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The puzzling three-player beauty contest game: play 10 to win.

Gisèle Umbhauer ♣

May 2023

Abstract

In this paper, we study the 3-player beauty contest game. This 3-player guessing game has the same Nash equilibrium than the usual (large) N-player beauty contest game but it has also nice specific properties. To highlight these properties, we study classroom experiments on 2-player, 3-player and large N-player guessing games, both from a theoretical and behavioral point of view. The spirit of the paper is the spirit of the French newspaper *Jeux et Stratégie* which, in the early eighties, proposed the beauty contest game to his fun of logic readers. As a matter of facts, we wonder if it is possible to win the 3-player guessing game. So we show that, despite the 3-player beauty contest game has no weakly dominant strategy, it is possible to play it in a way that leads to win with a large probability, provided the parameter a is lower than 0.75. And we argue that playing 10 for a=0.6 ensures a large probability to win.

Keywords: beauty-contest game, 3-player game, guess, nombre d'or, dominance, win area, behavioral heuristic.

JEL Classification : C72, C9

1. Introduction

In 1995, Nagel initiated a huge literature both on the N-player beauty-contest game and on level-k reasoning. In the classic version of the beauty-contest game, also called guessing game, N players simultaneously choose a number in the interval [0,100] and the winner is the player whose number is closest to a times the mean of all proposed numbers. In case of a tie, the winners share the prize. Many things have been written on the (large) N-player guessing game and some papers focused on the very special 2-player guessing game (among them Grosskopf & Nagel, 2008, and Costa-Gomez &Crawford, 2006). Yet, at least to our knowledge, only few authors, among them Ho et al (1998) and Breitmoser (2012), are interested in N-player guessing games with N small but different from 2.

This is quite astonishing, given the strong logical differences between the large N-player guessing game and the 2-player guessing game. For N=2, 0 is the weakly dominant strategy that always wins the game, given that $aX/2$ is always closer to 0 than to X, for parameters $a < 1$. So it is enough to play 0 to win the game, without guessing anything about the behavior of the other player. In other terms, the 2-player guessing game *is not a guessing game*, by contrast to the large N-player guessing game, where the winning strategy strongly depends on the way

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other players play the game. Usually, as is well known, in the large *N*-player beauty contest game, *guessing* leads to iterated dominance and level-*k* reasoning, at least if people play the game for the first time. In the large *N*-player guessing game, 0, which is both the Nash equilibrium strategy and the only strategy obtained by (infinite) iterated dominance, is generally not the winning strategy. The winning strategy, at least if players are not trained in this game, is often closer to $50a^2$, the level-2 behavior strategy, in that many players simply play $50a$, the level-1 behavior amount.

Given the gap between the very easy winning behavior in the 2-player guessing game and the uncertain winning strategy in the large *N*-player guessing game, one might be interested in a game in-between, namely the 3-player guessing game. This game reveals to be quite different both from the 2-player game and from the large N-player game. In facts, the 3-player beauty contest game has nice special properties, that can be exploited in a way to win the game with a large probability, provided the parameter a is not too large (lower than ¾).

So we work on this game, with a special focus on $a = 3/5$. The spirit of the paper is the spirit of the French newspaper "Jeux et Stratégie", already mentioned by Moulin (1984, 1986) and nicely detailed in Nagel et al. (2017). The newspaper - see Ledoux (1981)- proposed the large *N*-player guessing game to its readers, fun of logic, who all wanted to win the prize. And this is exactly what we want to do in this paper: we aim to win the 3-player beauty contest game, at least for $a = 3/5$.

To do so, we do a theoretical and a behavioral study. We start, in section 2, by presenting and analyzing two classroom experiments around the 3-player beauty contest game. Then we show theoretical properties of this game. In section 3, we fix a value *Y* for one opponent's guess and show that, for all the guesses *Z* of the other opponent, there exists an interval of winning guesses *X*. This interval draws attention to a threshold parameter value, $a = 3/4$. Section 4 puts into light the stronger impact of non-iterated dominance for $N = 3$ (by contrast to *N* large) and how this fact helps playing the 3-player guessing game. In section 5, like Breitmoser (2012), for each number *X*, we look for the area of couples of opponents' guesses such that *X* is a winning strategy. In section 6, we show why playing 10 is a good way to play in the 3-player game with $a = 3/5$, whether the players are trained or not in guessing games. In both classroom experiments, playing 10 leads to win with a probability exceeding 0.81. We conclude, in section 7, with a rather counterintuitive result: if one expects that the opponents play integers, then it is better to play an integer too, rather than playing an integer plus or minus a small quantity.

2. Two classroom experiments

In the classic large *N*-player beauty contest game with parameter a – now called Guess(*N*, a) throughout the paper-, a large number *N* of players simultaneously propose a number in $[0, 100]$, and the winner is the player who is closest to a times the mean of the *N* proposed numbers. The prize is shared among the winners in case of a tie.

The rules of 3- player guessing games and 2-player guessing games are those of Guess(N,a), with N=3 and N=2.

We study two classroom experiments on a 3-player guessing game with parameter a – called Guess(3,a) - in two different contexts. The first experiment was run at the University of

Strasbourg during the first lecture in a third-year class in game theory, in the academic year 2022-2023, so the students did not know the concepts of dominance and Nash equilibrium (at least in games with infinite pure strategy sets). In total 241 students participated to this first experiment. In a within-subject framework, the 241 students first played the usual large N-player guessing game with parameter a=3/5, Guess(241,3/5), before playing the 3-player guessing game with parameter $a = 3/5$, Guess(3,3/5). Several examples (with N=5) were explained to the students before the beginning of the experiment, to be sure that the rules of the games were understood. The students were invited to explain their choices by writing.

We ran a second experiment a few weeks later, in the same third-year class in game theory, when the students knew the concept of dominance and Nash equilibrium. They were also informed about the winning strategy (namely 19-which is not far from $50a^2$) in Guess(241,3/5), one of the two games they played a few weeks ago[1]. In this second experiment, 199 students, in a within-subject framework, first played the 2-player guessing game with parameter $a = 3/5$, Guess(2,3/5), before playing again Guess(3,3/5). In the 2-player guessing game, a player who plays 0 always wins the game (or is one of the two winners). In order to help the students to discover this fact, we informed them, before they played Guess(2,3/5), that this game has a winning strategy. By so doing, we wanted to avoid that the students simply replicate (or best replied to) the numbers chosen in Guess(241,3/5) a few weeks before (we namely wanted to avoid that they play 3/5.19). The students viewed this experiment as an exercise they had to solve. And they had enough time to find the winning strategy. So clearly the students were not in the same context than the players in Grosskopf & Nagel (2008), who were not informed on the existence of a winning strategy. And it worked: 20.6% of the students played 0, and many students played low numbers (27.14% played 1 or below, 30.65% played 5 or a lower number), in that they discovered, often after many numerical trials, that the winning number is always the lowest proposed one. These percentages are far from the 9.85% of participants playing 0 in Grosskopf and Nagel's (2008) experiment on Guess(2,2/3). After having played Guess(2,3/5), the students were invited to play again Guess(3,3/5), the game they played a few weeks before. We did not talk about the existence of a winning strategy -because it does not exist- but we hoped that, given the proximity of N=2 and N=3, the students aimed to find such a strategy. We also gave them two contrasted examples, one in which the winning number is the lowest one (the three proposed numbers were 17, 30 and 53) and one in which the winning number is the middle one (the three proposed numbers were 14, 31 and 72). With this second example, we wanted to avoid that the students who played 0 in Guess(2,3/5) simply replicate their behavior in Guess(3,3/5). The students had again enough time to make many numerical trials. We hoped that what they learned in Guess(2,3/5) would help them to better exploit some special properties of Guess(3,3/5), yet it only partly worked. The students were again invited to justify their choices by writing.

Figures 1a and 1b, respectively Figures 2a and 2b, display the way the students played in the first, respectively in the second experiment.

[1] Yet they were not informed about the Nash equilibrium in the guessing games and we gave no information about the 3-player beauty contest game Guess(3,3/5) they played a few weeks ago.

As expected, the students played Guess(3,3/5) differently in both experiments. This is confirmed by a Kolmogorov Smirnov test (p-value =1.786x10^{-8}) and all other tests. When they play Guess(3,3/5) after having played Guess(241,3/5), the mean proposed value is 31.71, whereas it shrinks to 23.76 when they play Guess(3,3/5) after Guess(2,3/5). A simple look at Figure 1*b* and Figure 2*b* shows that the distribution in Figure 2*b* shifts to the left in comparison to the distribution in Figure 1*b*.

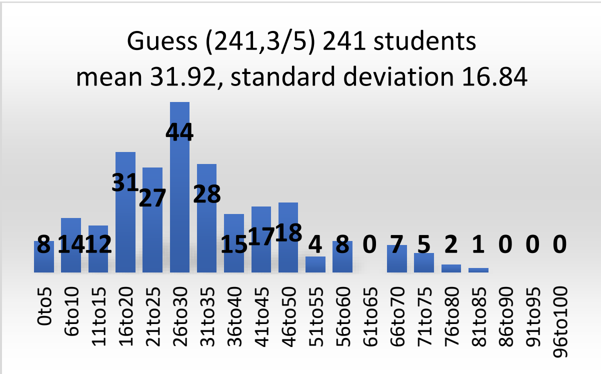

Figure 1a: Guess(241,3/5), 1st experiment

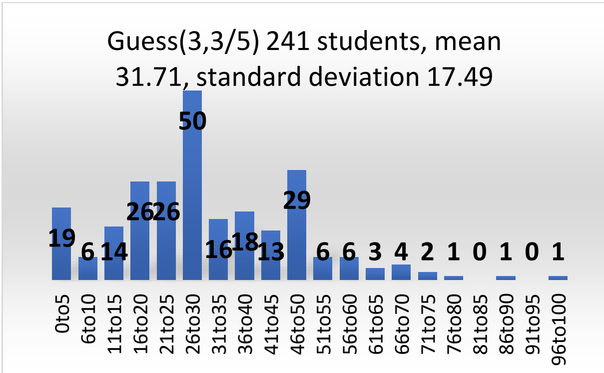

Figure 1b: Guess(3,3/5) 1stexperiment

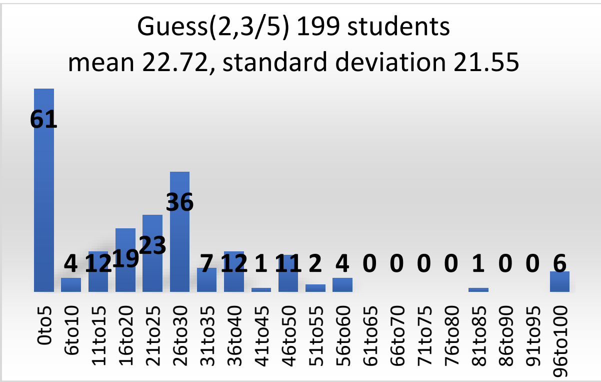

Figure 2a: Guess(2,3/5), 2nd experiment

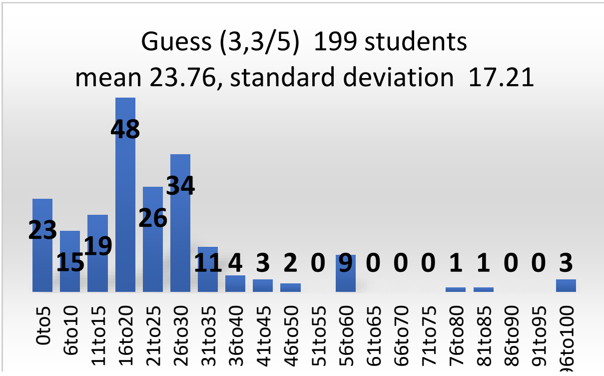

Figure *2b: Guess(3,3/5) 2nd experiment*

In both experiments, the students, while playing Guess(3,3/5), are influenced by the game they played just before. So, in the first experiment, the mean proposed value in Guess(241,3/5), 31.92, is close to the mean proposed value in Guess(3,3/5), 31.71. The same is true in the second experiment: the mean proposed value in Guess(3,3/5), 23.76, is only slightly larger than the mean value proposed in Guess(2,3/5), 22.72.

In the first one-shot experiment, both distributions are similar according to the Kolmogorov-Smirnov test (p-value 0.9262), and this is confirmed by a Khi-2 test and a Wilcoxon-Pratt signed rank test. Yet this fact does not mean that the students play Guess(3,3/5) and Guess(241,3/5) in the same way. Among the 241 students, only 17.43% choose the same number in both games, 42.74% choose a lower number and 39.84% choose a higher one. To put it more precisely, according to the students' choices and explanations, the students, when switching from Guess(241,3/5) to Guess(3,3/5), understand that their own choice has more impact on the mean in Guess(3, 3/5) than in Guess(241,3/5), but this leads to two opposite shifts. The students think that, to win, they have to propose a number between the two opponents' numbers (remember that we only gave examples with $N = 5$, so the students are not compelled to observe that their conjecture can be wrong). This induces some students playing low, respectively large numbers in Guess(241,3/5), to propose a larger number (up to 50 units larger), respectively a lower number, in Guess(3,3/5). So among the 106 players playing less than 30 in Guess(241,3/5), 66 choose a larger number in Guess(3, 3/5) and only 27 choose a lower number. And among the 105 students playing more than 30 in Guess(241,3/5), the behavior is reversed: 67 choose a lower number in Guess(3,3/5) and only 23 choose a higher one. There are also more students playing between 45 and 55 (17.43%) in Guess(3,3/5) than in Guess(241,3/5) (10.79%) and some students do not hesitate to shift from numbers below 10 in Guess(241,3/5) to 50 in Guess(3,3/5). In general, the students view Guess(3,3/5) as being more random and difficult to play than Guess(241,3/5).

In the second experiment, despite the means are close in Guess(2,3/5) and Guess(3,3/5), the distributions are different, as confirmed by a Kolmogorov Smirnov test (p-value 0.0014). Many students understand that that there is a strong difference between the 2-player game and the 3-player game, which explains that they do not adopt the same behavior in both games. Thanks to the two given examples and thanks to their own many trials, they observe that, in Guess(3,3/5), it is not enough- in contrast to what happens in Guess(2 3/5)- to play a very low number to win and that it is sometimes better to play a number between the two opponents' ones, but a not loo large one. This explains that among the 83 students playing less than 20 in Guess(2,3/5), 71.08% choose to increase their number – and only 6.02% choose to decrease it- when switching to Guess(3,3/5), but only 6.02% choose to increase it above 30. This also explains the shift from a mode on 0 in Guess(2,3/5) to a mode on 20 in Guess(3,3/5).

Anyhow, in the four games, the students' behavior is very dispersed. The standard deviations are large: 16.84 in Guess(241,3/5), 17.49 in Guess(3,3/5) in the 1st experiment, 21.55 in Guess(2,3/5) and 17.21 in Guess(3,3/5) in the 2nd experiment. In all games, there is a large range of played numbers. Almost all integers from 0 to 50 are played by at least one or two persons in the two games in the first experiment, and almost all integers from 0 to 40 are played by at least one or two persons in both games in the second experiment. The very strong standard deviation in Guess(2,3/5) is namely due to the fact that some students find the winning strategy (20.6% play 0 and 30.65% play a number lower than or equal to 5) and the other students do not (hence they rather play like in a large N-player guessing game-12.06% play 30). Moreover, some students play 100, trying to use their strong power to incite the opponent to play 100. To summarize, even if having some training in game theory, even if having enough time to make many trials, the students view 2-player and 3-player guessing games as being rather difficult. As said above, they perceive the 3 – player game as being more difficult than the classic large N-player guessing game: they say that the mean behavior of two opponents is more difficult to guess than the mean behavior of a large number of opponents which they often -wrongly- estimate around 50. And in some degree, they are right: as is well known in theory of mind, what matters in the N-player beauty contest game is not the truth, but what others think it is. So even if only 10.79% of the players play around 50 (between 45 and 55) in Guess(241,3/5), if a student believes that most students expect this fact and accordingly play 30 = 3/5.50, then, by best-replying to this fact, hence by playing around 3/5.30, he is not far from winning the game (19 is the winning value in the experiment). By contrast, the fact that the students know that the central limit theorem is inappropriate in front of two opponents[2], deprives them from a starting behavior they can best reply to. So guessing becomes indeed more difficult.

3. Intervals of winning strategies and threshold parameter $a = 3/4$

Before going into the 3-player guessing game, we make a clarification: throughout the paper, given a triple of propositions (X, Y, Z), we say that X wins the game if it is the number closest to $\frac{a(X+Y+Z)}{3}$, but also if it is *one of* the numbers closest to $\frac{a(X+Y+Z)}{3}$. So "winning proposition" does not mean "best-reply": as a matter of fact, if, for example, $X < Y < Z$ and X and Y are both at the same distance from $\frac{a(X+Y+Z)}{3}$, then we say that X and Y are winning strategies in that they share the prize, but a best reply would be to play $Y - \varepsilon$.

This is an important point. By saying that a player wins the game as soon as he is among the winners, we also give weight to the triples where the three players play a same number[3]. So some players may focalize on a possible common behavior, a fact that is not in the spirit of the guessing game. Yet this should not change a lot the general behavior in the game. For example iterated elimination of weakly dominated strategies is not affected by this change ($a100$ still weakly dominates all the numbers larger than $a100$, which generates the infinite iterated elimination process up to 0). And level-k reasoning is not much affected either. The triples where everybody plays a same number X different from 0 are very unstable, in that, as soon as one player switches to $X - \varepsilon$, the other players switch also away from X in order to not lose the game. In other terms, with a level-*k* reasoning, if one expects that all the others play *X*, playing *aX* is a much more stable best response than playing *X* (because *aX*, contrary to *X*, is also a best response to a context where most of the players play *X* except some of them who switch to $X - \varepsilon$). So iterated elimination of weakly dominated strategies and level-*k* reasoning should not be affected. Moreover, the set of triples where a player who plays X shares the prize with other players, is at the frontier of the set of triples where he is the only winner, and so it is of smaller dimension than the set of winning triples.

Let us see things from a pragmatic point of view. Do not forget that we want to give hints to win the game. When somebody announces the winner(s) of a game, the point that matters for a participant is to be among the winners, not so much to be the only winner. To say things differently, the gap between a loser and a winner is very huge, regardless of what is won by the winner. And this is particularly true when a player only shares the prize with at most 2 other players (which is the case in the 3-player game). What is more, given the dispersed behavior distributions in both experiments, the probability for a player to meet 2 players playing the same amount than himself is quite small. So the only context to take into account is the one in which a player may share the prize with one other player. Yet, in this context, if the prize is a sufficiently large amount of money -unless the game is not worth being played-, then, taking risk and Von Neumann Morgenstern utility into account, we can say that most players assign to the half of the prize a utility which is much larger than half of the utility assigned to the prize.

[2] The central limit theorem is also inappropriate in front of many students but the students do not know this fact.

[3] In the large *N* guessing game, this also leads to focalize on *N*-tuples where *K* players play a number *X* and the *(N-K)* other players play a number *Y* different from *X*, so that *X* and *Y* are at the same distance from $a(KX + (N - K)Y)/N$. In the 3-player game, these triples only exist for a strictly larger than ¾. If so, when one player plays *X* and two other players play the larger value *Y* equal to $X(3 - 2a)/(4a - 3)$, then the three players win the game.

And, to conclude on this point, should a player be only interested in the triples where he is the only winner, this is not problematic. Focusing on the triples $(X, Y\ Z)$ where X is the only winner just amounts to deleting the frontiers of the set where it is the only winner.

We now go into the 3-player guessing game.

A first observation is that the 3-player guessing game is a real guessing game. Despite 3 is close to 2, there is a main discontinuity between the 2-player guessing game and the 3-player guessing game. Whereas, in the 2-player game, 0 always wins the game, regardless of the opponents' choice and regardless of the value of the parameter a (<1), in the 3-player game the winning strategy depends on the numbers chosen by the two other players and also on the value of a. Insofar, in contrast to the 2-player game, the 3-player game is a true guessing game, like the usual large N-player game.
Yet, given the proximity of 3 and 2, many students, after having discovered the winning strategy in Guess(2,3/5), seriously hoped to also find a winning strategy in Guess(3,3/5), despite it does not exist. And they were right by so doing, because the 3-player game, in contrast to the large N-player game, has special properties that can be exploited to establish a strategy that often wins the game.

By playing Guess(3,3/5) in the 2nd experiment, the students, helped by the two given examples, often adopt one of the two following contrasted points of view: either they are convinced that playing a low number- namely 0-remains the best strategy to play, or they become convinced, like in the first experiment, that aiming to be between the two opponents' numbers is the best behavior. And all students are right in part. Let us call X a player's strategy and let us suppose that the 3 proposed numbers are X, Y and Z. We fix $Y \leq Z$ throughout the paper. X, with $X \leq Y$, is a winning strategy if Y is not too different from Z, even if Y and Z are quite large. By contrast, if the two opponents play distant numbers (like 14 and 72 in the given example), then it is better to play a number in-between (like 31 in the example).

These observations give rise to proposition 1:

Proposition 1:

a) *If* $Y < Z \leq (3-2a)Y/a$ *then any strategy in* $[max\ (0, \frac{2aZ}{3-2a} - Y), Y]$ *is a winning strategy.*
b) *If* $Z \geq \frac{(3-2a)Y}{a}$ *and Z>Y, then any strategy in* $\left[Y, min\left(\frac{2aZ}{3-2a} - Y, Z\right)\right]$ *is a winning strategy.*
c) *If Y=Z, then any strategy in* $[\max\left(0, \frac{(4a-3)Y}{3-2a}\right), Y]$ *is a winning strategy.*

Proof of proposition 1

For $X \leq Y < Z$, Z can never be the winning strategy.
Let us assume the contrary. So $\frac{(X+Y+Z)a}{3} - Y \geq Z - \frac{(X+Y+Z)a}{3}$
Hence $(Z+Y)(3-2a) \leq 2aX$, which is not possible, given that $a < 1$ and $X \leq Y < Z$.
So the winning strategy is either the lowest proposition or the middle proposition.
$X \leq Y < Z$: X is winning if $\frac{(X+Y+Z)a}{3} - X \leq Y - \frac{(X+Y+Z)a}{3}$, hence $Y \geq X \geq \max\left(0, \frac{2aZ}{3-2a} - Y\right)$, which requires that $\frac{2aZ}{3-2a} - Y \leq Y$ hence $Z \leq \frac{(3-2a)Y}{a}$.

$Y \leq X \leq Z, Z > Y$: X is winning if $X - \frac{(X+Y+Z)a}{3} \leq \frac{(X+Y+Z)a}{3} - Y$ hence $Y \leq X \leq \min\left(\frac{2aZ}{3-2a} - Y, Z\right)$, which requires $Z \geq \frac{(3-2a)Y}{a}$.
And when $Y = Z$, then X is a winning strategy if X≤Y and $\frac{(X+2Y)a}{3} - X \leq Y - \frac{(X+2Y)a}{3}$ hence $Y \geq X \geq \max\left(0, \frac{(4a-3)Y}{3-2a}\right)$. ■

Proposition 1 shows that, for a given couple (Y, Z), there is a range of values X that win the game. This fact is also true in the 2-player game, but completely new in comparison to the large N-player game. In Guess(N,a), a player, to win the game, has to be very close to a times the mean value, because many played numbers can be scattered around the winning value. By contrast, in the 2-player game, the whole interval $[0, Y]$ is the set of winning values if the opponent plays Y. So the notion of winning interval is a common point between the 3-player game and the 2-player game. Yet, by contrast to Guess(2,a), in Guess(3,a), the nature and the width of the interval of winning values both depend on the distance between Y and Z and on the value of the parameter a.
It derives from proposition 1 that $a =$ ¾ is a threshold parameter, as can be seen in Figures 3a, 3b and 3c. In these figures we draw the interval of winning values X, for a fixed value Y, when Z goes from Y to 100. The special case $a = 3/5$, to which the students are confronted, is illustrated in Figure 4.

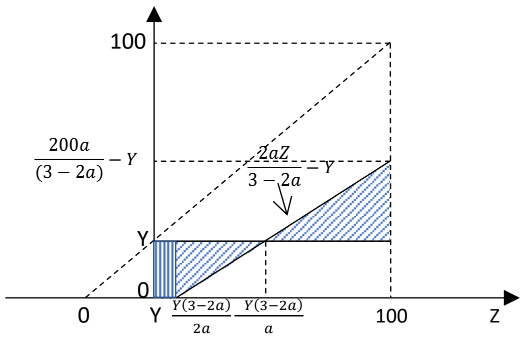

Figure 3a: a<3/4, areas of winning X

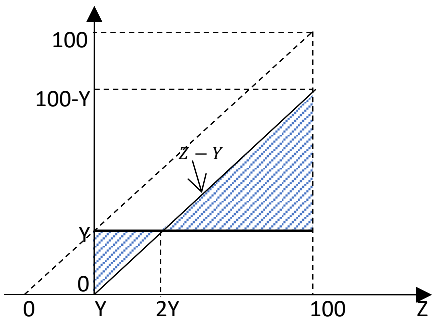

Figure 3b: a=3/4, areas of winning X

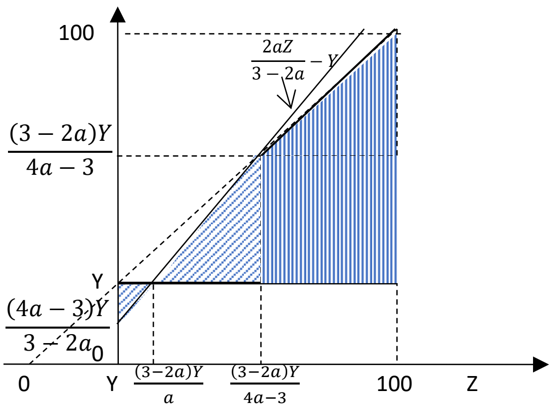

Figure 3c: a>3/4, areas of winning X

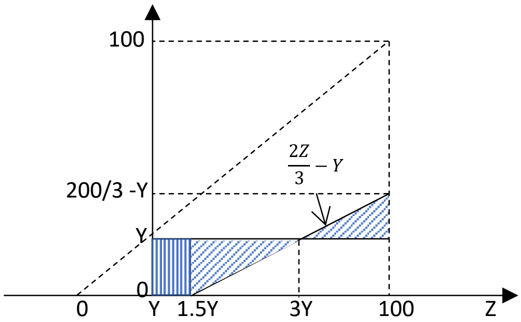

Figure 4: a=3/5, areas of winning X

For $a < 3/4$, there is an interval of values Z such that simply playing less than Y (the lowest opponent number), including 0, leads to win the game (see Figure 3a and Figure 4). This fact agrees with the students' conjecture that 0 is most often the winning strategy and with their conjecture that playing a number lower than the two opponents' numbers is the best way to win the game. Yet this is right only if the two opponents play sufficiently close values: for $a = 3/5$, as long as $Z \leq 1.5Y$, 0 remains a winning strategy and it is enough to play less than *Y* to win the game.

For a larger than ¾, by contrast, there is a range of values, the values *Y* and *Z* such that $Z > (3 - 2a)Y/(4a - 3)$, such that, to win the game, it is enough to choose a number between the two opponent numbers *Y* and *Z* (see Figure 3c).
This may explain that some students, in the first and in the second experiment, are convinced that they just have to play a number between the two opponents' guesses to win the game. This way of thinking is especially observed in the first experiment, where the students play Guess(3,3/5) after having been confronted to a large number of players, and without having experienced the strong power of 0 in Guess(2, 3/5). So, by playing Guess(3,3/5), many students simply think that, when there are 3 numbers, the middle one is surely closest to the winning strategy, given that the winning strategy is linked to the notion of mean (as it is the mean multiplied by a). Hence playing in the middle becomes a new heuristic of behavior. In the second experiment, the students who aim to propose a number between the opponents' guesses base their way of doing on the second given example (and many personal trials).

Yet, given that 3/5 <3/4, the students are only partly right. For $a < 3/4$, a player wins by playing a number *X* between *Y* and *Z* when *Z* is far from *Y*, $Z > \frac{(3-2a)Y}{a}$, but X has to be below $\frac{2aZ}{3-2a} - Y$, which is strictly lower than Z. By contrast, for $a > 3/4$, provided $Z > Y\frac{3-2a}{4a-3}$, all the numbers between *Y* and *Z* win the game. Yet this area only exists as long as $Y\frac{3-2a}{4a-3} < 100$, hence for $Y < \frac{(4a-3)100}{3-2a}$.

The above results show that $a = 3/4$ is a threshold value in the 3-player beauty contest game. As long as $a < 3/4$, the 3-player guessing game shares nice similarities with the 2-player game, namely the fact that 0 remains a winning strategy as long as *Z* is sufficiently close to *Y* ($Z \leq \frac{Y(3-2a)}{2a}$), but for $a > 3/4$, 0 is no longer a winning strategy, except if at least one opponent plays 0 too.

4. Dominance is more powerful

In the 2-player game, dominance is extremely powerful, given that 0 is the weakly dominant strategy. In the *N*-player game, by contrast, non-iterated dominance only allows to eliminate the strategies larger than a100, which is much less powerful. In the 3-player game, the result is in-between.

Proposition 2

All the strategies larger than a100/(3-2a) are weakly dominated.

Proof of proposition 2

We show that *X* weakly dominates any number *Y* larger than *X*, if and only if *X* is the winning strategy when confronted to any numbers *Y* and *Z* ($\geq Y$).
We first prove the necessary condition:

Suppose that X does not win when confronted to a couple of opponents' guesses (Y, Z) such that $X < Y \leq Z$. In that case X does not dominate Y, because Y does better, in that it shares the prize, when confronted to the couple (Y, Z). This is true regardless of the value $a < 1$.
Now suppose that X wins when confronted to any couple of opponents' guesses (Y, Z) with $X < Y \leq Z$. This means that $X > max(0, 2aZ/(3-2a) - Y)$, for any couple (Y, Z) such that $X < Y \leq Z$. So this condition must hold for the most restrictive values i.e. for Z=100 and Y almost equal to X, so it becomes $X \geq \frac{200a}{3-2a} - X$, i.e. $X \geq \frac{100a}{3-2a}$
We now show that the condition $X \geq \frac{100a}{3-2a}$ is sufficient for X to weakly dominate any larger number S. Suppose the contrary. This means that there exist couples (Y, Z), with $Y \leq Z$ (without loss of generality), such that X loses when confronted to (Y, Z), and S wins. S is necessarily lower than Z, unless it cannot win and $Y < X$, unless X wins (because X wins against any couple (Y, Z) with $X < Y \leq Z$). So there only remains one possible configuration for X to lose and S to win: $Y < X < S < Z$. Yet this means that
$X - (X+Y+Z)a/3 > (X+Y+Z)a/3 - Y$ and $S - (S+Y+Z)a/3 < (S+Y+Z)a/3 - Y$, hence $X(3-2a) > 2Za + Y(2a-3)$ and $S(3-2a) < 2Za + Y(2a-3)$ which is impossible for $S > X$. ■

This result is interesting for two reasons.

First, given that it allows to eliminate all the strategies larger than $\frac{100a}{3-2a}$ (100/3 for a= 3/5), non-iterated dominance is much more powerful than in the large *N*-player guessing game where it only allows to eliminate the numbers larger than 100a (larger than 60 for $a = 3/5$). The difference between $\frac{100a}{3-2a}$ and 100a is quite attractive, at least if a is lower than ¾, as illustrated in table 1.

a	Undominated strategies in Guess(*N*,*a*), without and after one iteration	Undominated strategies in Guess(3,*a*), without and after one iteration	Undominated strategy in Guess(2,*a*)
3/5	[0, 60] [0, 36]	[0, 33.33] [0, 11.11]	0
2/3	[0, 66.67] [0, 44.44]	[0, 40] [0, 16]	0
0.7	[0, 70] [0, 49]	[0, 43.75] [0, 19.14]	0
0.75	[0, 75] [0, 56.25]	[0, 50] [0, 25]	0

Table 1: intervals of undominated strategies, without iteration and after one iteration in Guess(N,a), Guess(3,a) and Guess(2,a).

Second, dominance becomes closer to *standard* level-*k* reasoning. The mean value played by *(standard)* level-0 players is usually supposed to be around 50. Standard level-1 players play a50, and more generally standard level-*k* players play a^k50. Standard level-1 behavior is often observed in beauty-contest games with untrained players, but standard level-1 and level-2 behavior are also often observed with more trained players. For example, in the first experiment, both in Guess(241,3/5) and in Guess(3,3/5), the mode is $a50 = 30$ (more than 15% of the students are standard level-l players in Guess(3,3/5)). But even more trained students adopt a standard level-1 and level-2 behavior: more than 13% of the students play 30 and more than 18% of them play 20 -a value close to $a^2$50- in Guess(3,3/5) in the second experiment. As argued by Breitmoser (2012) playing a50 or $a^2$50 makes no sense in Guess(3,a), because a *true* level-1 behavior leads to a whole range of winning strategies and not to the value a50 (see next section). Yet many students, when playing Guess(3,3/5), are unable to exploit the specificity of *N*=3, and therefore prefer behaving like standard level-1 or level-2 players. And other students, even if they understand the specificities of Guess(3,3/5), fear that their opponents are unable to catch them. So they still guess that many of their opponents just play $a50$, in which case the standard level-2 behavior is a good way to play.

Insofar, the stronger power of dominance helps by making the compared amounts more similar. For example, in Guess(3,3/5), the opponents' expected behavior is similar, whether they are supposed to be rational -hence able to discover that they should not play more than $100a/(3-2a)=33.33$-, or whether they are supposed to be standard level-1 players who play $a50=30$. Given that 33.33 is close to 30, for both kinds of conjectures, it becomes reasonable to not play more than a number around 11 (in that 11.11 weakly dominates any larger number when the opponents play a value in $[0, 33.33]$). This is not true in the usual large *N*-player guessing game where weak dominance leads to 60 and standard level-1 behavior to 30. In other terms, by playing around 11 in Guess(3,3/5), you win with a larger probability, because you win against opponents with different kinds of rationalities.

To summarize, in the large *N*-player beauty contest game, iterated dominance does the same iterations than level-k reasoning, but starts with 100, whereas level-*k* reasoning starts with 50 (iterated dominance leads to $100, 100a, 100a^2, 100a^3$ ... whereas level-*k* reasoning leads to $50,\ 50a,\ 50a^2,\ 50a^3$ ...). So a player who works with dominance in a context where many players do a level-*k* reasoning generally loses the game (he systematically plays too large numbers). By contrast, in the 3-player game, standard level-*k* reasoning still leads to the numbers $50, 50a, 50a^2, 50a^3$ ..., but dominance more quickly converges to similar numbers in that the multiplicative factor now decreases to $\frac{a}{3-2a}$. So, if many persons still do a standard level-*k* reasoning, a player who reasons with dominance in the 3-player game may more easily win than a player who reasons with dominance in the large *N*-player guessing game.

5. Uniform distribution and optimal guesses

In order to help a player to guess, we now turn to another way to exploit proposition 1. Proposition 1 gives the interval of winning guesses *X* when the opponents play a couple (Y, Z). Rather than looking for the winning *X* in front of a given couple of opponents' guesses (Y, Z), we now look for all the couples (Y, Z), with $Z \geq Y$, such that *X* is a winning guess. We call this area the win area of *X*. Insofar we follow Breitmoser (2012) and we get the same optimal win areas.

Let us stress the point that the notion of win area makes no sense in the large *N*-player game, given that the win area is very small for any number. And it makes no sense in the 2-player game because the win area of 0 covers the total area of the opponent's guesses. By contrast, it makes sense in the 3-player game, because, on the one hand, it differs in function of the played number and, on the other hand, it may be quite large for some guesses, especially if *a* is not too large (lower than ¾). So playing a number with a large win area may help winning the game with a large probability, and remember that this is exactly what we aim to do.

Breitmoser (2012) argues that a "serious" level -1 player should in fact play the value *X* that maximizes the win area. As a matter of fact, a level-1 player should best reply to a level-0 behavior, where a level-0 behavior does not consist in playing 50, but consists in playing

randomly on [0, 100]. It derives from this fact that a level-1 player has to suppose that the two opponents' guesses are randomly scattered on the set $[0,100] \times [0,100]$ and to choose the value *X* that maximizes its win area.
So we compute, for each amount *X*, the area of couples (Y, Z) in $[0,100] \times [0,100]$, that leads *X* to win the game (or to be among the winners).

Proposition 3 (identical to Breitmoser 2012)
Let us fix $Y \leq Z$. X *wins the game if :*

- *either* $X \leq Y \leq Z$ *and* $Y - \frac{(X+Y+Z)a}{3} \geq \frac{(X+Y+Z)a}{3} - X$

hence $Z \geq Y$ *and* $Y \geq X$ *and* $Z \leq X(3-2a)/2a + (3-2a)Y/2a$

- *or* $Y \leq X \leq Z$ *and* $X - \frac{(X+Y+Z)a}{3} \leq \frac{(X+Y+Z)a}{3} - Y$

hence $Z \geq X$ *and* $Y \leq X$ *and* $Z \geq \frac{X(3-2a)}{2a} + \frac{(3-2a)Y}{2a}$

For $a \leq 0.75$, *the value X* that leads to the largest win area is* $X* = \frac{4aM}{21-16a}$. *The corresponding win area is* $2M^2\left[\frac{4a}{21-16a} + \frac{3-4a}{6-4a}\right]$.

Proof of proposition 3: see below

The win areas differ depending whether a is lower or larger than ¾. And $a > 3/4$ gives rise to two different cases (we are not really interested in given that it becomes hazardous to give hints to win for $a > 3/4$).

Given that we want to win the game, it is surely not a good idea to suppose that the couples (Y, Z) are uniformly distributed on $[0,100] \times [0,100]$, in that the opponents also aim to win the games. So it makes sense to calculate the win areas by assuming that the opponents have some minimal rationality. For example, we can suppose that nobody plays more than $a100$, given that everybody- if trained a little- can understand that the mean cannot be larger than 100, so that it makes no sense to play more than $a100$ [4].

More generally, some behavioral observations allow to reduce the set of couples (Y, Z) to $[0, M] \times [0, M]$, where M can take different values. Mathematically, this changes nothing as regards the win areas except that 100 has to replaced by M and $\frac{200a}{3-2a} - X$ by $\frac{2Ma}{3-2a} - X$. Observe also that we only study the values X lower than $100a/(3-2a)$ – more generally $Ma/(3-2a)$- given that the other numbers are weakly dominated by $100a/(3-2a)$ -respectively $Ma/(3-2a)$-, so do less often win the game and therefore lead to a smaller win area.

It is also more or less inadequate to suppose that the opponents' guesses (Y, Z) are *uniformly* distributed on $[0, M] \times [0, M]$ but we come back to this fact later on. For the moment we comment the optimal win areas when the couples (Y, Z) are uniformly distributed on $[0, M] \times [0, M]$.

For $a \leq 3/4$, the win areas are given in Figure 5*a* and 5*b*. Given that Figures 5*a* and 5*b* only focus on the couples (Y, Z) with $Z \geq Y$, the win area is 2 times the area on the graphic, i.e.:

[4] Yet remember that some players may wish to bring the three players to the same value so that everybody wins the game. Insofar they can play 100, because 100 is a focal value. The number of players with this aim is however quite small in general.

$$2[\left(M-\frac{(3-2a)X}{2a}\right)X-\frac{X(3-2a)X}{4a}+\frac{(M-X)^2}{2}-\frac{\left(M-\frac{(3-2a)X}{a}\right)\left(\frac{2Ma}{3-2a}-2X\right)}{2}]$$

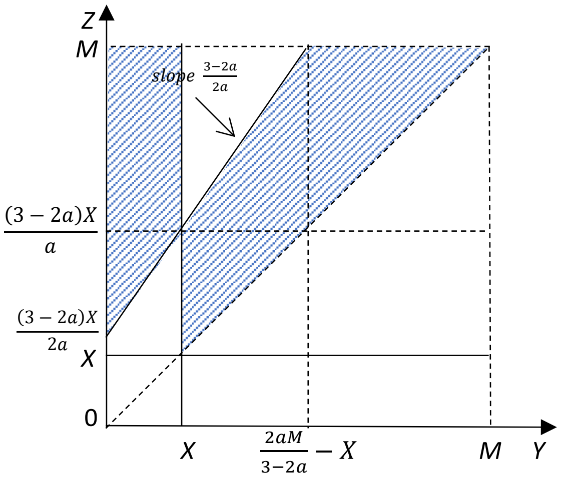

Figure 5a: win area for $a < 0.75$

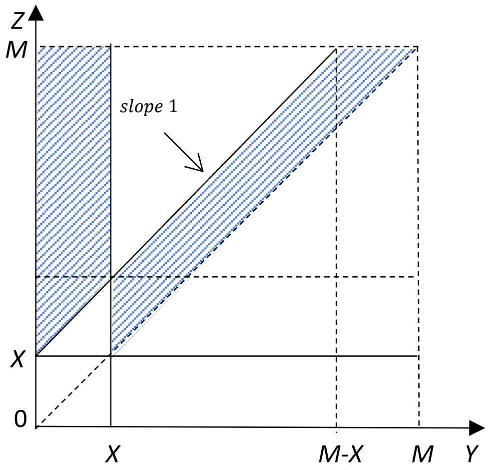

Figure 5b : win area for $a = 0.75$

Maximizing this expression leads to the optimal $X *= \frac{4aM}{21-16a}$. The optimal corresponding win area is $2M^2\left[\frac{4a}{21-16a}+\frac{3-4a}{6-4a}\right]$.

So the optimal win area decreases in a (it goes from 80.77% for $a = 0.5$ to 66.67% for $a = 3/4$). For $a = 3/5$, it covers 75.44% of the total area of the opponents' possible guesses, which is a large percentage of the total area, and $X *= M/4.75$. The division factor $\frac{21-16a}{4a}$ is fast decreasing in a, so goes from 6.5 for $a = 0.5$ to 3 for $a = 0.75$. The strong division factor 4.75 helps to focus on a small set of values, and the winning probability of 75.44% helps to be confident in the chosen value (under the uniformity assumption). As a matter of fact, according to the expected rationality of the two opponents, we may set $M = 60$, if the opponents are supposed to be able to apply *naïve* dominance (hence do not play more than $a100$); we may also set $M = 50$, if the opponents are supposed to not play more than the standard level-0 amount, $M = 30$ if the opponents are supposed to not play more than the standard level-1 amount 30, or, by contrast, $M = 100/3$, if the opponents are supposed to be rational enough to not play weakly dominated strategies. Given the strong division factor 4.75, for *M* going from 30 to 60, the set of optimal values is reduced to [6.32, 12.63] which is a rather small interval of values in comparison to the set of possibilities [0, 100]. So, if it can be assumed that the behaviors are uniformly distributed on [0, *M*], the good amount to play in Guess(3,3/5), leading to win with a probability around 3/4, is between 6.5 and 12.5.

Things are more complicated for *a* larger than ¾. For these values, the notion of win area, even if it can be assumed that the played values are uniformly distributed on [0, *M*], does not allow to prognostic a value to play, because the size of the optimal win area becomes small. Two configurations are possible, given in Figures 6a and 6b. The good configuration depends both on X and on a. Figure 6a holds for $X > \frac{(4a-3)M}{3-2a}$ (the two lines $Z = (3-2a)X/2a + (3-2a)Y/2a$ and $Z = Y$ do not intersect in $[0, M] \times [0, M]$), whereas Figure 6b holds for $X < \frac{(4a-3)M}{3-2a}$ (the two lines intersect in $[0, M] \times [0, M]$).

In Figure 6*a*, the win area is:

$$2[(M-X)X-\frac{(6-6a)(3-3a)X^2}{2a(3-2a)}+\frac{(M-X)^2}{2}-\left(M-\frac{(3-2a)X}{a}\right)\left(\frac{Ma}{3-2a}-X\right)]$$

This area holds for $X>\frac{M(4a-3)}{3-2a}$ and leads to X*=$\frac{2aM(3-2a)}{24a^2-57a+36}$. The optimal win area becomes $2[\frac{2aM^2(3-2a)}{24a^2-57a+36}+\frac{M^2(3-4a)}{2(3-2a)})=M^2(\frac{2X*}{M}+\frac{3-4a}{(3-2a)})$

$X*$ has to check $X*>M(4a-3)/(3-2a)$ so these results only hold for $a<0.9078$.

Observe that the range of optimal $X*$ becomes large for different values of M, given that the division factor $\frac{24a^2-57a+36}{2a(3-2a)}$ of M is decreasing, going from 3 for $a=0.75$ to 1.88 for $a=0.9078$. Moreover the optimal win area is strongly decreasing in a, going from 2/3 for $a=0.75$ to only 53.31% of the total area for $a=0.9078$.

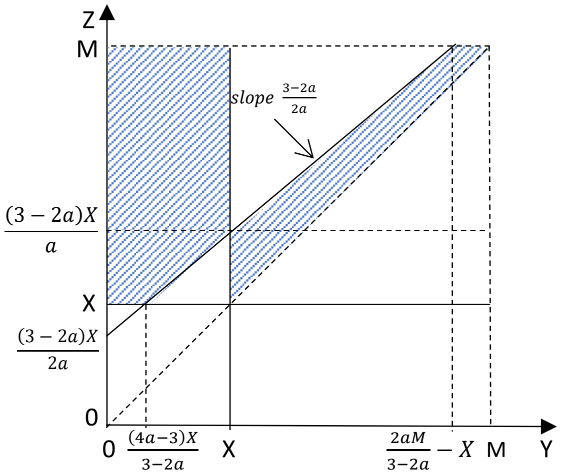

Figure 6a : win area for a>0.75, X>$\frac{(4a-3)M}{3-2a}$

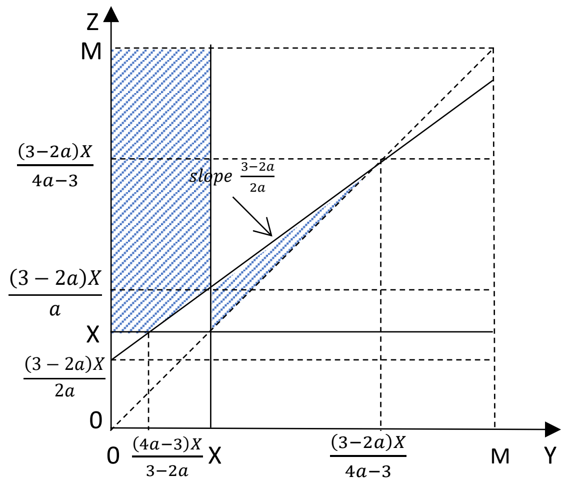

Figure 6b : win area for a>0.75, X<$\frac{(4a-3)M}{3-2a}$

For example, for $a=$ 0.8, the maximal area is obtained for $X*=0.389M$ but it is only equal to 63.49% of the total area. Moreover the division factor is only 2.57 so the range of optimal $X*$ becomes large for different values of M. If we take $M=80$ (because everybody should understand that it makes no sense to guess more than 80), then $X*$ becomes 31.11, an amount that is not far from the standard level-2 reasoning amount 32 and from the amount obtained with one step of iterated (true) dominance (32.65). But if we suppose that the two opponents play at most 50, because they aim to be between the two other opponents or will not play more than a standard level-0 player, then $X*$ shrinks to 19.44. The range [19.44, 31.11] is very large. Combined with the fact that the win area for the optimal $X*$ is only 63.49% of the total area, it clearly becomes difficult to give a hint to win.

This explains that we focus on the smaller class of games with *a* lower than ¾. This also explains that we give hereby the results for *a* larger than 0.9078 only for sake of completeness.

In Figure 6*b*, the win area is:

$$2[(M-X)X-\frac{(6-6a)(3-3a)X^2}{2a(3-2a)}+\frac{(6-6a)^2X^2}{2(4a-3)^2}-\frac{3-2a}{4a}.\frac{(6-6a)^2X^2}{(4a-3)^2}]$$

$=2(M-X)X+\frac{(6-6a)^3X^2}{4a(3-2a)(4a-3)}$

The optimal X* is equal to $X*=\frac{M}{2-\frac{(6-6a)^3}{2a(3-2a)(4a-3)}}$ and the optima win area is $MX*$

$X*$ has to check $X*< M(4a-3)/(3-2a)$ so these results only hold for $a>0.9078$.

Let us now talk about the uniformity of the behavior distributions, as the results in proposition 3 help guessing only if we can suppose that the opponents' guesses are uniformly distributed on $[0,M]\times[0,M]$. Clearly, the distributions for Guess(3,3/5) in our two classroom experiments- see Figures 1a and 2b- are not uniform, but they have some uniform characteristics. With untrained students, by setting $M=50$ (because only 11.2% students play more than 50), we observe that only 8 integers in [0, 50] are not played in Guess(3,3/5) and that 29 integers in [0, 50] are played by a number of persons going from 1 to at most 5. With trained students, by setting $M=40$ (in that only 9.55% students play more than 40), we observe that only 6 integers in [0, 40] are not played, and that 28 integers in [0, 40] are played by a number of persons going from 1 to at most 5. So, despite the distributions are not uniform, if we exclude some peaks on isolated values (20, 25, 30, 40 and 50 in the first experiment, 0, 15, 20, 25 and 30 in the second experiment) the students' guesses are rather uniformly distributed on the integers in $[0,M]$. This is namely due to the fact that Guess(3,3/5) is perceived as a complicated game: the behavior rules "I should play a number between the numbers proposed by the two opponents" and "I should rather play a low number" are very vague and therefore explain that the guesses are scattered on a whole interval in a rather uniform way.

6. 10, a "nombre d'or" for $a=3/5$

What can we deduce up to now? First, that is much too hazardous to give a hint when $a>0.75$. Fortunately, usually a is often below this threshold, one of the most standard value being 2/3. For *a<0.75*, it becomes possible to suggest a small range of values that can lead to a good probability to win the game, at least if it can be assumed that the guesses are rather uniformly distributed on an interval $[0,M]$. Given that is reasonable to expect that the two opponents should not play more than 100a, and that it would be a too strong assumption to suppose that both play less than the standard level-1 behavior a50, the guesses giving rise to the optimal win areas (proposition 3) are in the interval $[\frac{200a^2}{21-16a},\frac{400a^2}{21-16a}]$. The range of the interval is increasing in a (it goes from 3.85 (interval [3.85, 7.69]) for $a=0.5$ to 12.5 (interval [12.5, 25]) for $a=0.75$ and is equal to 6.32 (interval [6.32, 12.63]) for $a=3/5$.

Dominance, if the strategy set of the opponents is reduced to $[0,100a]$ would lead to play a number below $\frac{100a^2}{3-2a}$, yet $\frac{100a^2}{3-2a}$ (20 for $a=3/5$) is always larger than $\frac{400a^2}{21-16a}$ so does not help to reduce the interval and may even broaden it.

Moreover, uniformity of the distribution is a strong assumption. For $a=3/5$, with untrained students, 2 values are played by more than 15 persons, 30 (37 students) and 50 (25 students). With trained students, 2 values too are played by more than 15 persons, 20 (37 students) and 30 (26 students). So, in order to give a hint, we have to focus on more specificities of the game.

To do so, we now only focus on the specific parameter a= 3/5.

It appears that 10 is like a “nombre d’or” in the 3-player guessing game with parameter $a = 3/5$.

First, 10 is the mean of the set of values $[0{,}20]$ that weakly dominate the larger numbers when the two opponents are supposed to play less than 60 (the largest naïve dominance amount).

10 is only slightly larger than the mean of the interval $[\frac{200a^2}{21-16a}, \frac{400a^2}{21-16a}]$=[6.31, 12.63].

10 is only slightly lower than the number that maximizes the win area when it can be assumed that the population is uniformly distributed on $[0, 50]$, an assumption that makes sense in Guess(3,3/5), especially in the second experiment, where many students think that to win it is good to play a number between the two opponents’ ones, but not a too large one, in that the students also know that low numbers often win the game.

10 weakly dominates all the larger numbers if it can be assumed that the other persons will not play more the standard level-1 amount 30.

What is more, 10 wins the game each time at least one of the two opponents plays the standard level-1 amount 30, provided the other opponent does not play more than the naïve weakly dominating strategy 60. *So 10 performs nicely even in a nonuniform setting when we expect that at least one of the opponent plays like a standard level-1 player*.

This can be seen in Figure 7a (we adapt Figure 5a by setting $a = 3/5$, $X = 10$ and $M = 60$). For $X = 10$, $(3 - 2a)X/a$ is the standard level-1 behavior a50, that is to say 30. The blue lines are in the win area. They show that, as soon as one opponent plays 30, 10 wins the game regardless of the other opponents’ amount, between 0 and 60.

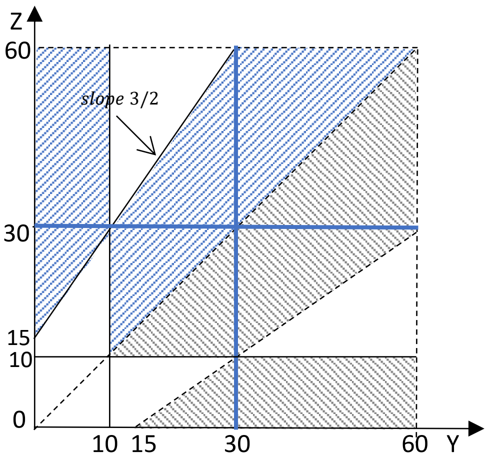

Figure 7a: win area for a=3/5, X=10 and M=60 The blue lines are in the win area: if one opponent plays 30, 10 wins the game regardless of the other opponents’ amount, between 0 and 60.

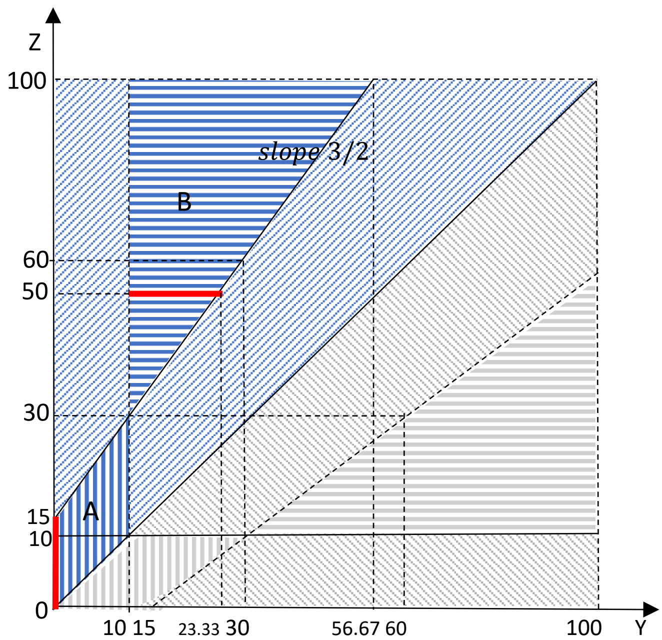

Figure 7b: areas A and B for a=3/5 and X=10

This is quite uncommon. Usually, the specific value $\tilde{X}$ that checks $(3 - 2a)\tilde{X}/a = a50$, i.e. $\tilde{X} = 50a^2/(3 - 2a)$, is seldom an integer, and the interval $[0, Z = (75 - 25a)]$ for which $\tilde{X}$ wins the game when one of the two opponents plays the standard level-1 behavior a50 and the other plays a value in the interval $[0, Z]$, is different from $[0, a100]$. This is very specific to $a = 3/5$, in that $a100 = 75 - 25a$ only for $a = 3/5$.

By the way let us observe that, for any parameter a lower than ¾, numbers around the number $\tilde{X}$ that equalizes $a50$ and $\frac{(3-2a)\tilde{X}}{a}$, hence around $\frac{a^2 50}{3-2a}$, are worth of interest. As a matter of fact, $\frac{a^2 50}{3-2a}$ is included in $[\frac{200a^2}{21-16a}, \frac{400a^2}{21-16a}]$ regardless of the value $a < 1$, and it is not stupid to expect that at least one of the opponent plays the standard level-1 amount. And if so, it is interesting to look for a value that ensures to win regardless of the other opponent's chosen number in a large enough interval with no hole, the interval [0, 75-25a].

Last but not least 10 is an integer. Moreover, 10 is an easy integer, a multiple of 5 and of 10; these multiples are always focal values and often played by the participants. As a matter of fact, whereas the multiples of 5 only constitute 20% of the integers in [0,100] and a set of dimension 0 in [0,100], in the first experiment, 59.75% students play a multiple of 5 in Guess(3,3/5), and, in the second experiment 67.84% students play a multiple of 5 in Guess(3,3/5) (similar observations hold for Guess(241,3/5) and Guess(2,3/5)). And the values played by 12 persons or more are all multiples of 5 (20, 25, 30, 40 and 50 with untrained students, 0, 15, 20, 25, 30 with trained students). This fact is not uncommon in experiments[5] and this matters because we will show in the next section that it is better to play an integer, rather than a little more or a little less than it, when we expect that the opponents also play integers. Moreover, given that 10 wins against any couple (Y, Z) with $10 \leq Y \leq Z \leq 1{,}5Y + 15$, 10 wins against the numerous couples of opponents playing (0, 15), (15, 20), (15, 30), (20, 30), (20, 40), (25, 30), (25, 40), (25, 50), (30, 40), (30, 50) and (40, 50) .

But for the moment let us study the true performance of 10 in our two experiments, with untrained students (first experiment) and trained students (second experiment). To do so we confront 10 to any couple of guesses that can be extracted from the students' behavior (i.e. 241x120 couples in the first experiment, 199x99 couples in the second experiment). We look for all the couples (Y,Z) such that 10 does not win against (Y,Z). These couples are either in area *A* or in area *B* (see Figure 7*b*); in area *A*, the two opponents choose low values so that 10 is too large to win. In area *B*, one opponent plays a large value, larger than 30, and the other plays a value larger than 10 (but not too large), so that 10 is too low to win the game. Depending on whether we play with trained or untrained students, the most "dangerous" area for 10 will not be the same.

In the first experiment, area *B* is more dangerous for 10 than area *A*. As a matter of fact, when players are untrained, they sometimes (wrongly) think that the best thing to do, in a 3-player guessing game-, is to play a number between the two opponents' guesses, which can lead them to play around 50, and more generally a number in [20, 50] (68.88% of the students play in this way). Namely 25 students among 241 play 50 (more than 10%) whereas only 11 among the same students play 50 in the large *N*-player guessing game (which proves that the students more feel the necessity to guess between the (two) opponents' guesses in a 3-player game than in a large *N*-player game). Yet these players may be problematic for 10. If an opponent plays 50, 10 loses the game as soon as the other opponent plays a number larger than 10 and lower than 100/3-10=23.33. Yet 1/5 of the students (49 students) play a number in this range of values; so

[5] See for example classroom experiments on the traveler's dilemma in Lefebvre and Umbhauer (Cairn International forthcoming)

10 loses the game in front of these 25x49= 1225 couples of opponents (the horizontal red line in Figure 7*b*). More generally, with untrained persons, area *B* corresponds to 4448 couples because there are 100 students among 241 (41.49%) who play Z larger than 30 and because 10 loses the game when these students are coupled with a student who plays Y with $10 < Y < 2Z/3 - 10$. These couples represent 15.38% of all possible couples. Fortunately, with untrained students, area *A* is rather deserted by the players. Untrained students do not often play less than 10 (only 9.54% of the students play less than 10), so area *A* only counts 837 couples (only 2.89% of all the possible couples). So, at the end, 10 performs well, given that it wins with probability 0.8173.

In the second experiment, area *B* is much less dangerous for 10, in that trained students do less often play more than 30: only 17.09% of the trained students play in this way in contrast to 41.49% of the untrained students. Namely only 2 students play 50, given that they know that they can win by playing a number that is not between the two opponents' amounts. So only 2 types of students are dangerous for 10. Those who play 60 (9 students stick to naïve dominance) and the 3 persons who play 100 (for wrong reasons). So only 1953 couples (9.91% of the possible couples) are in area *B*. By contrast, area *A* becomes more dangerous, in that there are now 14.07% of the students who play less than 10 and 21.11% who play less than 15. It becomes dangerous to play 10 namely because 12 students (6.03%) play 0. In front of an opponent playing 0, 10 loses the game if the other opponent plays a number lower than 15 and larger than or equal to 0 (see the vertical red line in Figure 7*b*). This line represents 426 couples. So area *A* counts 1523 couples (7.73% of the total couples). In facts, area *A* counts less couples than area *B* but the percentages are more equilibrated than in the first experiment (2.89% and 15.38% for areas *A* and *B* in the first experiment, respectively 7.73% and 9.91% in the second experiment). And, at the end, 10 performs very well in the second experiment, in that it wins with probability 0.8236.

So what can we deduce? First that playing 10 performs well in both experiments. The more the players are trained, the more area *A* grows but area *B* shrinks, and the reverse holds when players are untrained. At the end, the sum of the areas *A* and *B* does not much change and remains low.

But what about the performance of other numbers? Does 10 better perform than 8, 9, 11, 12, 13, …?

10 is the best way to play in the experiment with trained students.

Switching to a lower value like 9 leads to a success rate of 78.08% because the number of couples in area *B* strongly increases (+≃1000), and the number of couples in area *A* only smoothly decreases (-≃150). Area *B* namely strongly increases because it shifts to the left (see Figure 8*a*) so that 9 systematically loses against all couples with a player playing 10, and the other player playing from 28.5 to 100 (see the vertical red line in Figure 8a) which represents 630 additional couples. And things get worse for all numbers lower than 9.

By contrast, by playing more than 10, like 11 for example, area *A* strongly increases, namely because 11 loses against the 1325 couples where one player plays 10 and the other plays from 10 to 31.5 (see the vertical red line in Figure 8*b*). In facts, area *A* increases by 1650 couples – so does more than double when switching from 10 to 11-, whereas area *B* only shrinks by about 100 couples. It follows from these facts that 11 wins in only 74.48% of all possible confrontations. And things get worse for all the integers larger than 11 (for example 12 and 15 respectively win with probability 0.7321 and 0.6739).

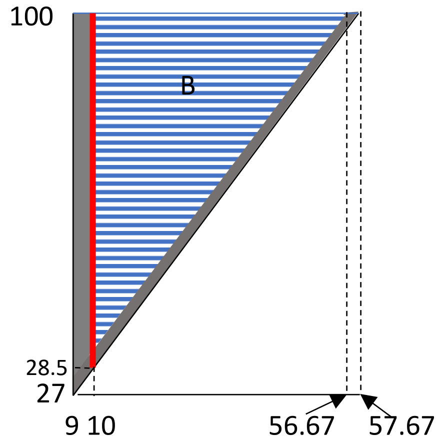

Figure 8a : area B for X=9 increases by the additional grey area

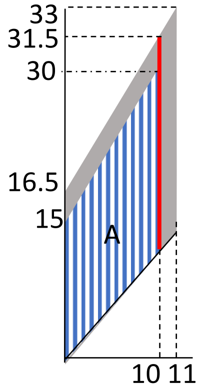

Figure 8b : area A for X=11 increases by the additional grey area

We give in table 2 the results obtained for integers from 6 to 17 (other integers do strictly worse than 10): we give the percentage of couples in area *A* and area *B* for the different integers, as well as the percentage to win the game.

	6	7	8	9	10	11	12	13	14	15	16	17	18	19
A %	2.68	3.60	5.07	6.96	**7.73**	**16.11**	***17.92***	21.43	25.30	26.52	37.62	39.90	41.81	45.07
B%	***20.88***	17.57	15.94	**14.96**	**9.91**	9.41	8.87	8.62	8.12	6.09	5.96	5.49	5.00	4.48
SR %	76.44	78.83	79.00	78.08	82.36	74.48	73.21	69.95	66.58	67.39	56.42	54.61	53.19	50.44

Table 2: percentages of couples in area A and area B and success rate for integers from 6 to 17 in the experiment with 199 trained students.

By construction, area *A* can only grow and area *B* can only decrease when the studied integer *X* increases, so the percentage of couples in area *A* increases from the left to the right of table 2 and the percentage of couples in area *B* decreases from the left to the right. Given that for 10, the sum of the percentages of couples in areas *A* and *B* is equal to 17.64, it derives that all numbers lower than or equal to 6 or larger than or equal to 12 do strictly worse than 10 and need not to be studied. For information, in this experiment with trained students, 0 wins in 51.80% of all possible confrontations, so has a better success rate than 19.

By contrast, with untrained students, 10 and 13 are in a pocket handkerchief (the success rate is 81.73% for 10 and 81.75% for 13), 14 performs a little better than 10 (success rate 81.96%) and 2 numbers win clearly more often than 10, the integers 12 and 15 (the success rates are respectively 83.69% and 83.71%). The high scores of 12 and 15 are namely due to the 25 students playing 50. 15, when confronted to 50, only loses when the third played value is larger than 15 and lower than 18.33 (only 9 students play such a number), whereas 10 loses when the third number is larger than 10 and lower than 23.33 (49 students play in this way). So 10 loses in 25x49=1225 confrontations while 15 only loses in 25x9=225 confrontations. In facts, area *B* decreases by 2686 couples when switching from 10 to 15 (4448 couples in area *B* for 10, only 1762 couples in area *B* for 15). Yet 10 performs much better than 15 when confronted to at least one player playing less than 15. So, for example, when confronted to one of the four players playing 12, 10 wins provided the third number is lower than 33.33, whereas 15 loses (because 12 wins) in all these confrontations, which represents 4x152=608 couples. This namely explains that area *A* increases by 2112 couples when switching from 10 to 15 (837 couples in area *A* for 10, 2949 couples in area *A* for 15). Yet, at the end, 15 performs better than 10.

All the other numbers do worse than 10. We give in table 3, for each integer from 6 to 19, the percentages of couples in areas *A* and *B* and the success rate.

	6	7	8	9	10	11	12	13	14	15	16	17	18	19
A %	0.97	1.24	1.68	2.46	2.89	4.20	4.68	7.63	9.44	**10.19**	**15.22**	16.69	17.46	***23.80***
B%	26.59	23.44	20.53	***18.28***	15.36	14.40	11.61	10.60	8.59	**6.09**	4.78	4.10	3.33	2.90
SR%	72.41	75.30	77.77	79.25	81.73	81.38	83.69	81.75	81.96	83.71	79.98	79.19	79.19	73.27

Table 3: percentages of couples in area A and in area B and the success rate for integers from 6 to 17 in the experiment with 241 untrained students.

For the same reasons than for table 2, given that the sum of percentages of couples in areas *A* and *B* is 18.27 for 10, all the guesses larger than or equal to 19 or lower than or equal to 9 can only do worse than 10 and therefore need not to be studied.

We can add that 0's success rate is 46.35%. This score is of course lower than in the experiment with trained students but it is larger than perhaps expected. As a matter of fact, if the distribution were uniform, 0's success rate would be 33.33%. Yet the distributions are not really uniform and we know that 0 is a best guess as long as the opponent's guesses Y and Z (with $Z \geq Y$)) are such that $Z \leq 1.5Y$. Many couples (Y, Z) share this property, among them (20, 20$\leq Z \leq 30$) but also (30,30$\leq Z \leq 45$) or (34, 34$\leq Z \leq 50$). This largely explains the success rate of 0.

We can also observe that the progression of the success rates in not regular. With untrained students for example, 10 does better than 11, 11 does worse than 12 but 12 does better than 13, 13 does worse than 14 and 14 does worse than 15, but 15 does better than 16 ….

This is partly due to the nonuniform features of the distributions, namely to the fact that some numbers are played by many students, and others are not played at all. This induces large or small changes of area *A* and/or area *B* (even if monotonically increasing and decreasing in *X*), so that the sum of couples in areas *A* and *B* may rise or decrease in *X*. So, if *X* is often played, one consequence is that area *B* for $X - 1$ contains many couples that do not exist in area B for *X*. As a matter of fact, area *B* for *X* contains the couples (Z, Y), with $Z > Y$ and $Z > 3X$, such that $X < Y < 2Z/3 - X$. So, for $X - 1$, it contains all the couples $(Y, Z), Y < Z$, with $Z > 3(X - 1)$ and $X - 1 < Y < 2Z/3 - X + 1$. So it contains many couples with $Y = X$ that do not exist in area *B* for *X*. In the same time, if *X* is often played, then the number of couples in area A may strongly increase when switching from X to $X + 1$, because area A for $X + 1$ contains all the couples (X, Z), with $X \leq Z < 1{,}5\,(X + 1) + 1{,}5X$, which may be numerous (because *X* is often played). This may help *X* to win more often than $X - 1$ and $X + 1$, and this fact is for example observed for 15 in both experiments with trained and untrained students, where 15 is respectively played by 15 and 8 students.

A similar phenomenon is at work when a number X is played by nobody. Then, when switching from X to $X+1$, area A will not be enlarged by the couples (X,Z) with $X \leq Z < 1{,}5(X+1)+1.5X$ given that nobody plays X. And when switching from X to $X-1$, area B, which contains the couples (Z,Y), with $Z > Y$ and $Z > 3(X-1)$, such that $X-1 < Y < 2Z/3 - (X-1)$ is not enlarged by the left because the couples checking $X-1 < Y < 2Z/3 - (X-1)$ and the couples checking $X < Y < 2Z/3 - (X-1)$ are the same. This may explain that X does not necessarily make a better score than $X-1$ and $X+1$.

The above observations for example (also) contribute to explain that 15 has a better success rate than 14 in both experiments, and that 10 has a better success rate than 9 in both experiments, 9 and 14 being the only integers not played in both experiments. It also explains why 9, respectively 14, has a worse success rate than 8 and 10, respectively 13 and 15, in the experiment with trained students.

At the end it derives from the two curves in Figure 9 that, as long as we only compare the performance of integers, 10 remains the good value to play (with a success rate larger than 81.7% in both experiments). 12 and 15 perform less well in the experiment with trained students (73.21% for 12 and only 67.39% for 15), namely because there are much less trained students who play 50. By contrast, 10 does well in both experiments despite the fact that trained students and untrained ones do not play in the same way. In other terms 10 resists to different kinds of logics.

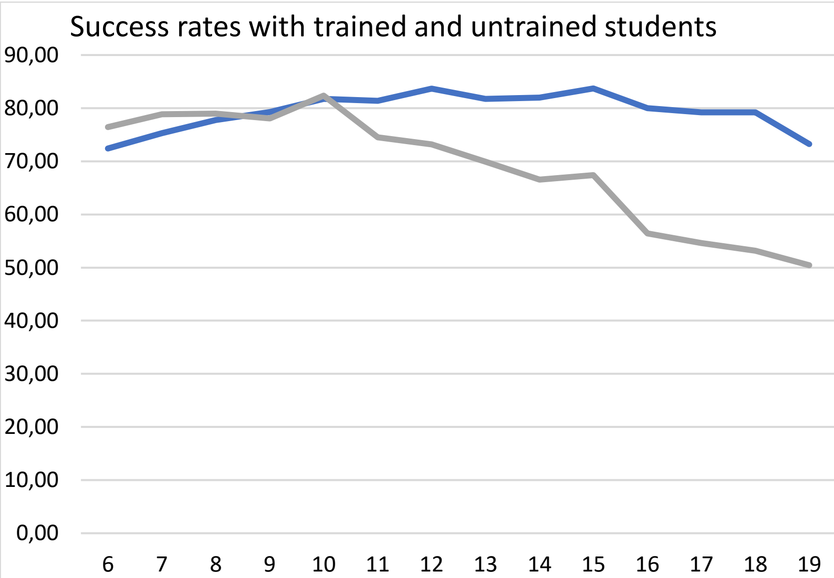

Figure 9: success rates for integers from 6 to 19 with the untrained 241 students (blue curve) and with the 199 trained students (grey curve).

7. Play an integer

In this concluding section, we draw attention to the following fact. Very often, the players playing the beauty contest game play an integer, despite they can choose any real in $[0,100]$. In the classroom experiment with trained students, all the 199 students play integers in Guess(2,3/5) and only two of them do not play integers in Guess(3,3/5). In the classroom experiment with untrained students, only four among the 241 students do not play integers, both in Guess(N,3/5) and in Guess(3,3/5). Moreover, we can add, as said in section 6, that many of them play multiples of 5, which is not uncommon in many experiments.

So intuitively, a spontaneous reaction would be to say that the best way to play is to not play an integer but rather a real just below or above this integer. Yet this would not be a good idea. Let us prove this rather counterintuitive result.

We show it for 10 in the experiment with trained students, but the same proof holds for any integer in both experiments.

First, it would be bad to play a little more than 10, say 10.01 for example. In facts, area *B* is the same for 10.01 and 10, but area *A* is larger for 10.01 than for 10.

We recall that area *B*, for 10, is the set of couples (Z, Y), with $Z > 30$, $Z > Y$, and $10 < Y < 2Z/3 - 10$. Area B, for 10.01, is the set of couples (Z, Y), with $Z > 30.03$, $Z > Y$, and $10.01 < Y < 2Z/3 - 10.01$. Yet, given that the students only (most often) play integers, the first played number larger than 30 is also the first played number larger than 30.03 and the set of integers *Y* (and even integers divided by 2) such that $10 < Y < 2Z/3 - 10$ is also the set of integers (and even integers divided by 2) such that $10.01 < Y < 2Z/3 - 10.01$. So area *B* counts the same number of couples, for 10 and 10.01.

By contrast, area *A* becomes larger by construction, when switching from 10 to 10.01, for two reasons. As a matter of facts, area *A* for 10 is the set of played couples (Y, Z) with $Y < 10$ and $Y \leq Z < 15 + 1.5Y$. So the first reason is that switching to 10.01 leads to switch to the couples $Y \leq Z < 15.015 + 1.5Y$. So *Z* can be equal to $15 + 1.5Y$, and there may exist students who play $15 + 1.5Y$ when *Y* and *Z* are mainly integers. Moreover, and this is the main reason, given that we now work with $Y < 10.01$, we have to add all the couples $Y = 10$ and $10 \leq Z < 30.015$, which can be numerous (as in our experiment). So area *A* will increase.

It follows from above that, in the second experiment with trained people, 10.01 wins the game only with probability 0.7495 (whereas 10 wins with probability 0.8236), so 10.01 does worse than 6, 7, 8, 9 and 10 in this experiment).

Perhaps more surprisingly, it is better to play 10 than a real just below 10, say 9.99 for example. This is due to the fact that area *A* is not affected by this change when most people play integers (or integers divided by 2) but area *B* increases.

As a matter of fact, as regards area *A*, we now switch from the couples (Y, Z) with $Y < 10$ and $Y \leq Z < 15 + 1{,}5Y$ to the couples $Y < 9.99$ and $Y \leq Z < 14.985 + 1{,}5Y$. Yet, given that the students mainly play integers, $Y < 9.99$ is equivalent to $Y < 10$, and the played *Z* lower than 14.985+1.5*Y* are the same than those lower than 15+1.5*Y*. So area *A* does not change when players mainly play integers.

By contrast, area *B* increases when switching from 10 to 9.99 for three reasons. Area *B*, for 10, is the set of couples (Z, Y), with $Z > 30$, $Z > Y$, and $10 < Y < 2Z/3 - 10$. Area *B*, for 9.99 is the set of couples (Z, Y), with $Z > 29.97$, $Z > Y$, and $9.99 < Y < 2Z/3 - 9.99$. So first we add a whole range of couples, those where one of the opponent plays 30, and the other plays a number *Y* with $9.99 < Y < 10.01$ which may contain numerous couples, namely those where one opponent plays 30 and the other plays 10. Second, for each *Z* from 30 to 100, we have to add all the couples where one opponent plays 10 and the other opponent plays *Z*. Third, the numbers lower than $2Z/3 - 9.99$ include the integer $2Z/3 - 10$, when Z is an integer divisible by 3, which leads to add additional couples. So area B increases.

It follows from above that 9.99 has a lower success rate than 10. In facts, in the second experiment with trained people, 9.99 leads to win with probability 0.7751, and so does worse than 7, 8, 9 and 10.

What we explained for 10 is true for any played integer, namely 12 and 15, for the same reasons. As long as one expects, in the beauty contest game with 3 players, that the two opponents play integers, it is better to play an integer too. And this fact also holds for a different from 3/5[6]. This is a rather unexpected result.

So what should we conclude? The 3-player beauty-contest game is a real guessing game and there is no strategy that always wins the game. So we still have to guess what guides the others in their choice and the kind of logic they may work with. The two classroom experiments namely show that the students do not adopt the same heuristics of behavior in large N-player guessing games and in 3-player guessing games. Yet the 3-player beauty contest game, in comparison with the *N*-player game, has some mathematical properties that may help some numbers to win with a large probability, even in a context where players follow different kinds of logics. The 3-player game namely gives rise to win areas that are large for some numbers and thin for others, and it brings dominance closer to standard level-*k* reasoning. So 10 becomes a good number to play in a 3-player beauty contest game with parameter $a = 3/5$. In our classroom experiments, whether the students are trained or not in game theory and guessing games, whether they just think that the good way to proceed consists in guessing a number between the two opponents' numbers or whether they also know that low numbers can win the game, 10 leads to win with a probability close to 0.82, which is a very nice performance. In facts, for $a = 3/5$, 10 is a kind of nombre d'or that cumulates nice properties that help to win with a large probability, with opponents sharing different logics. So let us play 10.

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[6] For a different from 3/5, the facts justifying that areas A or B do not change are the same than for a=3/5, and there always remains at least one of the reasons given for a=3/5 that justifies the enlargement of area A or B.

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