainder of the division of D by $(f+1)$*. The set of Subgame Perfect Nash Equilibria (SPNE) is given by:*
Player 1 : At her first turn of play (first round), she plays r if $r>0$*, and she plays any integer from 1 to f if* $r=0$*. At her other turns of play, she plays in such a way that the remaining distance after her turn of play is a multiple of* $f+1$*. If this is not possible (because the distance she observes at her turn of play is a multiple of* $f+1$*), she plays any integer from 1 to f.*

Player 2: At each potential turn of play, he plays in such a way that the remaining distance after his turn of play is a multiple of $f+1$. If this is not possible (because the distance he observes at his turn of play is a multiple of $f+1$), he plays any integer from 1 to f.[5]

We now suppose that one of the players is stronger than the other: one player is of strength *f*, and the other player is of strength *F*, with $F-f \geq 1$. In that case the strongest player always wins the game, except if $D \leq f$ and the weakest player begins the game; moreover, the strongest player can win in many ways (see Umbhauer, 2023).

This leads us to drawing attention to the notion of crossed cycle, which is central to any Nim game. Given that both payers play in turn, a crossed cycle is necessarily the sum of the millimeters played by both players in two successive rounds. To put it more precisely, a cycle is the number of millimeters that at least one of both players is able to complete regardless of what is played by the opponent at the previous round. In each game, there are two crossed cycles. So, when both players are of same high dexterity $e=1$ and same strength *f*, the two crossed cycles are $1+f$, given that, whatever is played by one player, the other can complete the number of millimeters to $f+1$ at the next round. This is illustrated in figure 2a. When both players are of high dexterity $e=1$ and of different strengths *f* and *F* (with $F>f$), the two cycles are $1+f$ and $1+F$ given that, whatever the number of millimeters played by the weaker player, the stronger player can complete it to $f+1$ or $F+1$. In contrast, the weaker player cannot complete the crossed cycles. These cycles are illustrated in figure 2b.

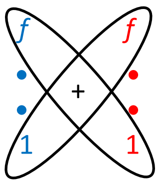


Figure 2a

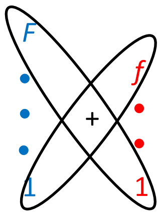


Figure 2b

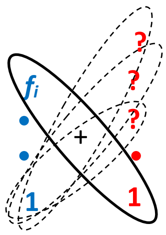


Figure 2c: the cycles in dashed lines are unknown by player i.

The strongest player, in front of any remaining distance $d>f$, can bring the weakest player in front of a multiple of $f+1$, simply by playing the remainder *r* of the division of *d* by $f+1$ if *r* is different from 0, or by playing $f+1$ if $r=0$. Yet in proposition 2 (Umbhauer, 2023), we exploited the crossed cycle $F+1$. We showed that, even in front of a remaining distance that is a multiple of $F+1$, say $k(F+1)$, $k \in \mathbb{N}$, $k \geq 1$, the strongest player wins the game by behaving as follows: she chooses 1, so that the opponent, who can only play from 1 to *f*, leads her to a remaining distance that is not a multiple of $F+1$, because this distance goes from $k(F+1)-1-f = (k-1)(F+1)+F-f$ to $k(F+1)-2=(k-1)(F+1)+F-1$. So she plays an integer from $F-f$ to $F-1$ so that the remaining distance is a multiple of $F+1$, which leads her to winning the game. This argument is illustrated in figure 3.

Proposition 2 (out of Umbhauer, 2023)
We note e=1 the dexterity of both players, f the weakest strength, F the strongest one, with $F \geq f+1$. A possible SPNE is characterized by:

[5] Given that 0 is a multiple of any number, the proposition includes that in front of a remaining distance lower than or equal to the strength, a player simply plays the remaining distance, i.e. fully drives the nail into its support.

The weaker player: if, at his turn of play, the remaining distance is lower than or equal to f, he plays the remaining distance. If not, he can play from 1 to f.
The strongest player: if, at his turn of play, the remaining distance is lower than or equal to F, she plays the remaining distance. Otherwise, at any turn of play, she plays so that the weaker player faces a remaining distance that is a multiple of $F+1$. If this is not possible (because she faces a remaining distance that is a multiple of $F+1$) she can play any integer from 1 to $F-f$.
The stronger player always wins the game in any SPNE, except if the weaker player starts the game, with D lower than or equal to f.

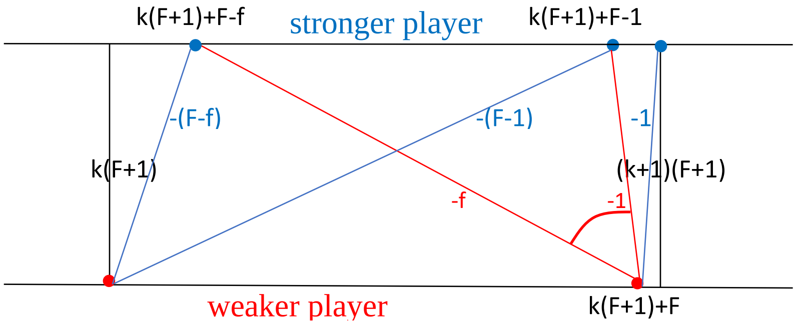


Figure 3

Legend of figure 3: the rectangle represents the distances from k(F+1) to (k+1)(F+1). The horizontal lines are the distances at which the stronger player and the weaker player are playing. The blue and red lines represent the number of millimeters respectively played by the stronger and the weaker player when called on to play.

Proposition 2 allows the stronger player, in front of a remaining distance that is a multiple of $F+1$, to play 1, but also to play any integer from 1 to $F-f$. Yet by playing 1, the player of strength *F* wins in front of any weaker player, so playing 1 is the good way to play even if the stronger player ignores the strength *f* of the weaker player. This fact is exploited in proposition 3 which gives the optimal behavior when both players ignore the strength of the opponent. We suppose that the strength of player i , $i=1{,}2$, belongs to a set of integers going from 1 to an upper bound f_i, with $f_i \geq 2$. Given that both players are of high dexterity ($e=1$), each player *I* only knows one crossed cycle of the game, f_i+1. This is illustrated in figure 2*c* for player i. The Perfect Bayesian Equilibrium given in proposition 3 exploits this knowledge given that each player only works with the crossed cycle he knows.

Proposition 3 (out of Umbhauer, 2023)
Consider two players of high dexterity (e=1) that ignore the strength of the opponent. A (Perfect Bayesian) Nash equilibrium of this game with high dexterity and incomplete information on strength goes as follows:
At each turn of play, a player of strength f, regardless of the value of f, plays as follows:
- *If the distance d he is confronted to is lower than or equal to f, he plays d.*
- *If $d>f$, he plays so as to put the opponent in front of a remaining distance that is a multiple of $f+1$. If this is not possible (because d is a multiple of $f+1$), then he plays 1.*

This result is robust, in that it holds even if the players have no information at all on the strength of the opponent. It does not need a distribution of prior probabilities on the types of the opponent. This explains that we do not introduce the player's beliefs at each turn of play, given that they play no role in the result. And this explains why we put (Perfect Bayesian) in brackets, given that we make no use of the bayesian rule to get the optimal way of behavior.

It follows from proposition 3 that a player behaves as if he were the strongest player, by playing 1 if the remainder *r* of the division of *d* by $(f+1)$ is 0, and *r* when *r* is different from 0.

3. Dexterity, enlargement of the crossed cycle and set of losing distances

We now allow players to not be of high dexterity. We recall that non dexterity – unskillfulness- just means the inability to hammer smoothly, so that a player of unskillfulness *e*, with one swing of the hammer, drives the nail at least *e* millimeters into the support. Dexterity matters only if the remaining distance *d* the player is confronted to is larger than *e*. If $d < e$, then lack of dexterity is not a problem, because hammering like a beast for sure fully drives the nail into the support. The main impact of unskillfulness *e* is to reduce the strategy set from below: the strategy set of a player of strength *f* and unskillfulness *e*, with $f > e \geq 1$, becomes $\{e, \ldots, f\}$.

In this section we suppose that both players are of same strength and of same dexterity. A first consequence of unskillfulness is the enlargement of the size of the crossed cycles: when the two players are of strength *f* and of unskillfulness *e*, the size of the two crossed cycles becomes $f + e$ (see figure 4*a*), because each player is able to complete to $f+e$ the number of millimeters played by the opponent at the previous round (as the opponent plays at least *e* millimeters and at most *f* millimeters). A second consequence is that unskillfulness reduces the probability to win the game for the player who starts the game. As a matter of facts, in the subgame perfect Nash equilibrium, a player, when starting the game at distance $k(f+e)+r$, with *k* an integer and *r* an integer from 1 to $f+e$, wins the game if *r* goes from 1 to *f*, and loses the game if *r* goes from $f+1$ to $f+e$. So unskillfulness reduces the number of winning distances by $e/(f+e)$ %, even if one starts the game (we recall that for $e=1$, the starting player loses the game only if the starting distance is a multiple of $f+1$).

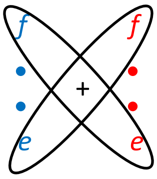
Proposition 4
Consider two players of strength f and unskillfulness e, with f >e ≥1. Each player, in front of the distance d= k(f+e)+r, with k an integer and r an integer from 1 to f+e, wins the game for r from 1 to f, and loses the game for r from f+1 to f+e. More precisely, when both players are of same strength f and same unskillfulness e, with f>e>1, then, at the SPNE, at any distance d he is called on to play, each player plays as follows:

- *He plays max(e,d) if d≤f* [6] *and if f<d≤f+e, he can play any integer from e to f.*
- *If d>f and d=k(f+e)+ r, with k an integer larger than or equal to 1, r an integer from 1 to f+e:*
  if r goes from 1 to e-1, he plays e
  if r goes from e to f, he plays r
  if r goes from f+1 to f+e, he can play any integer from e to f.

Example: For $f=5$ and $e=3$, a player, if starting the game, wins the game only if *D* is a multiple of 8 plus 1, 2, 3, 4 or 5. He loses the game, despite he starts the game, when *D* is a multiple of 8, or a multiple of 8 plus 6 or 7, i.e. in front of 37.5% of all possible distances.

[6] We recall that if *d<e*, playing *e* drives for sure the nail into its support.

Proposition 4 is illustrated in figure 4*b*.



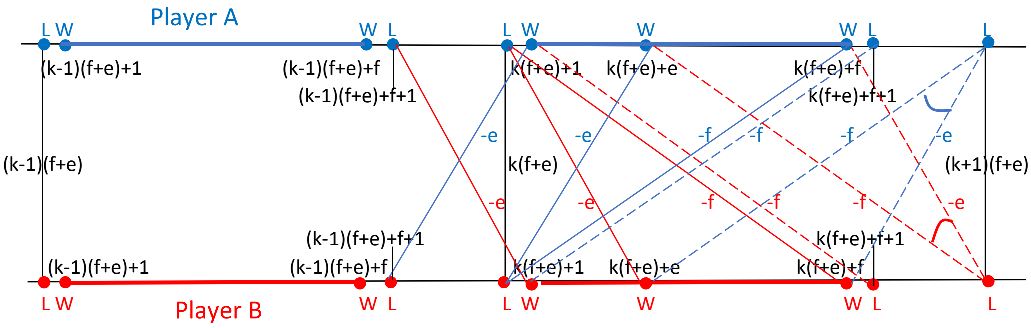


Figure 4a *Figure 4b : same strength f, same dexterity e, f>e>1*

Legend of figure 4b: The two horizontal lines represent the distances at which player A and player B are playing. L and W mean that the player loses and wins the game at the corresponding distance. Each rectangle is a crossed cycle (f+e). The horizontal blue -red- segments represent distances where player A -player B- wins the game. The blue -red- lines going from one horizontal line to the other are the number of millimeters played by player A -player B-. The dashed lines are optimal ways of playing but they lead to losing the game.

Proof of proposition 4

A player's behavior is optimal in front of a distance $d \leq f$ (she wins the game) and it is also optimal in front of a distance d, with $f < d \leq f + e$, given that she loses the game whatever she plays. So let us suppose that the players' behavior is optimal in *cycle k*, i.e. in front of all distances $d = (k-1)(f+e) + r$ with k an integer larger than 0 and r an integer from 1 to $f + e$, and that this behavior leads the player to winning the game for *r* from 1 to *f*, and to losing it for *r* from $f + 1$ to $f + e$. This is true for $k = 1$ (obvious)

So we just have to show that the players' behavior is optimal in *cycle k+1*, i.e. in front of the distances $k(f+e) + r$, with *r* from 1 to $f + e$.

If a player is at distance $k(f+e) + r$, with r from 1 to $e - 1$, then playing *e* is optimal because the remaining distance becomes $(k-1)(f+e) + f + r$, with *r* from 1 to $e - 1$, and so the opponent loses the game by assumption.

If a player is at distance $k(f+e) + r$, with *r* from *e* to *f*, then playing *r* is optimal because the remaining distance becomes $k(f+e)$ and so the opponent loses the game by assumption.

Finally, if a player is at a distance $k(f+e) + r$, with *r* from $f + 1$ to $f + e$, playing any integer from *e* to *f* is optimal, because the distance of the opponent becomes a number going from $k(f+e) + 1$ to $k(f+e) + f$, and so the opponent wins the game anyway, regardless of the chosen integer. ■

4. Different dexterity, same strength

We now suppose that both players are of same strength *f*, but of different unskillfulness, *e* and *E*, with $f > E > e > 1$.

The two crossed cycles of the game become $f + e$ and $f + E$ (see figure 5*a*). The more skilled player can complete both crossed cycles. As a matter of facts, if the less skilled player plays from *E* to *f*, she can play from $f + e - E$ to *e* in order to complete the crossed cycle $f + e$ and from *f* to *E* to complete the crossed cycle $f + E$. Inversely, the less skilled player can complete none of both crossed cycles. If the more skilled player plays *f*, he cannot complete the crossed cycle $f + e$, and if the more skilled player plays *e*, he cannot complete the crossed cycle $f + E$, given that $f + E - e > f$. So each time the more skilled player can lead the less skilled one to a distance that is a multiple of $(f + E)$ or $(f + e)$ she wins the game. Proposition 5 works with the smallest crossed cycle, i.e. the crossed cycle composed of the common strength (the strength of the opponent) and the unskillfulness of the more skilled player (e)[7].

In facts, the more the initial distance *D* grows, the less numerous are the situations where the more skilled player loses the game. Yet there are a certain number of distances where she will not win, even if she starts the game. As a matter of facts, in the first cycle (i.e. in front of distances from 1 to $f + e$), the more skilled player loses the game when she is in front of the distances $f + 1, f + 2, \dots\ f + e$. It follows that the less skilled player can win the game in front of the distances from $f + E + 1$ to $2f + e$. And, if $E < 2e$, there are additional distances in the second cycle (distances from $f + e + 1$ to $2(f + e)$), namely the distances from $2f + E + 1$ to $2f + 2e$, in front of which the more skilled player again loses the game, given that she can only lead the less skilled player to a distance where he is winning (because $2f + E + 1 - f = f + E + 1$ and $2f + 2e - e = 2f + e$). Yet there are less distances ($2e - E$) where she loses the game in the second cycle than in the first one (*e* distances), given that $e < E$. So there are less additional distances where the less skilled player can win the game. And so on: given that the number of losing distances of the more skilled player is decreasing in each cycle (*e* in the first cycle, $2e - E$ in the second, $3e - 2E$ in the third, more generally $ke - (k - 1)E$ in cycle *k*) the number of losing distances generally quickly goes to 0. And if $2e \leq E$, then the more skilled player wins in front of all distances except if she initially faces the distances from $f + 1$ to $f + e$.

Proposition 5
Consider two players of same strength f but different dexterities, e and E, with f>E>e. The subgame perfect Nash Equilibria lead the more skilled player to always winning the game, provided the initial distance she is confronted to is larger than or equal to K(f+e), with K the integer part of (E-1)/(E-e). For lower distances, the more skilled player wins the game at all distances except the distances from tf+(t-1)E+1 to t(f+e), with t from 1 to K.
A SPNE way of playing for the more skilled player consists, at distance d, in playing f when a simple swing is not sufficient to go to a multiple of f+e (lower than d), to play e when a simple swing leads below the closest multiple of f+e (lower than d), and to play the remainder of the division of d by f+e, if the remainder goes from e to f.

Proof of proposition 5
The main steps of the proof have been given in introduction to the proposition. It remains to prove the recurrence step.
We suppose that in cycle *k*, $k \geq 1$ (distances $(k - 1)(f + e) + r$, with *r* from 1 to $(f + e)$), the more skilled player loses the game at the distances from $kf + (k - 1)E + 1$ to $k(f + e)$ and wins the game at the distances from $(k - 1)(f + e) + 1$ to $kf + (k - 1)E$. And we suppose also that the less skilled player loses the game at the distances from $kf + (k - 1)e + 1$ to $k(f + E)$ – we go a little beyond cycle *k*- and wins the game at the distances from $(k - 1)(f + E) + 1$ to $kf + (k - 1)e$. These facts are true for $k = 1$.

[7] We show in the conclusion that working with the largest crossed cycle leads to the same optimal behavior.

We now turn to cycle $k+1$ (distances $k(f+e)+r$, with r from 1 to $(f+e)$).
Given that the more skilled player loses the game at the distances from $kf+(k-1)E+1$ to $k(f+e)$, the less skilled player wins the game at the distances from $k(f+E)+1$ to $(k+1)f+ke$, because, by playing from E to f, he brings the more skilled player in front of a distance going from $kf+(k-1)E+1$ to $k(f+e)$.
We now consider the more skilled player. Playing e at a distance from $k(f+e)+1$ to $k(f+e)+e-1$ is optimal because the remaining distance goes from $kf+(k-1)e+1$ to $k(f+e)-1$ where the opponent loses the game. Playing r, for r from e to f, at the distance $k(f+e)+r$, is optimal because it leads the less skilled player to the distance $k(f+e)$ where he loses the game. Playing f at a distance from $(k+1)f+ke+1$ to $(k+1)f+kE$ is optimal because the less skilled player will be at a distance from $k(f+e)+1$ to $k(f+E)$, where he loses the game. And when the more skilled player is at a distance from $(k+1)f+kE+1$ to $(k+1)(f+e)$, then she loses the game whatever she plays because the remaining distance necessarily goes from $k(f+E)+1$ to $(k+1)f+ke$, so the opponent wins the game. So playing f is an optimal action among others.
We now turn to the less skilled player. We have already shown that he wins the game at the distances from $k(f+E)+1$ to $(k+1)f+ke$. We now show that he loses the game in front of all distances from $(k+1)f+ke+1$ to $(k+1)(f+E)$. This follows from the fact that he can play at least E and at most f, which leads the more skilled player to a distance from $k(f+e)+1$ to $(k+1)f+kE$ where she wins the game.
The number of losing distances for the more skilled player in cycle k is $kf+ke-kf-(k-1)E$. To make sense, this number has to be larger than or equal to 1 so we need $k \leq \frac{E-1}{E-e}$.
Therefore, if the more skilled player is at a distance larger than $(f+e)K$, with K the integer part of $\frac{E-1}{E-e}$, she wins the game.■

The proof is partly illustrated in figure 5*b*.

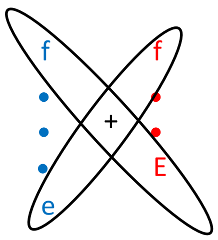


Figure 5a

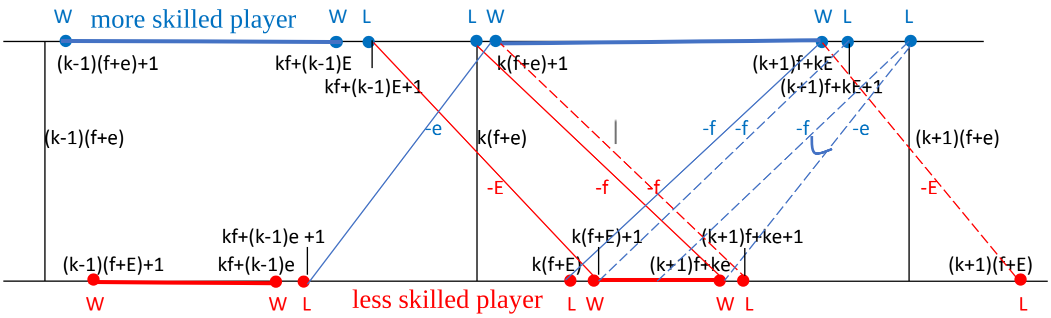


Figure 5b :same strength f, different dexterity, f>E>e

Legend of figure 5b: similar to the legend of figure 4b

Let us comment proposition 5, by first giving two examples that illustrate the advantage of the more skilled player. For $f=6, e=3$, $E=4$, we get $K=3$, so the more skilled player, if starting the game, only loses the game if the initial distance she is in front to is 7,8,9, 17,18 or 27. For $f=5, e=2$, $E=3$, we get $K=2$ and the more skilled player wins the game at all distances, except for 6, 7 and 14.

Yet this doesn't mean that the less skilled player wins seldom the game, especially when the initial distance D is not large. Assume $D \leq 25$. For $f = 5$, $e = 2, E = 3$, if the more skilled player starts the game, she wins at 22 among 25 possible distances. But the less skilled player, when he starts the game, wins when D goes from 1 to 5, from 9 to 12, and from 17 to 19. So he wins the game at 12 among 25 possible distances, i.e. almost half of time. And the possibilities to win the game can even be much larger. For example, if the less skilled player starts the game, if $f = 7$, $e = 2$, $E = 3$ and $D \leq 25$, then he wins at $18/25 = 72\%$ of all possible initial distances. But for larger initial distances these percentages fall and they also fall when the difference between e and E rises.

In other terms, when both players are of same strength, more dexterity gives a strong advantage when the initial distance is large (the more skilled player always wins the game if the first distance she is confronted to is larger than $K(f + e)$). But if the initial distance is not very large, if f is much larger than E and E is not much larger than e, then the most important fact becomes again the position in the game (first or second player). So more dexterity needs a sufficiently large initial distance to counterbalance the position in the game (first or second player), something we did not observe when only different strengths (with the same high dexterity) were at work (proposition 2).

An immediate consequence of proposition 5 is that, if one player is of high dexterity, i.e. $e = 1$, the other player being of unskillfulness $E > 1$, and if both players are of same strength f, with $f > E > 1$, then the player with high dexterity always wins the game except at distance f+1 (because $K = 1$ in this special case). So the less skilled player always loses the game, except if he starts the game at the distances from 1 to f and from f+1+E to 2f+1.

Corollary of proposition 5 (out of Umbhauer, 2023)
When both players are of same strength f, one player being of high dexterity e=1, the other being less skilled, with f>E>1, a SPNE way of playing is given by:
For the player with high dexterity at distance d:
If d ≤ f, she plays d. If $d > f$, she plays in order to bring the opponent in front of a remaining distance that is a multiple of $f + 1$. If this is not possible (because d is a multiple of $f + 1$), then she plays f.
For the player with unskillfulness E>1 at distance d: If d ≤ f, he plays max (E,d). If $2f + 1 \geq d \geq f + E + 1$, he plays from f to E so that the player with high dexterity is in front of distance $f + 1$. If $d \notin [0, f]$ and $d \notin [f + E + 1, 2f + 1]$, he can play any integer from E to f.

Observe that in front of a distance equal to $(k + 1)(f + 1)$, with $k \geq 1$, the player with high dexterity, as in proposition 5, simply plays f. By so doing she exploits the unskillfulness of the opponent, who is not able to play less than E, and therefore is unable to bring her in front of a multiple of $f + 1$ (namely $k(f + 1)$). As a matter of facts, given that the opponent is unable to play less than E and given that he cannot play more than f, the player with high dexterity will be in front of a distance going from $(k - 1)(f + 1) + 2$ to $(k - 1)(f + 1) + f + 2 - E$, and therefore she can lead the opponent to the multiple $(k - 1)(f + 1)$ and win the game. This way of playing is illustrated in figure 6.
This behavior is in sharp contrast with the behavior in propositions 2 and 3 which confront two players of different strengths (both of high dexterity $e = 1$), where the stronger player plays 1 to win when she is in front of a multiple of (her strength) $F + 1$. By playing 1, she exploits the fact that the weaker player is unable to play F, hence to bring her back to a multiple of $F + 1$.

In other terms, in propositions 2 and 3, the more gifted (stronger) player exploits the fact that the opponent cannot play large numbers, whereas in proposition 5 and its corollary, the more gifted (with higher dexterity) player exploits the fact that the opponent cannot play low numbers.

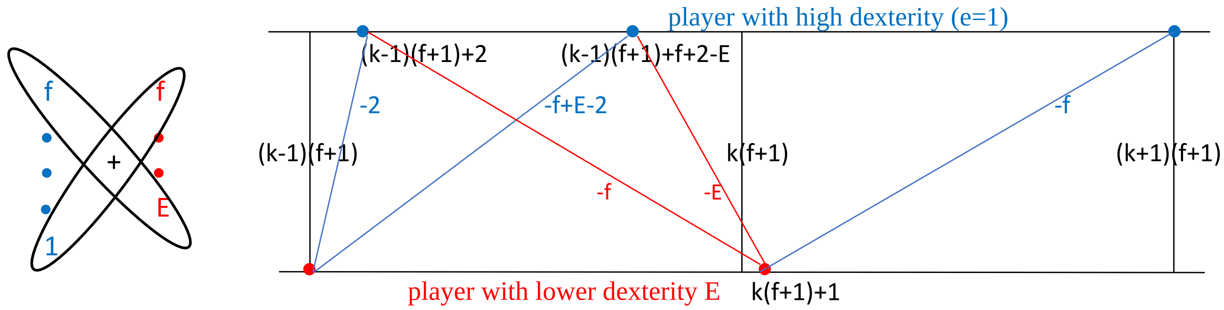


Figure 6 : same strength f, different dexterity, e=1 and f>E>1

The corollary can be exploited in a context with incomplete information. So assume that it is common knowledge of both players that they are of same strength *f*, but that both players ignore the dexterity of the other player. This amounts to saying that each player knows his own dexterity but that he only knows that the opponent's unskillfulness e goes from 1 to $f-1$. Then, if one player is of high dexterity, in a Perfect Bayesian Equilibrium, she can play as in the corollary of proposition 5.

Proposition 6
Suppose that it is common knowledge of both players that they are of same strength f, but that each player ignores the dexterity of the opponent, so that he only knows that the opponent's unskillfulness goes from 1 to f-1. If so, in any Perfect Bayesian Nash Equilibrium of the game, a player with high dexterity (e=1) plays optimally when she adopts the behavior given in the corollary of proposition 5.

Proof of proposition 6
The proof immediately follows from the fact that the strategy of the player with high dexterity in the corollary of proposition 5 is optimal regardless of the unskillfulness E of the opponent, when $E > 1$. So the only thing we have to show is that the strategy remains optimal when the opponent is also of high dexterity ($E = 1$). This is obvious. In front of an opponent of strength *f* and high dexterity $E = 1$, playing *f* in front of a distance that is a multiple of $f + 1$ is optimal because any behavior leads to losing the game (given that the opponent can always lead the player back to a multiple of $f + 1$). And playing the remainder of the division of the distance by f+1 is of course optimal when the remainder is different from 0.■

Observe that we get this optimal behavior without any information on the unknown value E of the opponent, because the player with high dexterity builds her strategy on the crossed cycle she is informed on (*f*=strength of the opponent +1=her own dexterity).

5. Strength is better than dexterity when the strategy sets are of same dimension

Let us now suppose that one of the player is stronger and that the other is of higher dexterity, but that both players have the same number of strategies, $F > f$, $e < E$ and $f - e = F - E$. So the two crossed cycles become $F + e$ and $f + E$ and both players can complete them (figure 7*a*). We should conjecture the existence of a subgame perfect Nash equilibrium that gives the same opportunities to win the game to both players. Yet this conjecture is wrong.

Proposition 7
Consider two players, one player with strength f and unskillfulness e, the other with strength F and unskillfulness E, with F>f, E>e and F+e=f+E. At any SPNE, when the more skilled player plays at distance d, she wins the game if the remainder of the division of d by f+E is 1 up to f, and loses the game if else. When the stronger player plays at distance d, he wins the game if the remainder of the division of d by f+E is 1 up to F, and loses the game if else.
The subgame perfect Nash equilibria of the game are given by:
For the more skilled player, in front of a distance k(f+E)+r, with k an integer and r an integer from 1 to f+E: for r from 1 to e-1, she plays e, for r from e to f, she plays r, for r from f+1 to f+E, she can play any integer from e to f.
For the stronger player, in front of a distance k(f+E)+r, with k an integer and r an integer from 1 to f+E: for r from 1 to E-1, he plays E, for r from E to F, he plays r, for r from F+1 to f+E, he can play any integer from E to F.

Proposition 7 amounts to saying that the stronger player, when starting the game, wins the game in $F/(f + E)\,\% = 1/(1 + e/F)$ situations whereas the player with better dexterity, when starting the game, only wins the game in $f/(f + E)\% = (1/(1 + E/f))$ situations. So a better dexterity is always to the advantage of the opponent, given that it lowers e/F.

If two players are very different (a very strong –non skilled- player and a high skilled -but weak- player) this may lead to very asymmetric probabilities of winning, which are always to the advantage of the stronger player. For example, for $e = 1$, $f = 3$, $E = 4$ and $F = 6$, the stronger player wins the game if he starts the game in front of any distance different from a multiple of 7. By contrast, the more skilled player, if she starts the game, wins the game only in front of a distance that is a multiple of 7 plus 1, 2 or 3. This amounts to saying that in front of a random distance, the stronger player wins the game with probability 0.86, whereas the high dexterity player only wins the game with probability 0.43 . So even if the more skilled player starts the game, she loses more than half of time, when the starting distance is randomly chosen.

Things are less extreme if $F > f > E > e$, in which case $f/(f + E)$ is larger than 0.5, so that the more skilled player, if starting the game, wins the game in front of more than half of the distances. Yet the stronger player, when starting the game, always wins more often. For example, in the more equilibrated context where $e = 1$, $E = 3$, $f = 5$ and $F = 7$, the more skilled player loses the game in front of a distance that is a multiple of 8, or a multiple of 8 plus r, with $r = 6$ or 7. By contrast, the stronger player only loses the game when he is in front of a multiple of 8. So, when starting the game, the high dexterity player loses the game in front of 37.5% possible distances whereas the stronger player, if starting the game, only loses in front of 12.5% possible distances.

Proof of proposition 7
$F + e = E + f$
We proceed by recurrence. We call *A* the more skilled player (she) and *B* the stronger player (he). Let us consider cycle k, i.e. the distances $(k - 1)(f + E) + r$, with r from 1 to $f + E$. We suppose that *A*, in front of a distance d, wins the game if d goes from $(k - 1)(f + E) + 1$ to $(k - 1)(f + E) + f$, loses the game if d goes from $(k - 1)(f + E) + f + 1$ to $k(f + E)$

and that B wins the game if d goes from$(k-1)(f+E)+1$ to $(k-1)(f+E)+F$ and loses the game if d goes from $(k-1)(f+E)+F+1$ to $k(f+E)$.
This is true for $k=1$ when A and B play as in proposition 7 and these actions are optimal (obvious).
We now consider cycle $k+1$ (distances $k(f+E)+r$ with r from 1 to $f+E$).
First, for A, playing e is optimal if she faces a distance from $k(f+E)+1$ to $k(f+E)+e$, because this leads B to a distance that goes from $(k-1)(f+E)+F+1$ to $k(f+E)$ where he loses the game by assumption. And playing r if she faces a distance k(f+E)+r with r from $e+1$ to f is optimal because it brings B in front of the distance $k(f+E)$ where he loses the game.
In a symmetric way, for B, playing E is optimal if he faces a distance from $k(f+E)+1$ to $k(f+E)+E$, because this leads A to a distance that goes from $(k-1)(f+E)+f+1$ to $k(f+E)$ where she loses the game by assumption. And playing r if he faces a distance $k(f+E)+r$ with r from $E+1$ to F is optimal because it leads A to the distance $k(f+E)$ where she loses the game.
Second, A, at any distance from $k(f+E)+f+1$ to $(k+1)(f+E)=k(f+E)+F+e$, loses the game whatever she plays because she can only bring B in front of a distance that goes from $k(f+E)+1$ to $k(f+E)+F$, where B wins the game. So any action is optimal given that all actions lead to losing the game.
In a symmetric way, B, at any distance from $k(f+E)+F+1$ to $(k+1)(f+E)$, loses the game whatever he plays because he can only bring A in front of a distance that goes from $k(f+E)+1$ to $k(f+E)+f$, where A wins the game. So any action is optimal given that all actions lead to losing the game.
So what was is true for cycle k is also true for cycle $k+1$ and we proved that the strategies in proposition 7 are optimal.■

The proof is partly illustrated in figure 7*b*.

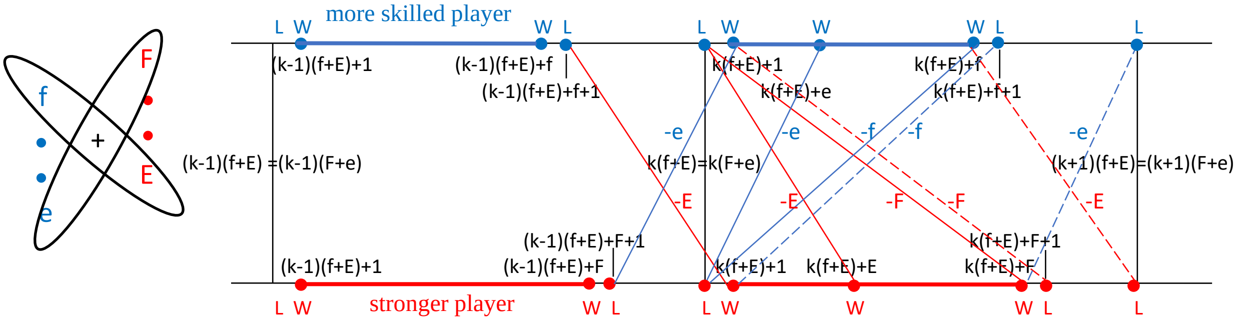


Figure 7a Figure 7b : different strength and dexterity, same number of strategies, F>f, E>e, f-e=F-E

Legend of figure 7b: similar to the legend of figure 4b.

6. Strength and dexterity when the strategy sets are of different sizes

In section 2 and section 4 we established that when two players are of same (high) dexterity but different strength or of same strength but different dexterity, then the player with the larger strategy set always wins the game, provided the initial distance he is confronted to is sufficiently large. These two contexts gave rise to two crossed cycles of different sizes that only the player with the largest strategy set was able to complete.

This leads us naturally to investigating the case where $f - e \neq F - E$, with $F > f$ and $E > e$, so that the more skilled player has a larger or a smaller strategy set than the stronger one (respectively $f - e > F - E$ and $f - e < F - E$). This context gives rise to two crossed cycles f+E and F+e of different sizes, and it is reasonable to conjecture that the player with the larger strategy set will win the game in front of more distances than the other player. This will indeed be the case, but strength keeps an advantage on dexterity.

To show this, we first study the case where the stronger player has a larger strategy set than the more skilled one, so $F - E > f - e$, which implies that only the stronger player is able to complete the two crossed cycles $f + E$ and $F + e$, with $F + e > f + E$. So each time the stronger player can lead the more skilled one to a distance that is a multiple of $(f + E)$ or $(F + e)$ he wins the game. Proposition 8 works with the smallest crossed cycle $f + E$.

Proposition 8
Consider two players, one player with strength f and unskillfulness e, the other with strength F and unskillfulness E, with F>f, E>e and F+e>f+E.
If F>f+E, in the SPNE, the stronger player wins the game in front of any distance if he has the opportunity to play. The more skilled player only wins the game if she starts the game at a distance lower than or equal to f.
If F<f+E, then, in any SPNE, if the stronger player is confronted to any distance larger than (f+E)K, with K the integer part of (e-1)/(F+e-(f+E)), he always wins the game. In each cycle k (i.e. at distances (k-1)(f+E)+r, with r an integer from 1 to f+E and k an integer such that 1≤ k ≤K), the stronger player only loses the game at the distances from kF+(k-1)e+1 to k(f+E); this number of distances is decreasing in k .
The more skilled player, if confronted to a distance larger than (f+E)(1+K) always loses the game. In each cycle k, with 1≤ k ≤ 1+K, she only wins the game at the distances from (k-1)(F+e)+1 to kf+(k-1)E; this number of distances is decreasing in k .

Proof of proposition 8
If $F > f + E$, then the stronger player (*B*, he) obviously wins the game at all distances of the first cycle (distances from 1 to $f + E$). He ends – hence wins- the game at all distances from $f + E$ to *F*. Then, at the distances from $F + 1$ to $f + 2E$, he can play *E*, so that the more skilled player (*A*, she) can only lose the game. And at the distances from $f + 2E + 1$ to $2(f + E)$, he plays such that the more skilled player is in front of the distance $f + E$, where she loses the game. So *B* wins the game at all distances from 1 to $2(f + E)$ and therefore also wins the game at larger distances, because *A* can never lead him to a distance where he loses the game. It follows from this fact that *A* can only win the game if she starts the game at a distance lower than or equal to *f*.
If $F < f + E < F + e$, we suppose that, in cycle *k* (distances $(k-1)(f + E) + r$, with *r* from 1 to $f + E$), the stronger player wins the game at the distances from $(k-1)(f + E) + 1$ to $kF + (k-1)e$ and loses the game at the distances from $kF + (k-1)e + 1$ to $k(f + E)$. We also assume that the more skilled player wins the game at the distances from $(k-1)(F + e) + 1$ to $kf + (k-1)E$ and loses the game at the distances from $kf + (k-1)E + 1$ to $k(F + e)$ (we go a little beyond cycle *k*).
These facts are true for $k = 1$ (obvious).

We now study cycle k+1(distances $k(f+E)+r$, with r from 1 to $(f+E)$).
The facts in cycle k imply that the stronger player B wins the game at the distances from $k(f+E)+1$ to $(k+1)F+ke$, because by playing adequately from E to F, he leads A in front of a distance going from $kf+(k-1)E+1$ to $k(F+e)$, where A loses the game. And these facts imply that the more skilled player A wins the game at the distances from $k(F+e)+1$ to $(k+1)f+kE$, because by playing adequately from e to f, she leads B to a distance going from $kF+(k-1)e+1$ to $k(f+E)$ where B loses the game.
In turn these new facts imply that the stronger player B loses the game at the distances from $(k+1)F+ke+1$ to $(k+1)(f+E)$, because, whatever he plays, from E to F, he brings A to a distance from $k(F+e)+1$ to $(k+1)f+kE$, where A wins the game. And these new facts also imply that the more skilled player A loses the game at the distances from $(k+1)f+kE+1$ to $(k+1)(F+e)$, because, whatever she plays, from e to f, she leads B to a distance from $k(f+E)+1$ to $(k+1)F+ke$, where B wins the game.
So the facts are true for all cycles k, but only make sense if the distances from $kF+(k-1)e+1$ to $k(f+E)$ exist, hence for $k(f+E)-kF-(k-1)e \geq 1$ i.e. $k \leq (e-1)/(F+e-(f+E))$. Observe that $k((f+E)-(F+e))+e$ and $(k-1)(f+E-(F+e))+f$ are decreasing in k given that $f+E < F+e$, so the number of B's losing distances, respectively A's winning distances, diminish from one cycle to the next, up to disappear. ■

The proof is illustrated in figure 8*b*.

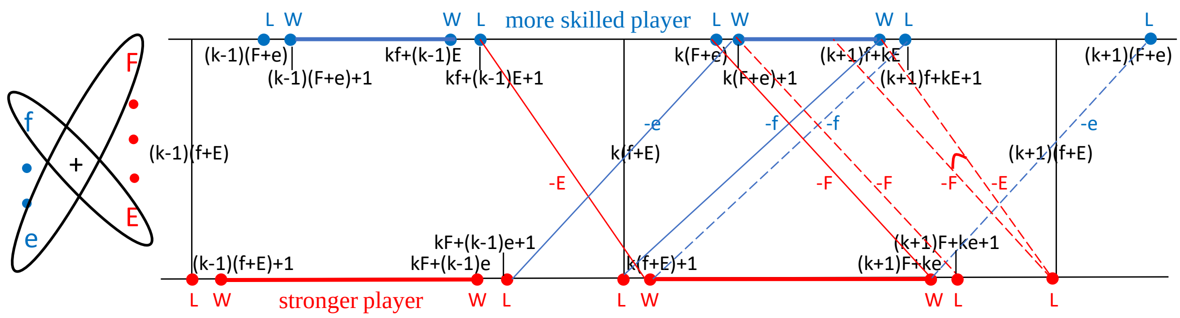


Figure 8a   Figure 8b : different strength and dexterity, different number of strategies, F>f, E>e, F-E>f-e

Legend of figure 8b: similar to the legend of figure 4b

We comment this proposition with two examples, $f=4$, $e=2$, $F=7$, $E=4$ and $f=6$, $e=4$, $F=8$, $E=5$.

In the first example, we get $K=1$ and, in the first cycle (distances from 1 to 8), the stronger player B loses the game when starting at the distances from $F+1$ to $f+E$, i.e. only at distance 8. At any other distance, he wins the game. And the more skilled player A, when starting the game, only wins the game in the first cycle at the distances from 1 to $f=4$, and in the second cycle at the distances from $8+e=10$ to $8+f=12$. At any other distance she loses the game.

In the second example, we get $K=3$. In the first cycle (distances from 1 to 11), the stronger player B loses the game when starting the game at the distances from $F+1=9$ to $f+5=11$. In the second, respectively in the 3rd cycle, he loses the game at the distances from $2F+e+1=21$ to $2(f+E)=22$, respectively at the distance $3F+2e+1 = 3(f+E)=33$. In front of all other distances he wins the game.

The more skilled player wins in front of the distances from 1 to 6 in the first cycle. In the second cycle, respectively in the 3rd and in the 4th cycle, she wins at the distances from $F + e + 1 = 13$ to $2f + E = 17$, respectively at the distances from $2F + 2e + 1 = 25$ to $3f + 2E = 28$, and at the distances from $3F + 3e + 1 = 37$ to $4f + 3E = 39$. In front of all other distances she loses the game.
So, when we focus on the three first cycles, i.e. at the distances from 1 to 33, where the stronger player may lose, we see that the stronger player can only lose the game in front of 6 distances, hence that he wins the game in front of 27 distances. By contrast, the more skilled player, when starting the game, can only win the game in front of 15 distances.

Things are different when the more skilled player has a larger strategy set, i.e. when $f - e > F - E$, with $f < F$ and $e < E$. This time, only the more skilled player is able to complete the two crossed cycles $f + E$ and $F + e$ (see figure 9*a*), so that by bringing the stronger player to a multiple of $(f + E)$ or $(F + e)$ she wins the game. But the winning process is slower than in the previous case. Proposition 9 again works with the shortest crossed cycle, $F + e$.

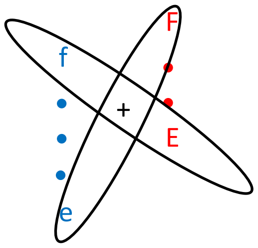
Proposition 9
Consider two players, one player with strength f and unskillfulness e, the other with strength F and unskillfulness E, with F>f, E>e and f+E>F+e. In any SPNE, when the more skilled player is confronted to any distance larger than (F+e)K, with K the integer part of (E-1)/(f+E-(F+e)), she always wins the game. In each cycle k (distances (k-1)(F+e)+r, with r from 1 to (F+e) and k an integer from 1 to K), she only loses the game at the distances from kf+(k-1)E+1 to k(F+e); this number of distances is decreasing in k .
The stronger player, if confronted to a distance larger than (F+e)(1+K) always loses the game. In each cycle k, with 1≤ k ≤ 1+K, he only wins the game at the distances from (k-1)(f+E)+1 to kF+(k-1)e; this number of distances is decreasing in k .

We observe that, for a same value $|(f + E) - (F + e)|$, the threshold cycle K (largest cycle where the player with the largest strategy set can lose the game) in proposition 8 is lower than in proposition 9 (because $e < E$). We also see that, in a cycle k (lower than or equal to the threshold cycle), when the more skilled player has the largest strategy set, she wins the game in front of $(k - 1)|(f + E) - (F + e)| + f$ distances, whereas the stronger player, in the same cycle, when he has the largest strategy set, wins at more distances given that he wins the game at $(k - 1)|(f + E) - (F + e)| + F$ distances. By constrast, when the stronger player has the largest strategy set, the more skilled player loses in front of $k|(f + E) - (F + e)| + E$ distances, whereas the stronger player, when the more skilled player has the largest strategy set, only loses in front of $k|(f + E) - (F + e)| + e$ distances. So clearly strength keeps an advantage on dexterity.

Proof of proposition 9
We suppose that in cycle k (distances $(k - 1)(F + e) + r$, with r from 1 to $F + e$) the more skilled player (A, she) wins the game at the distances from $(k - 1)(F + e) + 1$ to $kf + (k - 1)E$ and loses the game at the distances from $kf + (k - 1)E + 1$ to $k(F + e)$. We suppose also that the stronger player (B, he) wins the game at the distances from $(k - 1)(f + E) + 1$ to $kF + (k - 1)e$ and loses the game at the distances from $kF + (k - 1)e + 1$ to $k(f + E)$ (we go a little beyond cycle k).
These facts are true for $k = 1$ (obvious).
We now study cycle $k+1$ (distances $k(F + e) + r$, with r from 1 to $F + e$).

The facts in cycle k imply that the more skilled player A wins the game at the distances from $k(F+e)+1$ to $(k+1)f+kE$ because, by playing from e to f, she brings the stronger player in front of a distance going from $kF+(k-1)e+1$ to $k(f+E)$ where he loses the game. Symmetrically, the stronger player B wins the game at the distances from $k(f+E)+1$ to $(k+1)F+ke$, because by playing from E to F, he brings the more skilled player in front of a distance going from $kf+(k-1)E+1$ to $k(F+e)$, where she loses the game. These new facts imply that A loses the game at the distances from $(k+1)f+kE+1$ to $(k+1)(F+e)$, because playing any action from e to f can only lead the stronger player to distances going from $k(f+E)+1$ to $(k+1)F+ke$, where he wins the game. And B loses the game at the distances from $(k+1)F+ke+1$ to $(k+1)(f+E)$ because playing any action from E to F can only lead the more skilled player to distances going from $k(F+e)+1$ to $(k+1)f+kE$, where she wins the game.
So the facts are true for all cycles k, but only make sense if the distances from $kf+(k-1)E+1$ to $k(F+e)$ exist, i.e $k \leq \frac{E-1}{f+E-(F+e)}$. ■

The proof of proposition 9 is partly illustrated in figure 9b.



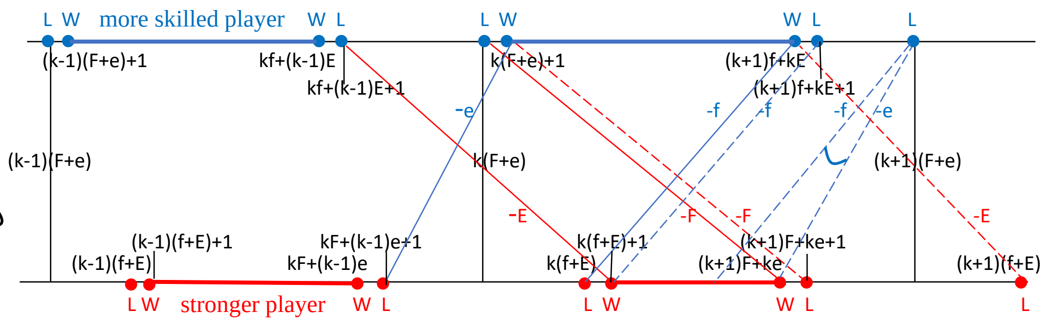


Figure 9a   Figure 9b: different strength and dexterity, different number of strategies, F>f, E>e, F-E<f-e

Legend of figure 9b: similar to the legend of figure 4b

We compare proposition 9 to proposition 8 thanks to the example: $f=6$, $e=3$, $F=8$ and $E=6$. This example is close to the second example we studied for proposition 8. For $f=6$, $e=3$, $F=8$ and $E=6$, we have again two crossed cycles of length 11 and 12 but this time $F+e=11<f+E=12$. We have also two strategy sets with 4 and 3 strategies, but this time the more skilled player has the largest strategy set.

According to proposition 9, the threshold cycle K checks $K=5$, instead of $K=3$ in the example for proposition 8, which clearly shows that the stronger player has more possibilities to win the game than the more skilled player in the symmetric configuration.

To put it more precisely, respectively in the 1st, 2nd, 3rd, 4th and 5th cycle, the more skilled player A loses the game respectively in front of 5 distances (from 7 to11), 4 distances (from 19 to 22), 3 distances (from 31 to 33), 2 distances (from 43 to 44) and 1 distance (55). She wins the game in front of all distances larger than 55. The stronger player B, respectively in the 1st, 2nd, 3rd, 4th, 5th and 6th cycle, wins the game in front of respectively 8 distances (from 1 to 8), 7 distances (from 13 to 19), 6 distances (from 25 to 30), 5 distances (from 37 to 41), 4 distances (from 49 to 52) and 3 distances (from 61 to 63). He loses the game at all distances larger than 63.

So when we focus on the three first cycles (distances from 1 to 33), we see that, when starting the game, both the more skilled player and the stronger player win the game in front of 21 distances, despite the more skilled player has a larger strategy set. This contrasts with the previous example for proposition 8, where the stronger player, when starting the game, won the game in front of 27 distances among 33, whereas the more skilled player, when starting the game, only won the game in front of 15 distances. So strength has a strong advantage on dexterity, when the starting distance *D* is not too large.

7. Conclusion, incomplete information and heuristics of behavior

Well, several facts emerge from our study.

First of all, we see that strength has a significant advantage on dexterity. This is obvious when the more skilled player and the stronger player have the same number of strategies (section 5), but keeps true when the size of both strategy sets is different. In the latter situation, the player with the larger strategy set always wins the game when the starting distance *D* is larger than a given threshold value. Yet, on the one hand, this threshold value is larger when the more skilled player has a larger strategy set than in the opposite situation. And, on the other hand, in front of distances lower than this threshold value, the winning distances of the stronger player (when the more skilled one has a larger strategy set) are more numerous than the winning distances of the more skilled player (when the stronger one has a larger strategy set). For the hammer-mail game, this is clearly not good news, but can be tempered by the observation that the dexterity we introduce is a quite limited notion of dexterity. We can reasonably expect that a more sophisticated notion of dexterity would help us nuancing this result! Yet more generally, the results in the paper establish that, in a Nim game, reducing the strategy set from below, i.e. eliminating small strategies -which amounts to decreasing dexterity-, is not as damaging as reducing the strategy set from above, i.e. eliminating large strategies – which amounts to decreasing the strength.

Another fact is that we chose in the proofs and the propositions to only work with one crossed cycle, most often the smallest one (from proposition 4 to proposition 9). This is of no importance as regards the winning and losing distances for the players. Yet it is interesting to do a study with the largest crossed cycle in that this will help us to give some hints of behavior in presence of incomplete information on strength or dexterity (with $F-E>f-e$ or $F-E<f-e$).

So, without loss of generality, we suppose that $f-e\ >F-E$, hence $f+E>F+e$, and call player *A*(she), respectively player *B*(he), the player with strength *f* and unskillfulness *e*, respectively the player with strength *F* and unskillfulness *E*. From now on, *F* may be larger or smaller than *f* and *e* may be smaller or larger than *E* , but we suppose that $f<F+e$, to exclude the case where player *A* wins the game in front of any distance. Player *A* can complete both crossed cycles $f+E$ and $F+e$ and we now focus on the largest crossed cycle $f+E$.
We suppose that in cycle k (distances $(k-1)(f+E)+r$, with r from 1 to $f+E$) player *A* loses the game at the distances from $kf+(k-1)E+1$ to $k(F+e)$ and wins the game at the distances from $k(F+e)+1$ to $(k+1)f+kE$ (so we go a little beyond cycle k).We also suppose that player *B* wins the game in front of the distances from $(k-1)(f+E)+1$ to $kF+(k-1)e$ and loses the game at the distances from $kF+(k-1)e+1$ to $k(f+E)$. These facts are true for $k=1$. For $k=1$, *A* also wins the game at the distances from 1 to *f*.

We now consider cycle $k+1$ (distances $k(f+E)+r$, with r from 1 to $f+E$).
The facts in cycle k imply that player B wins the game at the distances from $k(f+E)+1$ to $(k+1)F+ke$, because, by playing from E to F, he can bring A to distances going from $kf+(k-1)E+1$ to $k(F+e)$ where A loses the game. By contrast, B loses the game at the distances from $(k+1)F+ke+1$ to $(k+1)(f+E)$, because any action from E to F leads A to distances going from $k(F+e)+1$ to $(k+1)f+kE$ where A wins the game. In turn, it follows that player A loses the game at the distances from $(k+1)f+kE+1$ to $(k+1)(F+e)$, because any of her action from e to f leads B to distances going from $k(f+E)+1$ to $(k+1)F+ke$ where B wins the game. By contrast, A wins the game at the distances from $(k+1)(F+e)+1$ to $(k+2)f+(k+1)E$, given that, by playing from e to f, she can lead B to distances going from $(k+1)F+ke+1$ to $(k+1)(f+E)$ where he loses the game.
So the reasoning is true for all cycles k, as long as $k(F+e)-kf-(k-1)E \geq 1$ hence $k \leq \frac{(E-1)}{f+E-(F+e)}$.

This way of reasoning is illustrated in figure 10.

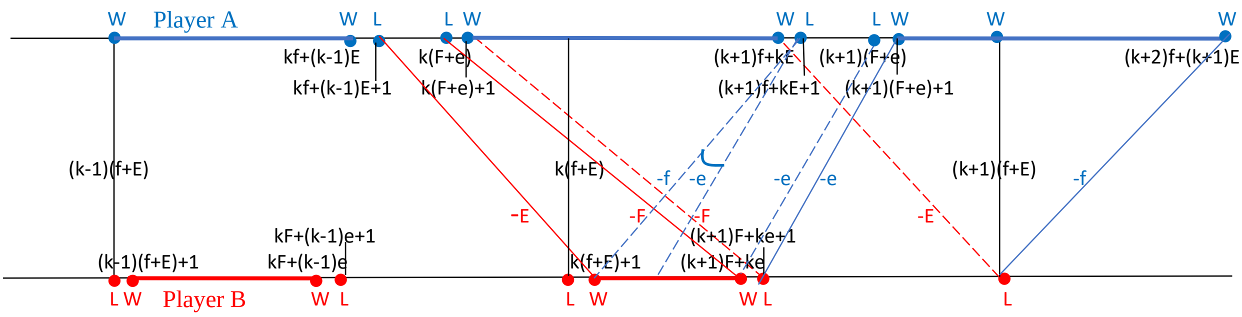


Figure 10: different strength and dexterity, different number of strategies, f-e>F-E

Legend of figure 10: similar to the legend of figure 4b

It follows from the reasoning, as expected, that figures 9*b* and 10 are identical except for the size of the chosen cycle. Yet figure 9*b* and figure 10 will be useful in different contexts of partial incomplete information.

As a matter of facts, it derives from our study that to build a strategy, each player has to know at least one crossed cycle, so has to know either the strength of the opponent or his dexterity. It is not possible to propose a behavior when a player both ignores the strength and the dexterity of the opponent. Sometimes, like in proposition 3 and in proposition 6, only knowing the dexterity or the strength is enough to build a Perfect Bayesian Equilibrium. But most often, this is not sufficient. Yet, when a player knows one crossed cycle, we can propose a heuristic of behavior that fits well if luckily the player has the largest strategy set: anyhow, if he has the smallest strategy set, he has few chances to win the game, at least if the other player follows the heuristic. In other terms, as long a player knows the lowest strategy of the opponent or his largest one, we can propose a way to play the game.

So let us again consider player A (with strength f and dexterity e) and player B (with strength F and dexterity E). F may be larger or smaller than f and e may be smaller or larger than E. For player A, there are only two possible configurations of partial incomplete information given in figure 11*a* and figure 11*b*. In configuration 1, player A knows the opponent's strength F but not his dexterity E, in configuration 2, player A knows the opponent's dexterity E but not his strength F.

We first study configuration 1. In this configuration, if player A expects to be the player with the largest strategy set, hence $f - e > F - E$, the crossed cycle she knows, $F + e$, is the smallest one, and we can build a strategy (a heuristic of behavior) that uses this crossed cycle.

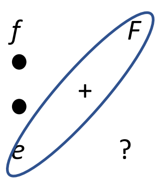


Figure 11a Configuration 1

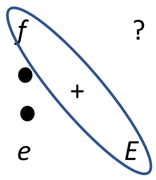


Figure 11b Configuration 2

The heuristic is: at any distance d, complete to a multiple of F+e if possible; if this is not possible, play e when the remainder r of the division of d by (F+e) goes from 1 to e-1, and play f for r=0 and r going from f+1 to F+e-1.

Observe that this is what we proposed to a player of high dexterity ($e = 1$) when she meets a player of same strength but unknown dexterity (proposition 6). This heuristic immediately follows from the behavior in figure 9b. In figure 9b, which illustrates the optimal behavior behind proposition 9, there is a degree of liberty: when being at the distances from $(k+1)f + kE + 1$ to $(k+1)(F+e)$, player A loses the game whatever she plays, so she can play f. By contrast, when she is at distance $(k+1)f + kE$, she has to play f to make player B lose the game, and at all distances $k(F+e) + r$, with r from $f + 1$ to $(k+1)f + kE - k(F+e)$, playing f is a good way of playing.

So by playing f at all distances from $k(F+e) + f$ to $(k+1)(F+e)$, player A is sure to play f at all the distances she has to do so (namely at the distance $(k+1)f + kE$) even if she does not know E. So she plays well, even in a context of incomplete information on E, when $f - e > F - E$. Moreover, by so playing, she leads B faster to lower distances. So if E, luckily, is large, she brings B faster to distances where he loses the game (the green segment in figure 12a is growing in E). So, given that player B, when $f - e > F - E$, cannot win more often than at his winning distances in figure 9b, even if he knows player A's strategy, player A wins at least as often as in a context of complete information. These facts are illustrated in figure 12a.

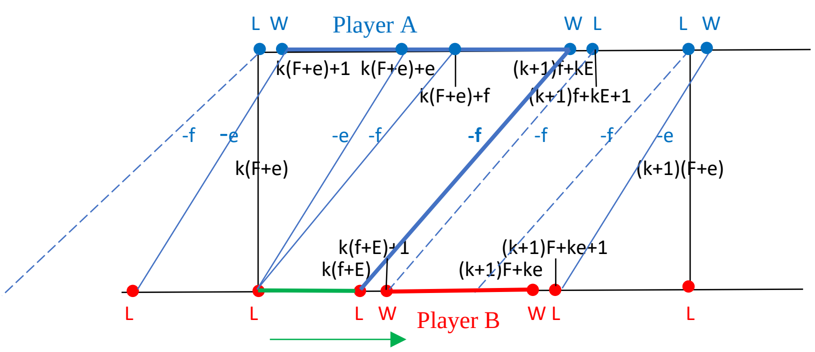


Figure 12a : player A does not know E

We now study configuration 2. In this configuration, if player A expects to be the player with the largest strategy set, hence $f - e > F - E$, the crossed cycle she knows, $f + E$, is the largest one, and we can build a strategy (heuristic of behavior) that uses this crossed cycle (so that works with figure 10).

The heuristic is: at any distance d complete to a multiple of $f+E$ if possible; if this is not possible, play e.

Observe that this is what we proposed to a player of high dexterity ($e=1$) who faces a player of same dexterity but unknown strength (proposition 3). Here, we exploit optimally the degree of liberty possible in the context of complete information studied in figure 10. In this context, player A (the player with the largest strategy set) loses at the distances from $(k+1)f+kE+1$ to $(k+1)(F+e)$ whatever she plays, so she can play e. But when she is at distance $(k+1)(F+e)+1$ she has to play e to make player B lose the game. And playing e at the distances from $k(f+E)$ to $k(f+E)+e$ and from $(k+1)(F+e)+1$ to $(k+1)(f+E)$ leads player B to losing the game. So, by playing e each time she cannot bring player B to a multiple of $(f+E)$, player A, on the one hand, is sure to play e when it is necessary, namely at distance $(k+1)(F+e)+1$, even if she does not know F. So she plays well even in a context of incomplete information on F. Moreover, on the other hand, by playing e, when F, luckily, is low, player A brings player B faster to the green segment (see figure 12b) where B is losing the game. So, given that player B, when $f-e>F-E$, cannot win more often than at his winning distances in figure 10, even if he knows player A's strategy, player A wins at least as often as in a context of complete information. These facts are illustrated in figure 12b.

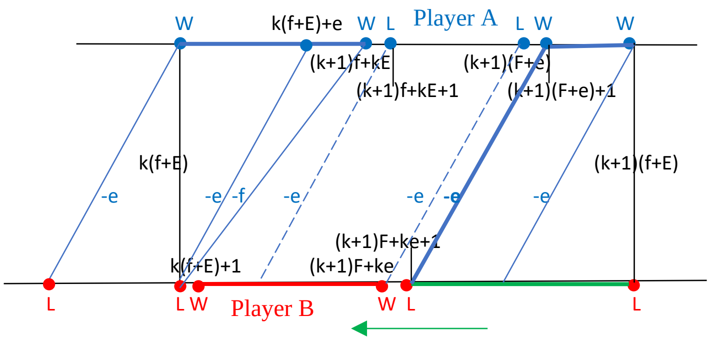


Figure 12b : player A does not know F

In some way, these two heuristics of behavior, that exploit e and f, may partly explain why so many players in classic Nim games choose to either play f or e when they are unable to discover the optimal strategy.

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