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Schedule Situations and their Cooperative Games[1]

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Abstract

We introduce a new problem of cost allocation resulting from a scheduling problem, and we study it by a new class of cooperative games, the schedule situations and the associated games. In a schedule situation several players share a non-rival common-pool infrastructure. Its consumption is possible during several periods. The consumption needs of each player are described by the set of minimal schedules satisfying this player. The use of this infrastructure induces a fixed per-period cost normalized to one unit. Therefore, one objective is to minimize the overall total number of consumption time periods in order to satisfy all players. For this purpose, the schedule game gives for each coalition of players the minimal number of time periods needed to satisfy the consumption needs of all its members. We provide a characterization of the class of schedule games: a game is a schedule game if and only if it is monotonic, sub-additive, integer-valued and all nonempty coalitions have positive worths. Moreover, specific schedule games can be linked to other classes of operational research games: the airport games and the carpool games. We also introduce Equal pooling allocations, which in some cases coincide with the Shapley value. Next we develop a natural sufficient condition to guarantee the non-emptiness of the core of a schedule game. Finally, we provide an application of the the schedule situations and the associated games to the allocation of cost of the mail carrier route in France.

Keywords: Schedule, OR-game, Cost allocation, Equal pooling allocations, Core.
JEL codes: C71, L87.

1. Introduction

In this article we introduce a new scheduling cost allocation problem called a schedule situation. Several players share a non-rival common-pool infrastructure whose consumption is possible during several periods but is costly. The per-period cost is normalized to one unit. The needs of each player are expressed in the form of consumption schedules, i.e. each schedule specifies a minimum set of periods that meets the player's needs. Consumption schedules can take a variety of forms. In some cases, the identity of the periods included in a consumption schedule is important, while in others only the number of periods matters.

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The objective is to find the combination of consumption schedules for all the players that minimizes the overall cost, i.e. the one that leads to the use of the infrastructure for the smallest number of periods in order to satisfy all the needs, in particular through potential mutualisation. Once this total cost has been determined, the natural next step is to allocate it among the participating players, taking into account how they managed to jointly use the infrastructure. We investigate this cost allocation problem by means of the theory of cooperative games with transferable utility. The resulting game specifies, for each coalition of players, the minimum number of periods of use of the infrastructure necessary to meet the needs of all members of the coalition.

Our approach originates from the concrete problem of allocating the common cost of the mail carrier route in France, which is an ongoing challenge for La Poste, the postal operator in charge of the postal universal service in France, and Arcep[2], the French national regulatory authority. The European directive 97/67/CE in article 14-3 states that the universal service providers shall keep separate accounts within their internal accounting systems between the postal products belonging the universal service scope and the other. For that reason, the common cost of the mail carrier is allocated between the different postal products that are delivered. In addition, this article states that " whenever possible, common costs shall be allocated on the basis of direct analysis of the origin of the costs themselves; [...]". Therefore, to allocate the common cost of the mail carrier route two cost drivers are taken into account, the delivery speed and the format/volume of the postal products. Currently, the common cost of the mail carrier route is allocated in two steps. In the first step, postal products are grouped into three categories according to their delivery speed: D^7, D^3 and D^1 with a delivery target on the 7^{st}, 3^{rd} and 1^{th} business day after posting, respectively. Given that La Poste must organize the delivery network in order to be in capacity to visit all recipients' addresses six days a week and given the logistical constraints, a theoretical delivery frequency of one, three, and six days per week would be required to respectively deliver D^7, D^3 and D^1. Arcep's decision 2008-0165 states that the common cost of the six weekly mail carrier routes is allocated to the three categories in proportion to their aforementioned delivery frequency: 60% of the delivery costs to D^1, 30% to D^3 and 10% to D^7. In the second step, the share of the cost previously calculated for each category is then allocated to the postal products belonging to this category according to their format/volume.

The schedule situations provide a good insight into the first step of this process (a detailed description of the second step can be found in Munich and Bohorquez Suarez, 2022). The infrastructure is the mail carrier route, which can be used six days/periods per week, and the players are the three postal product categories. The minimal consumption schedules for the three categories are as follows. For D^1, the unique consumption schedule is the set of all six days of the week (or equivalently a mail carrier route every business day) since the postal products in this category must be delivered on the next business day. On the contrary, for D^7, there are six singleton possible alternative consumption schedules, one for each day of the week (one mail carrier route is enough, no matter which day), since a postal product belonging to this category must be delivered not later than 7 days after being posted. For D^3, due to the logistic

[2] The french's electronic communications, postal and print media distribution regulatory authority. It has various responsibilities with respect to the postal sector. Notably exercising accounting and price supervision over the postal products in the universal service scope and monitoring the quality of the service provided. https://en.Arcep.fr/

constraints, the set of minimal consumption schedules contains all the triple of days which are not consecutive two by two such as, for example, $\{\text{day } 1, \text{day } 4, \text{day } 6\}$. In Section 6, we explain how our model of schedule situations can lead to a relevant alternative allocation.

On top of introducing the new model of schedule situations, we make two types of contributions to the literature, on the structure of schedule games and on the cost allocations. Regarding the first type, Proposition 1 is a characterization of the class of schedule games: a game is a schedule game if and only if it is monotonic, sub-additive, integer-valued and all nonempty coalitions have (strictly) positive worths. One of the particularities of this new class of games is that specific schedule situations can be linked to other classes of operational research games. First, a schedule situation is called anonymous if for each player only the number of consumption time periods matters but not their timing. Proposition 2 shows that the games associated to anonymous schedule situations are airport games (Littlechild and Owen, 1973). This result follows Munich and Bohorquez Suarez (2022) in which the postal allocation problem is addressed by an airport game. The latter article draws an analogy with airport games, but does not deepen or generalize the analysis as in the present article. Second, a schedule situation is called singleton if for each player there is a unique minimal schedule satisfying its needs. Proposition 4 proves that the class of games associated to singleton schedule situations coincides with the class of carpool games (Naor, 2005).

Regarding the second type of contributions, we provide natural allocations for schedule situations called the Equal pooling allocations. These allocations can be computed in two steps as follows: first, select an optimal consumption time schedule for the grand coalition, and second share the cost of each time period equally among the players who use it. On the subclass of anonymous schedule situations, Proposition 3 shows that there is always at least one optimal consumption time schedule for the grand coalition such that the corresponding Equal pooling allocation coincides with the Shapley value of the associated schedule game. On the subclass of singleton schedule situations, there is a unique Equal pooling allocation and, similarly, Proposition 6 demonstrates that it coincides with the Shapley value of the associated schedule game. Since airport games are concave, anonymous schedule games are concave too. Moreover, we also show in Proposition 5 that singleton schedule games are concave as well. In these types of schedule games the core is nonempty. However, Proposition 1 reveals that some schedule games have an empty core. Nevertheless, Corollary 1 provides a natural sufficient condition to ensure the non-emptiness of the core of a schedule game. This condition states that within each coalition, each members is guaranteed to consume the least amount of time periods among all consumption schedules satisfying its needs. In particular, this condition is weak enough to be satisfied by the classes of anonymous and singleton schedule situations.

This article is in line with the growing literature on operations research (OR) games in which the players wish to minimize total joint costs and then must distribute these joint costs among them. Borm et al. (2001) and Fiestras-Janeiro et al. (2011) provide a general view of the literature of OR problems and applications of cooperative games to cost allocation in transportation, connection, sequencing/queuing, production and inventory issues, among others. To the best of our knowledge, these models are different from ours. The Chinese postal problem is also different from our schedule situations for at least two reasons, even in the context of the postal application of Section 6. Firstly, in the Chinese postal problem the first step is to determine the cost of an optimal mail delivery route on a graph whereas the cost in this article is the optimal number of route per week. Secondly, in the Chinese postal problem the total cost is shared among the postal consumers whereas the cost in this article is shared among the postal product categories. A prominent example of OR problems is the airport problem studied, among others, by Littlechild (1974), Littlechild and Owen (1976), Littlechild and Thompson (1977), Tijs and Driessen (1986). There are also models where a structure similar to that of airport problems is applied to other contexts (Graham et al., 1990; Dehez and Ferey, 2013; Hou et al., 2018). Our model can be considered as a generalization of airport games and thus is in line with the other generalizations of the class of airport games proposed in Fragnelli et al. (1999), Kuipers et al. (2013) and Rosenthal (2017), among others.

The rest of the article is organized as follows. After giving the preliminaries on cooperative games in Section 2, we introduce the schedule situations and the associated games in Section 3. The equal pooling allocations are also presented in this section. In Section 4, we link schedule games to airport games and carpool games. In Section 5, we provide the sufficient condition for the non-emptiness of the core. Section 6 comes back to the application of allocating the cost of the mail carrier route in France. Section 7 concludes with possible extensions of this article.

2. Preliminaries on cooperative games

Let N be a nonempty and finite set of players. Each subset $E \in 2^N$ is referred to as a coalition of cooperating players. The grand coalition N represents a situation in which all players cooperate. Coalition $\emptyset$ represents a situation in which no player cooperates, it is called the empty coalition. For each $E \in 2^N$, the integer $|E| \in \mathbb{N}$ denotes the cardinality of coalition E.

A transferable utility game, or simply a TU-game, is a couple (N, v) consisting of a finite players set N and a characteristic function $v : 2^N \to \mathbb{R}$, with the convention that $v(\emptyset) = 0$. The real number $v(E)$ can be interpreted as the worth the players in E generate when they cooperate. This worth can be perceived by the players as desirable (like profits) or, on the contrary, undesirable (like costs). We will focus on the second case: the players share cost. Thus the game (N, v) is a cost game. For ease of writing the game (N, v) will be designated by its characteristic function v where N is fixed. A game v may satisfy some interesting properties:

Monotone For each $E \subseteq S \subseteq N$, $v(E) \leq v(S)$.

Adding a player to a coalition does not reduce its cost.

Sub-additive For each couple of coalitions $E, S \subseteq N$ such that $E \cap S = \emptyset$, $v(E \cup S) \leq v(E) + v(S)$.

When two disjoint coalitions come together, the resulting joint cost is at most equal to the sum of their initial costs. Merging two coalitions is not detrimental to their members.

Concave For each $i \in N$ and each $E \subseteq S \subseteq N \backslash \{i\}, v(E \cup \{i\}) - v(E) \geqslant v(S \cup \{i\}) - v(S)$.

This property indicates that the incremental cost due to the arrival of a new player in a coalition does not increase if this coalition grows.

Strictly positive For each $E \subseteq N$, $v(E) > 0$.

Finally, we say that a game v is **integer-valued** if $v(E)$ is an integer for each $E \subseteq N$.

The basic issue in the theory of cooperative games is to divide fairly the cost of the grand coalition among its members. This issue may be addressed using allocations for TU-games. An allocation $x \in \mathbb{R}^{|N|}$ is a $|N|$-dimensional vector that assigns a share of the cost $x_i \in \mathbb{R}$ to each player $i \in N$.

An efficient allocation shares exactly $v(N)$ among the players and it is called coalitionally rational if no coalition would be better off by splitting from the grand coalition and paying its cost. The core of a game v, is the set $Core(v)$ of efficient and coalitionally rational allocations:

$$Core(v) = \big\{x \in \mathbb{R}^N : \sum_{i\in N} x_i = v(N) \text{ and for each } E \subseteq N, \sum_{i\in E} x_i \leqslant v(E)\big\}.$$

The core of a game can be empty. However, Shapley (1971) demonstrates that the core of a concave game is nonempty. The core can contain often several allocations from which it can be difficult to choose one and only one. Alternatively, the Shapley value assigns to each game v a unique allocation $Sh(v)$ such that for each $i \in N$:

$$Sh_i(v) = \sum_{E\subseteq N\setminus\{i\}} \frac{|E|!(|N| - |E| - 1)!}{|N|!} \Big(v(E \cup \{i\}) - v(E)\Big).$$

Shapley (1971) proves that the Shapley value of a concave game lies in its core.

3. Schedule situations and schedule games

A group of players share a common-pool resource whose consumption is possible during several periods. The use of this resource induces a fixed per-period cost normalized to one unit. The players have different demands represented by the subsets of periods allowing to satisfy their needs. Let us formalize this framework and illustrate its features.

3.1. Schedule situations and schedule games

Let N be a fixed finite set of n players. A **schedule situation** on N is a tuple $M = (T, (T_i)_{i\in N})$ where

- $T = \{1, \ldots, |T|\}$ is a finite set of **time periods**.
- for each $i \in N$, $T_i \subset 2^T\setminus\{\emptyset\}$ is the nonempty set of minimal (w.r.t. inclusion) **time configurations** satisfying the needs of player i.

The previous minimality condition implies that if $S, E \in T_i$, then neither $S \subset E$ nor $E \subset S$.

In words, each player needs a schedule for the consumption of a common-pool resource. Such a schedule specifies the needed subset of consumption time periods. The set T_i collects all minimal (with respect to set inclusion) schedules or time configurations satisfying the consumption needs of player i. The use of the common-pool resource is costly, so that the objective is to minimize the overall total number of consumption time periods while satisfying the needs of all players. In order to do so, we introduce an associated cooperative game called the **schedule**

game. If $T_E = \prod_{i \in E} T_i$ denotes, for each $E \subseteq N$, the time configurations for E, then the associated schedule game is defined by

$$v_M(E) = \min_{R \in T_E} \Big| \bigcup_{Q \in R} Q \Big|.$$

The integer $v_M(E)$ is the minimal number of time periods needed to satisfy the consumption needs of all the members of E.

Example 1. *Set $N = \{A, B, C\}$, $T = \{1, \ldots, 8\}$, $T_A = \{\{1,2\}, \{3,4,5\}\}$, $T_B = \{\{1,2\}, \{7,8\}\}$ and $T_C = \{\{3,4,5\}, \{6,7,8\}\}$. Then*

E	$\{A\}$	$\{B\}$	$\{C\}$	$\{A,B\}$	$\{A,C\}$	$\{B,C\}$	$\{A,B,C\}$
$v_M(E)$	2	2	3	2	3	3	5

As an example, consider coalition $\{B, C\}$. We have

$$T_{\{B,C\}} = T_B \times T_C = \big\{(\{1,2\}, \{3,4,5\}), (\{1,2\}, \{6,7,8\}), (\{7,8\}, \{3,4,5\}), (\{7,8\}, \{6,7,8\})\big\}.$$

Hence,

$$v_M(\{B,C\}) = min\big\{|\{1,2,3,4,5\}|, |\{1,2,6,7,8\}|, |\{3,4,5,7,8\}|, |\{6,7,8\}|\big\} = 3,$$

which means that player B can completely pool its two-period demand $\{7,8\}$ with the demand $\{6,7,8\}$ of player C. □

The first result below characterizes the class of schedule games.

Proposition 1. *The class of all schedule games on N coincides with the class of monotone sub-additive integer-valued strictly positive TU-games on N.*

Proof. It is obvious that v_M is monotonic, integer-valued and strictly positive for each schedule situation M. Furthermore, for a schedule situation M on N, consider any pair of coalitions $E, S \subseteq N$ such that $E \cap S = \emptyset$. Pick any time configurations R^1 and R^2 such that $v_M(E) = |\bigcup_{Q \in R^1} Q|$ and $v_M(S) = |\bigcup_{Q \in R^2} Q|$. Since $(R^1, R^2) \in T_{E \cup S}$, i.e., the time configurations R^1 and R^2 for E and S are still available, when combined, as a time configuration for $E \cup S$, we immediately get $v_M(E) + v_M(S) \geq v_M(E \cup S)$, proving that v_M is sub-additive.

Conversely, let v be any monotonic, subadditive integer-valued strictly positive game on N. To show: there is a schedule situation M on N such that $v_M = v$. Consider any ordering π of the $2^n - 1$ nonempty coalitions on N, where, for each nonempty coalition E, $\pi(E)$ stands for the position of E according to π. Moreover, for each nonempty E, define $a_E = \sum_{S \subseteq N : \pi(S) < \pi(E)} v(S)$ and $A_E = \{a_E + 1, \ldots, a_E + v(E)\}$. Remark that, for each $E, S \subseteq N$ with $E \neq S$,

$$A_E \cap A_S = \emptyset. \tag{1}$$

From v, we construct the schedule situation $M = (T, (T_i)_{i \in N})$ such that $T = \{1, \ldots, |T|\}$ with $|T| = \sum_{E \subseteq N, E \neq \emptyset} v(E)$ and, for each $i \in N$, $T_i = \{A_S : S \ni i\}$. Equation (1) implies that the minimality condition imposed in the definition of set T_i is satisfied. Hence, any time configuration

R for N is of the form $R = (A_{S_i})_{i \in N}$ where, for each $i \in N$, S_i is a coalition containing player i. From now on, we focus on an arbitrary nonempty coalition E in order to prove that $v_M(E) = v(E)$. From E and any time configuration $R \in T_E$, $R = (A_{S_i})_{i \in E}$, define $x^E(R) = \{S \subseteq N : A_{S_i} = A_S \text{ for some } i \in E\}$ and $y^E(R) = |\bigcup_{Q \in R} Q|$. Hence, $v_M(E) = \min_{R \in T_E} y^E(R)$ or equivalently, from (1), $v_M(E) = \min_{R \in T_E} \sum_{S \in x^E(R)} v(S)$. Remark that $R^E := (A_E, \ldots, A_E)$ belongs to T_E and that $y^E(R^E) = |A_E| = v(E)$ since $x^E(R^E) = \{E\}$. It remains to show that if $R \in T_E$, $R = (A_{S_i})_{i \in E}$, then $y^E(R) \geqslant v(E)$. Given R and $S \in x^E(R)$, define $E_R(S) = \{i \in E : A_{S_i} = S\}$ and note that

$$E_R(S) \subseteq S. \tag{2}$$

From R, construct the collection $\bar{R} = (A_{E_R(S_i)})_{i \in E}$, which implies that $\bar{R} \in T_E$. By definition, $x^E(\bar{R})$ is a partition of E. We can write that

$$y^E(R) = \sum_{S \in x^E(R)} v(S) \geqslant \sum_{S \in x^E(R)} v(E_R(S)) = \sum_{S' \in x^E(\bar{R})} v(S') = y^E(\bar{R}) \geqslant v(E),$$

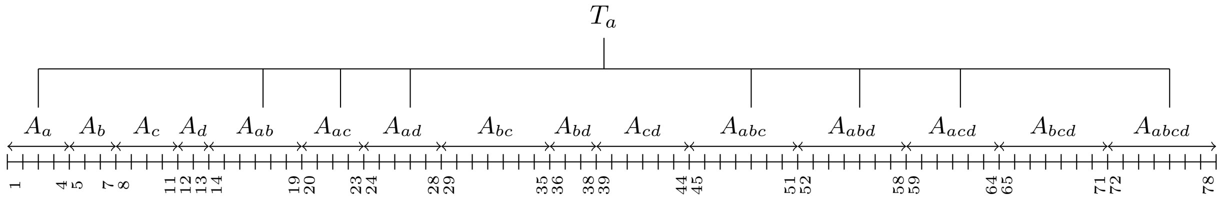
where the first inequality comes from the monotonicity of v and equation (2), and the second inequality comes from the subadditivity of v and the fact that $x^E(\bar{R})$ is a partition of E. We conclude that $v_M(E) = v(E)$, as desired. ■

Two remarks are in order. First, it is not difficult to get rid of the condition of strict positivity in Proposition 1. The only slight change needed in the definition of a schedule situation is to allow the sets T_i to be empty. Second, Proposition 1 implies that not all schedule games have a nonempty core as pointed out in the introduction.

In order to illustrate the proof, we consider the following four-players game in which brackets and commas are omitted in order to save space.

E	a	b	c	d	ab	ac	ad	bc	bd	cd	abc	abd	acd	bcd	$abcd$
$v(E)$	4	3	4	2	6	4	5	7	3	6	7	7	6	7	7

It is easy to check that v satisfies the conditions imposed in Proposition 1. As a start, let us build the schedule situation M as in the proof. The sum of all coalitions' worth is equal to 78 so that $T = \{1, \ldots, 78\}$. Let us also use the order of all 15 nonempty coalitions from left to right in the previous table, i.e., coalitions are ordered by size and lexicographically within a given size. The first coalition in the order is $\{a\}$ for which we create set $A_{\{a\}} = \{1, \ldots, 4\}$ since $a_{\{a\}} = 0$ and $v(\{a\}) = 4$. The second coalition in the order is $\{b\}$, $a_{\{b\}} = 4$ and $v(\{b\}) = 3$, so that $A_{\{b\}} = \{5, 6, 7\}$. Continuing in this fashion, we get for coalition $\{a, b, c\}$ that $a_{\{a,b,c\}} = 44$, which implies that $A_{\{a,b,c\}} = \{45, \ldots, 51\}$ since $v(\{a, b, c\}) = 7$. All sets A_E are disjoint two by two as illustrated by the following picture (in which brackets and commas are omitted as well).



We now focus on the previous coalition $E = \{a, b, c\}$ in order to show that $v_M(\{a, b, c\}) = v(\{a, b, c\})$. The proof proceeds in three steps.

Firstly, we prove that a specific time configuration for E costs exactly $v(E)$. In fact, since $A_{\{a,b,c\}}$ belongs to T_a, T_b and T_c, $R^{\{a,b,c\}} := (A_{\{a,b,c\}}, A_{\{a,b,c\}}, A_{\{a,b,c\}}) \in T_{\{a,b,c\}}$. Obviously, $y^E(R^{\{a,b,c\}}) = |A_{\{a,b,c\}}| = v(\{a, b, c\}) = 7$. In the final two steps, we show that no other time configuration $R \in T_{\{a,b,c\}}$ can do better. We only illustrate these steps with $R = (A_{\{a,b,c,d\}}, A_{\{a,b,c,d\}}, A_{\{c,d\}})$.

Secondly, step two is a "reduction" step in which the individual time configurations in R are reduced by eliminating unnecessary needs in some sense. From R and E, we have $x^E(R) = \{\{a, b, c, d\}, \{c, d\}\}$ so that $y^E(R) = v(\{a, b, c, d\}) + v(\{c, d\})$. We drop from coalition $\{a, b, c, d\}$ the two players c and d that do not choose $A_{\{a,b,c,d\}}$ in R and similarly, we drop d from $\{c, d\}$. The resulting coalitions, called $E_R(\{a, b, c, d\}) = \{a, b\}$ and $E_R(\{c, d\}) = \{c\}$ in the proof, are subsets of the original coalitions and $\bar{R} := (A_{\{a,b\}}, A_{\{a,b\}}, A_{\{c\}})$ is also in $T_{\{a,b,c\}}$. The monotonicity of v then yields that $|A_{\{a,b,c,d\}}| > |A_{\{a,b\}}|$ and $|A_{\{c,d\}}| > |A_{\{c\}}|$ so that $y^E(\bar{R}) < y^E(R)$. Thus, $\bar{R}$ is already better than R for coalition E.

Thirdly, step 3 is a "partition" step in which non-pooled consumption in $\bar{R}$ are compared to the fully pooled consumption in $R^{\{a,b,c\}}$. To see this, note that $\{a, b\}$ and $\{c\}$ form a partition of E so that the sub-additivity of v yields that $v(\{a, b, c\}) < v(\{a, b\}) + v(\{c\})$. Thus, we conclude that $y^E(R^{\{a,b,c\}}) < y^E(\bar{R})$, proving that $R^{\{a,b,c\}}$ from step one is even better than $\bar{R}$ for coalition E.

Next, we introduce three specific schedule situations. In the first one only the number of consumption time periods matters but not their timing.

Definition 1. *A schedule situation is called* **anonymous** *if for each player all time configurations of a certain size satisfy the needs of a player. Formally, for each $i \in N$, there is $p_i \in \{1, \dots, t\}$ such that $T_i = \{Q \subseteq T : |Q| = p_i\}$.*

This definition can be illustrated by the following example:

Example 2. *The set of players is $N = \{A, B, C\}$ and the set of time periods is $T = \{1, ..., 6\}$ and*

$$T_A = \{\{q\}, q \in T\},$$
$$T_B = \{E \subseteq T : |E| = 4\},$$
$$T_C = \{T\}.$$

Then the resulting game is:

E	$\{A\}$	$\{B\}$	$\{C\}$	$\{A,B\}$	$\{A,C\}$	$\{B,C\}$	$\{A,B,C\}$
$v_M(E)$	1	4	6	4	6	6	6

□

In the second type of schedule situation, on the contrary, the players have no flexibility: each seeks a unique particular minimal time configuration.

Definition 2. *A schedule situation is called* **singleton** *if for each player there is a unique minimal time configuration satisfying the needs of this player. Formally, for each $i \in N$ we have $|T_i| = 1$. In this case let us denote by A_i the unique element of T_i, for each $i \in N$.*

Example 3. *The set of players is $N = \{A, B, C\}$ and the set of time periods is $T = \{1, ..., 6\}$. The players' time configuration is the following $T_A = \{\{1\}\}$, $T_B = \{\{5\}\}$ and $T_C = \{\{1, 3, 5\}\}$. This is a singleton schedule situation in the sense of definition 2. Then,*

E	$\{A\}$	$\{B\}$	$\{C\}$	$\{A,B\}$	$\{A,C\}$	$\{B,C\}$	$\{A,B,C\}$
$v_M(E)$	1	1	3	2	3	3	3

□

In the two previous situations, within a coalition the players need not consume more than what they individually need for pooling reasons. The third type of schedule situation is build around an optimal time configuration for the grand coalition with a similar property. Formally, for each nonempty coalition E, let $O(E)$ be the set of all optimal time configurations, i.e. those which minimize the overall total number of consumption time periods in order to satisfy the needs of E:

$$O(E) = \big\{R \in T_E : \big|\bigcup_{Q \in R} Q\big| \leq \big|\bigcup_{Q \in R'} Q\big|, \forall R' \in T_E\big\}.$$

An optimal time configuration $R^* \in O(N)$ with $R^* = (\{A_i^*\})_{i \in N}$ is called **coherent** for M if for each $E \subseteq N, E \neq \emptyset$ $R_E^* \in O(E)$, where R_E^* is the restriction of R^* to E. Hence, a time configuration for N is coherent if no player has an incentive to change its consumption schedule in smaller coalitions.

Definition 3. *A schedule situation is called* **coherent** *if it admits a coherent optimal time configuration.*

These types of schedule situations are illustrated in the example below.

Example 4. *Let $N = \{A, B, C\}$ and $T = \{1, ..., 6\}$. We have the following time configurations:*

$$T_A = \{\{1, 2\}, \{2, 3, 4\}, \{2, 5\}\},$$
$$T_B = \{\{1, 3, 4\}, \{1, 5\}\},$$
$$T_C = \{\{2, 4, 5\}, \{3, 4, 5\}\}.$$

Then the resulting game is:

E	$\{A\}$	$\{B\}$	$\{C\}$	$\{A,B\}$	$\{A,C\}$	$\{B,C\}$	$\{A,B,C\}$
$v_M(E)$	2	2	3	3	3	4	4

□

There are two optimal time configurations for N:

$O(N) = \{\{R^1\}, \{R^2\}\}$ where $R^1 = (\{1, 2\}, \{1, 5\}, \{2, 4, 5\})$ and $R^2 = (\{2, 5\}, \{1, 5\}, \{2, 4, 5\})$.

This is also the case for several smaller coalitions as illustrated in the following table:

E	$v_M(E)$	$O(E)$
$\{A\}$	2	$(\{1,2\})$ $(\{\mathbf{2},\mathbf{5}\})$
$\{B\}$	2	$(\{\mathbf{1},\mathbf{5}\})$
$\{C\}$	3	$(\{\mathbf{2},\mathbf{4},\mathbf{5}\})$ $(\{3,4,5\})$
$\{A,B\}$	3	$(\{1,2\},\{1,5\})$ $(\{\mathbf{2},\mathbf{5}\},\{\mathbf{1},\mathbf{5}\})$
$\{A,C\}$	3	$(\{\mathbf{2},\mathbf{5}\},\{\mathbf{2},\mathbf{4},\mathbf{5}\})$
$\{B,C\}$	4	$(\{1,3,4\},\{3,4,5\})$ $(\{2,3,4\},\{2,4,5\})$ $(\{1,5\},\{3,4,5\})$ $(\{\mathbf{1},\mathbf{5}\},\{\mathbf{2},\mathbf{4},\mathbf{5}\})$
$\{A,B,C\}$	4	$R^1=(\{1,2\},\{1,5\},\{2,4,5\})$ $\mathbf{R^2}=(\{\mathbf{2},\mathbf{5}\},\{\mathbf{1},\mathbf{5}\},\{\mathbf{2},\mathbf{4},\mathbf{5}\})$

Table 1: The set of all optimal time configurations

The time configuration R^2 is the unique coherent time configuration (this is highlighted in bold characters in Table 1).

Examples 2 to 3 yield games with nonempty cores but this is not the case for all schedule games. The example 1 illustrates that the core of a schedule game can be empty. Let an allocation x be a candidate to belong to the core. Note that $x_A + x_B \leq 2$ and efficiency leads to $x_C \geq 3$. Similarly the use of efficiency together with $x_A + x_C \leq 3$ and $x_B + x_C \leq 3$ yields that $x_B \geq 2$ and $x_A \geq 2$. Summing these 3 inequalities, we get $x_A + x_B + x_C \geq 7$, which is incompatible with the efficiency constraint. Thus, $Core(v) = \emptyset$.

3.2. Equal pooling allocations

For schedule situations a natural allocation can be formulated in two steps. Firstly, we compute the set of all optimal time configurations for the grand coalition. The periods belonging to this set are called **active**. Secondly, if a player is the only one who consumes the common-pool resource during an active time period, it alone incurs the unit cost of this time period. Nevertheless, if there are several players which consume the common-pool resource during the same active time period, they pool the cost and share it equally among them. For these reasons, we call this allocation **Equal Pooling allocation**.

Definition 4. *Fix any schedule situation M. Let $R^* = (A_1^*, ..., A_n^*)$ be an element of $O(N)$. The* **Equal Pooling allocation** *EP^{R^*} on M associated with R^* is such that, for each $i \in N$,*

$$EP_i^{R^*}(M) = \sum_{t \in T : t \in A_i^*} \frac{1}{|\{j \in N : t \in A_j^*\}|}.$$

Since there can be more than one optimal time configuration for N, there can be several Equal pooling allocations as underlined in the following example. For the Example 4 we obtain the following results:

$$EP^{R^1}(M) = (1;1;2),$$
$$EP^{R^2}(M) = (\frac{5}{6};\frac{8}{6};\frac{11}{6}).$$

4. Particular schedule situations

4.1. Anonymous schedule situations and airport games

An **airport situation** (Littlechild and Owen, 1973) on N is a tuple $A = ((N_j, c_j)_{j\in\{1,...,m\}})$ where N_j denotes the set of n_j aircrafts of type j, for $j = 1, ..., m$, and $N = \cup_{j=1}^{m} N_j$ and $n = \sum_{j=1}^{m} n_j$. The cost associated with an aircraft of type j is given by c_j. These types of aircraft are ordered so that $c_0 < c_1 < ... < c_m$, where $c_0 = 0$ by convention. Any airport situation A gives rise to an **airport game** v_A such that for each $E \subseteq N$,

$$v_A(E) = \max_{j\in\{1,...,m\}:E\cap N_j\neq\emptyset} c_j.$$

Note that the schedule game in Example 2 coincides with the airport situation in which $N_1 = \{A\}$, $N_2 = \{B\}$, $N_3 = \{C\}$ and $c_1 = 1$, $c_2 = 4$, $c_3 = 6$. We show below that this property holds for any anonymous schedule situation. To see this, from any **anonymous schedule situation** M (recall definition 1 page 8) it is possible to construct a specific airport situation $A^M = ((N_j^M, c_j^M)_{j\in\{1,...,m\}})$ such that

$$c_1^M = \min_{i\in N} p_i \quad \text{and} \quad N_1^M = \{i \in N : p_i = c_1^M\},$$

and for each $j = 2, ..., m$,

$$c_j^M = \min_{i\in N\setminus\left(\cup_{k=1}^{j-1} N_k^M\right)} p_i \quad \text{and} \quad N_j^M = \{j \in N : p_j = c_j^M\},$$

The next result shows that anonymous schedule games are airport games.

Proposition 2. *If M is an anonymous schedule situation on N, then the associated schedule game v_M coincides with the airport game v_{A^M}.*

Proof. Let $M = (T, (T_i)_{i\in N})$ with $T_i = \{Q \subseteq T : |Q| = p_i\}, p_i \in \{1, ..., |T|\}$ be any anonymous situation on N and let E be any nonempty coalition in N. We will prove that,

$$v_M(E) = \max_{i\in E} p_i = v_{A^M}(E).$$

We proceed in two steps.
Step 1. Let

$$p^* = \max_{i\in E} p_i.$$

To show that $v_M(E) = p^*$ denote by i one of the players in E such that $p_i = p^*$. Then since the schedule situation M is anonymous, any $A_i \in T_i$ is such that $|A_i| = p^*$. We immediately get that $v_M(E) \geqslant p^*$. Next, for each $j \in E, \{1, ..., p_j\} \in T_j$ and $\{1, ..., p_j\} \subseteq \{1, ..., p^*\}$. Hence,

$$\left| \bigcup_{j \in E} \{1, ..., p_j\} \right| = \left| \{1, ..., p^*\} \right| = p^*,$$

which implies that $v_M(E) \leqslant p^*$. Thus $v_M(E) = p^*$.

Step 2. Consider the airport situation A^M. By definition of an airport game, for each nonempty E, we have:

$$v_{A^M}(E) = \max_{j \in \{1,...,m\}: E \cap N_j^M \neq \emptyset} c_j^M.$$

It is clear that player $i \in E$ such that $p_i = p^*$ is the player belonging to the group N_j with the greatest index j among E, from which one gets,

$$v_{A^M}(E) = p^*.$$

This completes the proof. ■

It is well-known that airport games are concave. According to the Proposition 2, anonymous schedule games are concave too.

Proposition 3. *If M is an anonymous schedule situation on N, then there is $R^* \in O(N)$ such that the Equal pooling allocation $EP^{R^*}(M)$ coincides with the Shapley value of game v_M.*

Proof. As a reminder, in the specific airport situations A^M each player with the same needs in time periods $p_i \in \{1, ..., t\}$ are grouped in the same subset of players. Formally, $N_j^M = \{i \in N : p_i = c_j^M\}$ with $j = 1, ..., m$ and c_j^M the costs associated to the group of players of type j. These types of players are ordered so that $c_1^M < c_2^M < ... < c_m^M$.

Littlechild and Owen (1973) give the following expression for the Shapley value of an airport game:

$$Sh_i(v_{A^M}) = \sum_{q=1}^{j} \frac{c_q^M - c_{q-1}^M}{\sum_{g=q}^{m} |N_g^M|}, \tag{3}$$

with $i \in N_j^M$. The Equal pooling allocation associated to some $R^* = (\{A_i^*\})_{i \in N} \in O(N)$, according to definition 4, is given by:

$$EP_i^{R^*}(M) = \sum_{t \in T: t \in A_i^*} \frac{1}{|\{j \in N : t \in A_j^*\}|}. \tag{4}$$

To prove Proposition 3, we first rewrite the Equal pooling allocation in the context of an anonymous schedule situation M. Proposition 2 implicitly demonstrates that $(\{1, ..., p_i\})_{i \in N}$ is a coherent optimal time configuration. So let $R^* = (\{1, ..., p_i\})_{i \in N}$ in (4). In particular, if player k is such that $p_k \leqslant p_i$ then $\{1, ..., p_k\} \subseteq \{1, ..., p_i\}$. We can rewrite (4) as follows:

$$EP_i^{R^*}(M) = \sum_{t=1}^{p_i} \frac{1}{|\{k \in N : t \leqslant p_k\}|}.$$

Secondly, we rewrite the previous expression in the form of (3). For each $i \in N_j^M$, we have $p_i = c_j^M$ and $|\{k \in N : t \leqslant p_k\}|$ corresponds to $\sum_{q=\{1,\ldots,m\}:t\leqslant c_q^M} |N_q^M|$. Hence,

$$EP_i^{R^*}(M) = \sum_{t=1}^{c_j^M} \frac{1}{\sum_{q=\{1,\ldots,m\}:t\leqslant c_q^M} |N_q^M|}.$$

Summing over types of players instead of summing over periods, we get:

$$EP_i^{R^*}(M) = \sum_{q=1}^{j} \frac{c_q^M - c_{q-1}^M}{\sum_{g=q}^{m} |N_g^M|} = Sh_i(v_{A^M}).$$

This specific Equal pooling allocation is equal to the Shapley value, as desired. ■

4.2. Singleton schedule situations and carpool games

A **carpool situation** (Naor, 2005) is a situation in which players form a carpool and decide to use it on different days. Formally, a carpool situation on N is a tuple $D = (D_k)_{k=1,\ldots,l}$ where each $D_k \subseteq N$ corresponds to the nonempty set of players who showed on day $k \in \{1, \ldots, l\}$. The use of the carpool system is costly: any carpool situation D gives rise to a **carpool game** v_D such that, for any subset $E \subseteq N$, $v_D(E)$ associates for each coalition E a cost measured by the number of days on which at least one player of the coalition E shows up, i.e.,

$$v_D(E) = \big|\{1 \leqslant j \leqslant l : D_j \cap E \neq \emptyset\}\big|.$$

From any singleton schedule situation M it is possible to construct a specific carpool situation $D^M = (D_k^M)_{k=1,\ldots,l}$, such that $l = |T|$ for each $k \in \{1, \ldots, l\}$, $D_k^M = \{i \in N : k \in A_i\}$, where A_i is the unique element in T_i. It is easy to get the correspondence between the carpool situation and the singleton schedule situation. In Example 3, the set of periods T can represent the set of days where the players A, B and C "showed up" or must be distributed. The following table gives the relationship between the carpool and the schedule situations:

$k \backslash i$	A	B	C	D_k^M
1	X		X	$\{A, C\}$
2				$\emptyset$
3			X	$\{C\}$
4				$\emptyset$
5		X	X	$\{B, C\}$
6				$\emptyset$
A_i	$\{1\}$	$\{5\}$	$\{1, 3, 5\}$	

Table 2: Relationship between carpool and schedule situations

Proposition 4.

If M is a singleton schedule situation on N, then the associated schedule game v_M coincides with the carpool game v_{D^M}.

Proof. Let $M = (T, (T_i)_{i\in N}$ with $T_i = \{A_i\}, i \in N$ be any singleton schedule situation. For any E subset of N, $v_M(E)$ can be rewritten as:

$$v_M(E) = \Big|\bigcup_{i\in E} A_i\Big|. \tag{5}$$

Let us show that $v_{D^M}(E) = v_M(E)$. Since $l = |T|$, we have

$$\begin{aligned} v_{D^M}(E) &= |\{1 \leq j \leq |T| : D_j^M \cap E \neq \emptyset\}| \\ &= |\{1 \leq j \leq |T| : \{i \in N : j \in A_i\} \cap E \neq \emptyset\}| \\ &= |\{1 \leq j \leq |T| : j \in A_i \text{ for some } i \in E\}| \\ &= |\textstyle\bigcup_{i\in E} A_i| \\ &= v_M(E), \end{aligned}$$

as desired. ■

Reciprocally it is easy to figure out that each carpool game coincides with the schedule game associated with some singleton schedule situation. In other words, the class of singleton schedule games on N coincides with the class of carpool games on N. The games in the latter class are concave as demonstrated below.

Proposition 5. *If M is a singleton schedule situation on N, then the associated schedule game v_M is concave.*

Proof. Let M be a singleton schedule situation on N with $T_i = \{A_i\}$ for each $i \in N$. We will prove that for each $E \subseteq S \subseteq N\backslash\{i\}$, $v_M(E \cup \{i\}) - v_M(E) \geqslant v_M(S \cup \{i\}) - v_M(S)$. From (5), we can rewrite both parts of the inequality as follows:

$$v_M(E \cup \{i\}) - v_M(E) = \big|A_i \backslash \big(\cup_{j\in E} A_j\big)\big|,$$
$$v_M(S \cup \{i\}) - v_M(S) = \big|A_i \backslash \big(\cup_{j\in S} A_j\big)\big|.$$

Next, we can write that:

$$\begin{aligned} & \cup_{j\in E} A_j \subseteq \cup_{j\in S} A_j \\ \iff\ & A_i \backslash \big(\cup_{j\in E} A_j\big) \supseteq A_i \backslash \big(\cup_{j\in S} A_j\big) \\ \iff\ & \big|A_i \backslash \big(\cup_{j\in E} A_j\big)\big| \geqslant \big|A_i \backslash \big(\cup_{j\in S} A_j\big)\big|. \end{aligned}$$

Hence,

$$v_M(E \cup \{i\}) - v_M(E) \geqslant v_M(S \cup \{i\}) - v_M(S),$$

for each $i \in N$ and $E \subseteq S \subseteq N\backslash\{i\}$. ■

In a carpool game Fagin and Williams (1983) and continued by Ajtai et al. (1998) proposed an equal share of the cost of each days between the players who used it. The resulting allocation rule α is as follows, for each D and each i:

$$\alpha_i(D) = \sum_{j \in \{1,\ldots,l\}: i \in D_j} \frac{1}{|D_j|}.$$

Naor (2005) demonstrated that $\alpha(D)$ is the Shapley value of the game v_D, i.e. $\alpha(D) = Sh(v_D)$. Note that there is trivially a unique optimal time configuration $R^* \in O(N)$ such that $R^* = (\{A_i\})_{i \in N}$ for each singleton schedule situation M.

Proposition 6.
If M is a singleton schedule situation on N, then the Equal pooling allocation $EP^{R^*}(M)$ coincides with the allocation $\alpha(D^M)$ of the carpool situation D^M.

Proof. The claim follows from viewing the allocation rule $EP^{R^*}(M)$ as the sum of the inverse of the number of players who consume the active time period t simultaneously. In the carpool situation the active time period t is expressed by a set of days $D_k^M \subseteq N$ corresponding to the players who showed on day k and $|D_k^M|$ is the number of these players. Hence, $|D_k^M|$ and $|\{j \in N : t \in A_j\}|$ express the same thing and $EP_i(M) = \alpha_i(D^M)$. See table 2. Therefore, $EP_i^{R^*}(M)$ is the Shapley value of the game v_M when M is a singleton schedule situation on N. ■

5. Non-emptiness of the core of coherent schedule situations

Proposition 1 reveals that the core of a schedule game can be empty (see Example 1). In this section, we provide a sufficient condition for the non-emptiness of the core of a schedule game. If we focus on Example 4, the time configuration R^2 is the unique coherent time configuration. The presence of such a coherent time configuration is sufficient to guarantee that the core is nonempty as a corollary of the next result.

Proposition 7. *If M is coherent, then $v_M = v_{M'}$ for some singleton schedule situation M'.*

Proof. Let $R^* = (A_i^*)_{i \in N}$ be any coherent time configuration on M. From M and R^*, construct the schedule situation M^{R^*} such that $M^{R^*} = (T^{R^*}, (T^{R^*})_{i \in N})$ with $T^{R^*} = T$ and $T_i^{R^*} = \{A_i^*\}$. Consequently, M^{R^*} is a singleton schedule situation. In addition, R^* is coherent for M^{R^*}. Therefore, $v_{M^{R^*}} = v_M$ follows from the fact that R^* is coherent for both M^{R*} and M. ■

From Propositions 5, 6 and 7, we get the following corollary.

Corollary 1. If R^* on M is coherent, then the Equal pooling allocation $EP^{R^*}(M)$ is in the core of v_M and coincides with the Shapley value $Sh(v_M)$.

The condition in Proposition 7 is sufficient but not necessary. In the following example an allocation $EP^{R*}(M)$ is in the core of v_M even if R^* is not coherent for a schedule situation M.

Example 5. *Let* $N = \{A, B, C\}, T = \{1, \ldots, 5\}, T_A = \{\{1, 2, 3\}, \{1, 4\}, \{2, 4\}\}, T_B = \{\{1, 5\}, \{2, 5\}\}$ *and* $T_C = \{\{1, 2, 3, 5\}\}$*. Then we obtain the following table,*

E	$v_M(E)$	$O(E)$	$\sum_{i \in E} x_i$
$\{A\}$	2	$(\{1,4\})$ $(\{2,4\})$	1
$\{B\}$	2	$(\{1,5\})$ $(\{2,5\})$	1
$\{C\}$	4	$(\{1,2,3,5\})$	2
$\{A,B\}$	3	$(\{1,4\},\{1,5\})$ $(\{2,4\},\{2,5\})$	2
$\{A,C\}$	4	$(\{1,4\},\{1,2,3,5\})$ $(\{2,4\},\{1,2,3,5\})$	3
$\{B,C\}$	4	$(\{1,5\},\{1,2,3,5\})$ $(\{2,5\},\{1,2,3,5\})$	3
$\{A,B,C\}$	4	$(\{1,2,3\},\{1,5\},\{1,2,3,5\})$ $(\{1,2,3\},\{2,5\},\{1,2,3,5\})$	4

Table 3: The set of all optimal time configurations

□

As in Example 1, the time configuration of player A used to compute $v_M(N)$ is not its smaller time configuration. More specifically, the selected time configuration for A is $(\{1, 2, 3\})$ whereas on its own its smaller time configurations are $(\{1, 4\})$ or $(\{2, 4\})$. Hence, none of the two optimal time configurations on N is coherent. However, contrary to Example 1, the core of this example is nonempty since it contains allocation $x = (1; 1; 2)$ as shown by the above table. Remark that the two Equal pooling allocations are also in the core.

6. An application to the French postal case

Below, we present once again the problem of allocating the cost of the mail carrier route in France which was already mentioned in the introduction. To meet its universal service obligations, La Poste must organize the delivery network in order to be in capacity to visit all recipients' addresses six days a week and meet the **delivery speed** of the postal products. The delivery speed refers to the time period within which a particular postal product must be delivered, from the moment between it is posted until its actual delivery at the customers' location choice. In its decision 2008-0165 the French national regulatory authority Arcep, in charge of defining the allocation rules of universal products' costs, distinguished three delivery speed categories: D^7 for a delivery target on the 7^{th} business day after posting, D^3 for a delivery target on the 3^{rd} business day after posting and D^1 for a delivery target on the 1^{st} business day after posting. Considering logistical constraints, a **delivery frequency** of one day per week would be enough to satisfy D^7, delivery frequency of three days per week would be enough to satisfy D^3 and delivery frequency of six days per week would be required to satisfy D^1. Arcep's decision states that the common cost of the six weekly mail carrier routes, first, is allocated to the three categories in proportion to their aforementioned delivery frequency, i.e. 10% of the delivery costs to D^7, 30% to D^3 and 60% to D^1. Secondly, the share of the cost previously calculated for each category is then allocated to the postal products belonging to this category according to their format/volume. We will only focus on the first part of this process which can be apprehended by an anonymous schedule game.

We can use the rich possibilities offered by the schedule situations in order to model the cost sharing of the mail carrier route as the following schedule situation M^1. The infrastructure is the mail carrier route which can be used once during six days per week, so that $T = \{1, 2, 3, 4, 5, 6\}$. Period 1 represents the delivery day Monday and so on. The players are the three postal product categories, i.e. $N = \{D^7, D^3, D^1\}$. For category D^7, there are six singleton possible alternative consumption schedules, one for each day of the week, since a postal product in this category must be delivered not later than 7 days after being posted. On the contrary, for category D^1, the unique consumption schedule is the set of all six days of the week since the postal products in this category must be delivered on the next business day. For category D^3, the set of minimal consumption schedules contains all the triple of days which are not consecutive two by two[3]. Therefore:

$$\begin{gathered} T_{D^7} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\}, \\ T_{D^3} = \{\{1, 3, 5\}, \{1, 3, 6\}, \{1, 4, 6\}, \{2, 4, 6\}\}, \\ T_{D^1} = \{\{1, 2, 3, 4, 5, 6\}\}. \end{gathered}$$

We obtain the associated schedule game v_{M^1} below, where superscript 1 is here added to distinguish the two schedule situations presented in this section.

E	$\{D^7\}$	$\{D^3\}$	$\{D^1\}$	$\{D^7, D^3\}$	$\{D^7, D^1\}$	$\{D^3, D^1\}$	$\{D^7, D^3, D^1\}$
$v_{M^1}(E)$	1	3	6	3	6	6	6

The content of decision 2008-0165 only considers the number of delivery days. This process could naively be formulated as the following anonymous schedule situation M^2 where $T = \{1, 2, 3, 4, 5, 6\}$, $N = \{D^7, D^3, D^1\}$ and:

$$\begin{gathered} T_{D^7} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\}, \\ T_{D^3} = \{E \subseteq T : |E| = 3\}, \\ T_{D^1} = \{\{1, 2, 3, 4, 5, 6\}\}. \end{gathered}$$

However, the set T_{D^3} does not reflect correctly the constraints imposed on category D^3. As an example, $\{1, 2, 3\} \in T_{D^3}$ which means that postal products in category D^3 can, a priori, be distributed in the three consecutive days Monday, Tuesday and Wednesday. However, this would prevent postal products posted on Wednesday to be delivered on time. This is legally not possible because La Poste has to meet the delivery speed of each postal product. This may seem inconsequential from the point of view of cost sharing since the resulting anonymous schedule game v_{M^2} is identical to v_{M^1}. This will no longer be the case if further changes are incorporated to this problem. As an illustration, imagine that we add a new category of postal products that must be delivered on some specific days, such as newspapers or advertisements,

[3]Time periods 1 and 6 belonging to T_{D^3} are not consecutive due to Sunday which is not a delivery day.

the mutualization of the time configurations can be different from v_{M^1} to v_{M^2}. More specifically, suppose that we add a fourth category of postal products D^2 corresponding to direct marketing mail that must be distributed in two specific consecutive days $\{2,3\}$. Consider for instance a first advertisement sent Tuesday that proposes discounts on Wednesday and, to encourage consumers, the store send back coupons on Wednesday, as a reminder. We get the following set of minimal time configurations for category D^2:

$$T_{D^2} = \{\{2,3\}\}.$$

Denote by $v_{M^{1\prime}}$ and $v_{M^{2\prime}}$ the two four-player schedule games obtained by adding player D^2 to the schedule situations M^1 and M^2, respectively. These two games are distinct. To see this consider coalition $\{D^2, D^3\}$. In the schedule game $v_{M^{1\prime}}$, the delivery category D^2 pools only one of its two days with D^3, which yields that $v_{M^{1\prime}}(\{D^2, D^3\}) = 4$, i.e. four routes per week are needed. On the contrary, in the anonymous schedule game $v_{M^{2\prime}}$ the label of the delivery days does not matter, hence the delivery category D^2 can completely (but inaccurately) pool its time periods with category D^3, which implies that $v_{M^{2\prime}}(\{D^2, D^3\}) = 3$.

To conclude this application, let us back to the original three-player problem. To determine allocations of the schedule game v_{M^1} we will apply the Equal pooling allocation to a coherent optimal time configuration R^* and to a non-coherent optimal time configuration R. Let $R^* = \big\{\{1\}, \{1,3,5\}, \{1,2,3,4,5,6\}\big\}$ and $R = \big\{\{1\}, \{2,4,6\}, \{1,2,3,4,5,6\}\big\}$. According to proposition 3 the Equal pooling allocation of a coherent optimal time configuration is the Shapley value. It gives the following percentages: D^7 incurs 5.56% of the costs, D^3 incurs 22.22% and D^1 incurs 72.22% of the costs. This corresponds to the efficient allocation $(\frac{1}{3}, \frac{4}{3}, \frac{13}{3})$ in the game v_{M^1}, as calculated in Munich and Bohorquez Suarez (2022). The Equal pooling allocation on R gives the following percentages: D^7 incurs 8.33% of the costs, D^3 incurs 25% and D^1 incurs 66.67% of the costs. This corresponds to the efficient allocation $(\frac{1}{2}, \frac{3}{2}, 4)$ in the game v_{M^1}. Compared to the two previous allocations the current allocation incurs less costs to D^1. Although, the three allocations are close to each other, they rely on distinct principles. The Shapley value is based on the incremental costs of each category to coalitions, the Equal pooling allocation takes into account the routes needed by each category and the current allocation shares the costs according to a proportional principle. Hence, the Equal pooling allocations can be considered as an alternative to the current allocation.

7. Concluding remarks

We conclude briefly with two remarks. Firstly, in Section 5 we provide a sufficient condition on a schedule situation of the non-emptiness of the core. It remains an open question to find a condition which would be both necessary and sufficient. Secondly, an axiomatic analysis of some solutions for schedule situations in order to underline, for instance, the fairness consideration underlying the Equal pooling allocations, is left for future work.

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