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Document de Travail n° 2023 – 05

Janvier 2023

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Show your strength in the hammer-nail game: a Nim game with incomplete information

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January 2023

Abstract

We study the hammer-nail game, a game played in the French TV show "Fort Boyard", by transforming this game into a Nim game with incomplete information. In this game, two players are in front of a nail slightly driven into a wooden support. Both have a hammer and in turn hit the nail. The winner is the first player able to fully drive the nail into the support. A player is of strength f if he is able, with one swing of the hammer, to drive the nail at most f millimeters into the support. We study the perfect Bayesian Nash equilibria of this game with incomplete information on the players' strength, and we also look at the equilibrium behavior when strength is combined with dexterity.

Keywords: Nim game, incomplete information, subgame perfect Nash equilibrium, perfect Bayesian equilibrium, Fort Boyard

JEL Classification: C72

1. Introduction

In this short paper we study a game played in the French TV show "Fort Boyard", the hammer-nail game. The game goes as follows: two players, player 1 and player 2, are in front of a nail slightly driven into a wooden support (see figure 1). Both have a (same) hammer and in turn hit the nail. At the beginning of the game, the head of the nail is at a distance D from the support. Player 1 begins the game. The winner is the first player able to fully drive the nail into the support. So of course, the closer the head of the nail is to the support, the more the suspense is growing and this is what makes the game attractive. Let us add that in the French TV show Fort Boyard, player 1 is a candidate who wants to win the game, in order to get more time to catch the "Boyards" (coins) and player 2 is a "Maître du temps", a person who belongs to the Fort Boyard TV team, who wants to impede the candidate from winning.

We suppose, throughout the paper, except in section 6, that each player knows his strength and is able to control it, which is not necessarily true in reality (it may be true for the "Maître du temps", who is trained to play the game, but not necessarily for the candidate). The strength is measured in numbers of millimeters. So we say that a player is of strength 5 if he is able, with

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I thank the third-year class students (year's class 2020/2021 and year's class 2022/2023) at the Faculté des Sciences Economiques et de Gestion (Faculty of Economic and Management Sciences) of the University of Strasbourg, who played traditional Nim games with stones. I especially thank one of these students, Bastien Landwerlin, who mentioned the hammer-nail game, and thereby gave me the idea to work on it.

one swing of the hammer, to drive the nail at most 5 millimeters into the support. This means that a player of strength 5 is able, with one swing of the hammer, given that he perfectly controls his strength, to drive the nail 1 millimeter (mm), 2mm, 3mm, 4mm or 5mm into the support. So we opt for a discrete approach of the game but we could choose, if needed, a much smaller increment of distance than the millimeter; this would not change the results. We also require that each player, at each turn of play, at least drives the nail 1mm into the support, which means that he cannot simulate hammering the nail. As a matter of fact, once the head of the nail is close to the support, but too far from it to be fully driven into with one swing of the hammer, a player may be incited to not drive the nail further into the support, in order to impede the opponent from winning the game at his next turn of play[1].

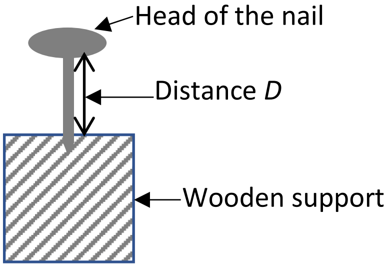

Figure 1 : the nail at the beginning of the play

So the game becomes a Nim game (see Bouton, 1901). TV shows sometimes work with true Nim games (see for example the sticks game also played in the Fort Boyard TV show (Umbhauer, 2016) or the Game of 21, played in an American TV show (Dufwenberg et al. 2010, Gneezy et al., 2010)). But Nim games are usually games with complete information. The hammer-nail game is a game with incomplete information because both players are not informed on the strength of the opponent. Of course, in the TV show, both players are usually of rather strong strength, yet none of them perfectly knows the strength of the other. That is why in this paper we focus on the game with incomplete information, by assuming that each player ignores the strength of the opponent. A nice feature of this game is that, to solve it, we do not have to introduce prior probability distributions on the unknown strengths; this not usual when dealing with games with incomplete information.

In section 2, we briefly recall the subgame perfect Nash equilibria when both players are of same strength and when this strength is common knowledge of both players. In section 3, we give the subgame perfect Nash equilibria when both players are of different strengths but the strengths are still common knowledge. We namely show that the strongest player always wins the game, except if D is lower than or equal to the weakest strength and the weakest player starts the game. In section 4 we turn to incomplete information and we establish a perfect Bayesian Nash equilibrium which has a surprising property: it is of now use to try to hide one's strength to the opponent. In some way, a player has to show his strength to win. Section 5 comments this equilibrium by comparing it to an easier behavior often observed in classroom experiments on Nim games. Section 6 concludes on the impact of dexterity, where by lack of

[1] This requirement does not exist in the TV Fort Boyard game, but it is mathematically necessary to avoid that the game never stops.

dexterity we mean that a player is unable to dose his strength: so, with one swing of the hammer, he systematically drives the nail more than 1 millimeter into the support.

2. Same strengths and complete information

In this section, we suppose that the strength of both players is *f*, so that each player is able to drive the nail from 1 to *f* millimeters into the support with one swing of the hammer. Right through the paper, we simply say that a player plays *s* to say that he drives the nail *s* millimeters into the support.

Proposition 1
We call f the strength of both players and r the remainder of the division of D by $(f+1)$.
The set of Subgame Perfect Nash Equilibria (SPNE) is given by:
Player 1 :
At her first turn of play (first round), she plays r if $r > 0$*, and she plays any integer from 1 to f if* $r = 0$.
At her other turns of play, she plays in such a way that the remaining distance after her turn of play is a multiple of $f+1$. *If this is not possible (because the distance she observes at her turn of play is a multiple of* $f+1$*), she plays any integer from 1 to f.*
Player 2:
At each potential turn of play, he plays in such a way that the remaining distance after his turn of play is a multiple of $f+1$. *If this is not possible (because the distance he observes at his turn of play is a multiple of* $f+1$*), he plays any integer from 1 to f.*[2]

Proof of proposition 1: well-known.

As is well-known, the strategies in these SPNE are weakly dominant ones.

As is well-known too, the equilibrium path of these SPNE leads player 1 to win the game when the starting distance *D* is not a multiple of $f+1$, and it leads player 2 to win otherwise.

Let us illustrate the equilibrium path for *f* = 4 and *D* =17, and for *f* = 4 and *D* = 15.

Case 1: $f=4, D=17$

Given that 17= 3x(4+1)+2, player 1 wins the game. She begins the game by playing 2, so that the remaining distance is 15 mm, a multiple of 5. It derives that player 2 plays 1, 2, 3 or 4. Hence player 1, in turn, plays 4, 3, 2 or 1, respectively, so that the remaining distance is 10 mm. So player 2 plays 1, 2, 3 or 4, which leads player 1 to play 4, 3, 2 or 1 mm, respectively, and the remaining distance is 5 mm. Hence player 2 plays 1, 2, 3 or 4, and player 1 wins the game, by playing 4, 3, 2 or 1 respectively, thereby fully driving the nail into its support.

It is immediate that player 2 can never win the game, because, whatever he does, player 1 plays in a way to put player 2 in front of a distance that is a multiple of $f+1$. The situation is reversed when the starting distance *D* is a multiple of $f+1$, like in the following example.

[2] Given that 0 is a multiple of any number, the proposition clearly says that in front of a remaining distance lower than or equal to the strength, a player simply plays the remaining distance, i.e. fully drives the nail into its support.

Case 2: $k = 4, D = 15$

Player 1, at the beginning can play any number from 1 to 4, because player 2, in turn, will play what is needed to put player 1 in front of a remaining distance of 10 mm. So player 1 can play any number from 1 to 4, because player 2, in turn, plays so that player 1 faces a remaining distance of 5 mm. And finally player 1 can play any number from 1 to 4, because player 2, in turn, wins the game by fully driving the nail into the support.

3. Heterogenous strengths and complete information

Now we switch to the situation where one of the players is stronger than the other. We still suppose, in this section, that the strength of both players is common knowledge. So we suppose that one player is of strength *f*, and that the other player is of strength *F*, with $F - f \geq 1$.[3] In that case the strongest player always win the game, except if $D \leq f$ and the weakest player begins the game.

We first give an example with $f = 3, F = 4$, and $D = 11$

Suppose that player 1 is the weakest player. Whatever she plays at her first turn of play, she loses the game.

- If she plays 2 or 3, then player 2, in turn, respectively plays 4 and 3, in order to put player 1 in front of a remaining distance of 5 mm. So player 1 loses the game because, whatever she plays, player 2 completes her chosen number to 5 and wins the game.
- If she plays 1, then player 2 is in front of a remaining distance of 10 mm (a multiple of 4+1), but this is not a problem for him.
  He can play 1, in order to put player 1 in front of a remaining distance of 9 mm. So, given that player 1 cannot play less than 1 and more than 3, player 2, at his turn of play, adapts his behavior so that player 1 faces a remaining distance of 5 mm. And thereby player 2 wins the game.
  But he can also play 2, in order to make player 1 face a remaining distance of 8 mm, a multiple of 3+1. Then, whatever player 1 plays, player 2 completes her chosen number to 4 and also wins the game in the next round (given that player 1 cannot play more than 3).

The only configuration where the strongest player loses the game is the trivial one where the weakest player begins the game and $D \leq f$.

In all other cases the strongest player always wins. The reason we will exploit in proposition 2, already given in the example, is that, even in front of a remaining distance that is a multiple of $F + 1$, say $k(F + 1)$, $k \in \mathbb{N}$, $k \geq 1$, the strongest player wins the game by behaving as follows: he chooses 1, so that the opponent, who can only play from 1 to *f*, leads the strongest player in front of a remaining distance that is not a multiple of $F + 1$, because this distance goes from $k(F + 1) - 1 - f = (k - 1)(F + 1) + F - f$ to $k(F + 1) - 2 = (k - 1)(F + 1) + F - 1$. So he plays a number from $F - f$ to $F - 1$, respectively, and the remaining distance is a multiple of $F + 1$. This allows him to win the game.

[3]We work with millimeters but we could work with any (smaller) measure of distance.

So we get proposition 2.

Proposition 2
We note f the weakest strength, F the strongest one, with $F \geq f + 1$.
A set of SPNE is characterized by:
The weakest player: if, at his turn of play, the remaining distance is lower than or equal to f, he plays the remaining distance. If not, he can play from 1 to f.
The strongest player: if, at his turn of play, the remaining distance is lower than or equal to F, she plays the remaining distance. Otherwise, at any turn of play, she plays so that the weakest player faces a remaining distance that is a multiple of $F + 1$. *If this is not possible (because she faces a remaining distance that is a multiple of* $F + 1$*) she can play any number from 1 to* $F - f$.

Proof of proposition 2:

The main argument has been given in the example above. If the weakest player, at his turn of play, is in front of a remaining distance that is a multiple of $F + 1$, then he loses the game: whatever he plays, from 1 to *f*, the strongest player completes to $F + 1$, so that the weakest player is again in front of a remaining distance that is a multiple of $F + 1$. And at the end, the strongest player wins the game.
Now, if the strongest player is in front of a remaining distance that is a multiple of $F + 1$, say $k(F + 1)$, with $k \in \mathbb{N}$ and $k \geq 1$, she can play any number from 1 to $(F - f)$. By so doing the weakest player faces a remaining distance going from $(k - 1)(F + 1) + F$ to $(k - 1)(F + 1) + 1 + f$. So, regardless of the number he chooses, from 1 to f, the strongest player will be in front of a remaining distance going from $(k - 1)(F + 1) + 1$ to $(k - 1)(F + 1) + F - 1$. Hence she plays in such a way that the remaining distance is a multiple of $F + 1$, which leads her to win the game. ■

Observe that the proposition allows the strongest player, in front of a remaining distance that is a multiple of $F + 1$, to play 1, but also to play any number going from 1 to $F - f$.
It follows from proposition 2 that the role of each player (being player 1 or player 2) is no longer of any importance if one excludes the trivial case where the initial distance is lower than or equal to *f*. Except in this case, the strongest player wins the game, regardless of the role he plays.
We can observe that the strongest player's strategy is a weakly dominant one.

Yet proposition 2 is too restrictive. In the example, we already exploited the fact that the strongest player can win the game by playing in such a way that the remaining distance after his turn of play is either a multiple of F+1 or a multiple of f+1. In facts, the set of SPNE in proposition 2 is strictly included in a much larger one, given in proposition 3:

Proposition 3
We note f the weakest strength, F the strongest one, with $F \geq f + 1$.*The set of SPNE is given by:*
The weakest player:
- *if, at his turn of play, the remaining distance is lower than or equal to f, he plays the remaining distance. If not, he can play from 1 to f.*

The strongest player:

- *if, at his turn of play, the remaining distance is lower than or equal to F, she plays the remaining distance. Otherwise:*
- *If the remaining distance d is strictly larger than* $f + 2$*, she can play any number s from 1 to F, that checks* $d - s \geq f + 1$
- *If the remaining distance d is equal to* $f + 2$*, hence* $F = f + 1$*, then she plays 1.*

The proof of proposition 3 is obvious: If the initial distance *D* is strictly larger than *f*, whoever begins the game, the strongest player will be called on to play. If the remaining distance *d* she faces is lower than or equal to *F*, he plays *d* and wins. If *d* is strictly larger than *F*, hence strictly larger than $f\ + 1$, there are only two possibilities. Either *d* is strictly larger than $f\ + 2$, and the only thing that matters for the strongest player is that she is again called on to play in the future; for that, it is enough to play s such as $d - s \geq f + 1$, so that the weakest player cannot play the remaining millimeters. Or $d = f + 2$, in which case the strongest player has to play 1, so that the weakest player cannot finish the game (and she finishes the game when again called on to play).

Proposition 3 clearly reveals two points:

- When the strength of both players is common knowledge and one player is stronger than the other, then the game, from a strategic point of view, is of no interest. The strongest player, if he faces a remaining distance that is strictly larger than *F*, just has to play in a way that impedes the opponent from finishing the game at the next turn. This is not really stimulating from a logical point of view.
- Proposition 3 may also explain why many persons, when asked to play a Nim game with two players of same strength *f*, have the impression that the way of playing is not strategic before the remaining distance is small, larger than but close to *f*. As a matter of facts, in classroom experiments on Nim games with stones[4], where the two players are of same strength, many students are wrongly convinced that they have to be careful only once the number of remaining stones becomes small. This may be due to the fact that, in real life, people are seldom of same strength (or have exactly the same set of strategies). So the students, who observe heterogeneous strengths (sets of strategies) in real life, mistakenly bring their true life observations into the classroom experiment, and therefore keep mistakenly convinced that they have to be careful only in front of small remaining distances (few remaining stones).

So what is the use of proposition 2? Proposition 2 gives a set of SPNE that display a kind of careful behavior. By bringing, each time it is possible, the weakest player in front of a remaining distance that is a multiple of $F + 1$, the strongest player is sure to win the game, when *f*, the weaker strength, is strictly lower than *F*, but also when *f* is equal to *F*. So, if the strongest player fears that the opponent may be of strength *F*, proposition 2 ensures that he wins even in this case (provided he is at least one time in front of a remaining distance that is not a multiple of $F + 1$). In other terms, the optimality of proposition 2's behavior resists to the introduction of some "trembles" on the lower strength, that may allow this strength to become equal to the

[4] Traditional Nim games with stones have for example been played by the third-year class students (year's class 2020/2021 and year's class 2022/2023) at the Faculté des Sciences Economiques et de Gestion (Faculty of Economic and Management Sciences) of the University of Strasbourg.

strongest one. Thus far proposition 2 introduces to the next section, where both players ignore the strength of the opponent.

4. Reveal your strength if you ignore the other's one

We now turn to the more realistic situation where both players ignore the strength of the opponent. Contrary to what happened in the previous section, a strong player can no longer afford to play in any way till the distance is close to her strength (proposition 3), because she does not know if the opponent is weaker, stronger, or of equal strength than himself. In other terms, we can reasonably expect that the players play in a strategic way, each time they are called on to play, and this will be the case.

Another fact we may expect is that, given the lack of information on the strength of the opponent, a player should perhaps hide his own strength for a while, in order to impede a potentially stronger player to exploit his weakness or in order to suddenly show his own strength in the last rounds of play. Yet the result we get is rather surprising in that it shows the opposite. At equilibrium, players will not try to hide their own strength.

In this section, we suppose that the strength of player *i* belongs to a set of integers going from 1 to an upper bound $T_i, i = 1,2$, with $T_i \geq 2$. Yet, in contrast to the usual literature on games with incomplete information, we do not need more information. We namely do not need to introduce prior probability distributions on the strength of the opponent, that usually express a kind of minimal information each player has on the strength of his opponent. These prior probabilities are useless in the proposed equilibrium. And each player *i* may even ignore the value of the upper bound $T_j, j \neq i, i, j = 1,2$. This makes the obtained equilibrium in proposition 4 quite robust.

Proposition 4
A (Perfect Bayesian) Nash equilibrium of the game with incomplete information goes as follows.
At each turn of play, a player of strength f, regardless of the value of f, plays as follows:

- *If the remaining distance d is lower than or equal to f, he plays d.*
- *If $d > f$, he plays so as to put the opponent in front of a remaining distance that is a multiple of $f + 1$. If this is not possible (because d is a multiple of $f + 1$), then he plays 1.*

Proof of proposition 4:

Consider a player (player *i*) of strength *f* facing a remaining distance d. If $d \leq f$, she of course plays d and wins the game. So we focus on the case $d > f$. We set $d = k(f + 1) + r$, with *r* the remainder of the division of *d* by $(f + 1)$ and *k* an integer larger than or equal to 1.

Case 1: *r* is different from 0.
By playing *r*, player *i* leads the opponent, player *j* (with $j \neq i$), to the remaining distance $k(f + 1)$, a multiple of $f + 1$.
So, if the opponent's strength is weaker than or equal to her strength, player *i* wins regardless of the opponent's strategy. As a matter of fact, the opponent can only play from 1 to *s*, with $s \leq f$. It derives that player *i* responds by playing from f to $f + 1 - s$ respectively. So she wins the game if $k = 1$, or, if $k > 1$, she again puts the opponent in front of a remaining distance

that is a multiple of $f + 1$; hence she wins the game at the end of the process. So playing r is an optimal strategy for player *i* if the opponent's strength is weaker than or the same as hers.
In front of a stronger opponent, of strength $F > f$, either $k(f + 1)$ is a multiple of $F + 1$, or it is not. If it is not, by playing the remainder of the division of $d - r$ by $F + 1$, the opponent will win the game because he will bring player i in front of a remaining distance that is a multiple of $F + 1$. So the opponent wins the game (given what we showed above)[5]. If the remainder of the division of $d - r$ by $F + 1$ is equal to 0, the opponent plays 1 and also wins the game (see below). So, given that the opponent wins regardless of the fact that $d - r$ is or is not a multiple of $F + 1$, the opponent wins regardless of what is played by player i. Hence playing r, the remainder of the division of *d* by $f + 1$, in order to bring the opponent in front of a multiple of $f + 1$, is an optimal strategy for player *i* even in front of a stronger opponent.

Case 2: $r = 0$, hence $d = k(f + 1)$, with $k \geq 1$
So $d = (k - 1)(f + 1) + f + 1$. By playing 1, the opponent will be in front of a remaining distance equal to $(k - 1)(f + 1) + f$.
If the opponent is weaker, he cannot play *f* (he can only play a lower number), so that player *i* can again bring him to the remaining distance $(k - 1)(f + 1)$ at her next turn of play. And so player *i* will win the game.
If the opponent is stronger or of the same strength, say of strength *F* (with $F \geq f$), it is useless to play differently. Let us show what happens if player *i* plays *y* = 1, 2, … up to *f*. The opponent is in front of a remaining distance of $d - y$.
If $d - y$ is different from a multiple of $F + 1$, the opponent plays in such a way that player *i* is in front of a remaining distance that is a multiple of $(F + 1)$. Hence he wins the game, regardless of the further actions played by player *i*. Observe that if $F = f$, $d - y$ is necessarily different from a multiple of $F + 1$.
Now, if $d - y$ is a multiple of $F + 1$ (which may only happen for $F > f$), then the opponent plays 1, so that player i is in front of the remaining distance $d - y - 1 = (k - 1)(f + 1) + f - y = k'(F + 1) - 1 = (k' - 1)(F + 1) + F$ (with $k' \in \mathbb{N}, k' \geq 1$). It follows that, regardless of what she plays, from 1 to *f*, the opponent will be in front of a remaining distance that goes from $(k' - 1)(F + 1) + (F - f)$ to $(k' - 1)(F + 1) + F - 1$, which is not a multiple of $F + 1$. So the opponent plays the remainder of the division of this distance by $F + 1$ and, at the end, wins the game.
So playing 1 is an optimal strategy in front of a player of same strength or of stronger strength, because playing any number from 1 to *f* will lead to lose the game.
It follows that playing 1 is an optimal strategy if *d* is a multiple of $(f + 1)$ regardless of the strength of the opponent. ■

Let us discuss this equilibrium:

We first observe that this equilibrium is at least partially revealing. Given that a player of strength *f* plays the remainder of the remaining distance divided by $(f + 1)$, the opponent can deduce that $f + 1$ is a divisor of $d - r$, and so gets some information on *f*. That is why we say that players do not try the hide their strength, which is not intuitive a priori. In other terms, when we say, in the title of the paper, that a player shows his strength to win, this does not mean

[5] This includes the case where $k(f + 1) \leq F$, given that 0 is a multiple of any number.

that he hammers as hard as he can, but that he is not afraid of the fact that the opponent is informed on his strength.

Observe that the result is robust, in that it holds even if the players have no information at all on the strength of the opponent. It does not need a distribution of prior probabilities on the types of the opponent. This explains that we do not introduce the player's beliefs at each turn of play, given that they play no role in the result. And this explains why we put (Perfect Bayesian) in brackets, given that we make no use of the bayesian rule to get the optimal way of behavior. We do not even need to know the value of T_i, i =1,2. The only fact we need is that $T_i \geq 2$ for i =1,2.

Observe that the equilibrium strategies are best responses to the opponent's strategy, but that they are not weakly dominant. Let us show this fact with an example.

Set $D = 10$ mm, $f = 3$ and $F = 4$. Suppose that the weakest player is player 1 and that she begins the game. Then the equilibrium path goes as follows: player 1 plays 2, so the remaining distance is 8 (a multiple of $f + 1$). Then player 2 plays 3, so the remaining distance is 5 (a multiple of $F + 1$). It follows from this fact that player 1 plays 1, and that player 2 wins the game by playing 4. So player 1's strategy leads to lose the game.

No imagine that the strongest player, player 2, does not play the equilibrium strategy, but plays as follows. If player 1 plays 1 or 2, he plays 3, if player 1 plays 3 or more, he plays 4, till the remaining distance *d* is lower than or equal to 4, in which case he plays *d*. If so, if player 1 plays the equilibrium strategy and starts by playing 2, then player 2 plays 3, then player 1 plays 1 and player 2 plays 4 and wins the game. But if player 1, instead of playing the equilibrium strategy, starts by playing 3, then player 2 plays 4, and then player 1 wins the game by playing 3. That is to say, a player's equilibrium strategy is no longer weakly dominant because another strategy may do better in front of a stronger opponent who plays in a non equilibrium way.

5. The probability of meeting an opponent of same strength matters

Rather surprisingly, incomplete information on the opponent's strength does not lead to a strategy that exploits the lack of information or the additional amount of information revealed by the opponent's behavior. In some way, proposition 4, like proposition 2, just says that the strongest player always wins the game, in that the strategy proposed in proposition 4 allows a player to win against any weaker opponent, regardless of the way the opponent behaves. So a player can reveal his strength because not showing it would not change anything. Yet let us draw attention to the fact that the strategies in proposition 4 are mainly due to the fact that a player may meet an opponent of same strength. As a matter of fact, if a player is sure that the opponent is either of strictly lower strength or of strictly stronger strength, then he can adopt the behavior in proposition 5, which is easier and does not reveal the player's strength.

Proposition 5
If both players can be of any strength front 1 to T but cannot be of same strength, then a (perfect Bayesian) Nash equilibrium of the game with incomplete information goes as follows.
At each turn of play, a player of strength f, regardless of the value of f, plays as follows:

- *If the remaining distance d is lower than or equal to f, he plays d.*
- *If the remaining distance d is strictly larger than f, he can play any number s, from 1 to f, such that* $d - s \geq f$.

Proof of proposition 5:

Playing d is obvious if $d \leq f$. So we only focus on a player's behavior, say player i's behavior, when she faces a remaining distance $d \geq f+1$. We first show that her strategy leads her to winning the game if she is in front of a strictly weaker opponent. As a matter of fact, by playing s so that $d - s \geq f$, the weaker opponent, in front of the distance $d - s$, cannot win the game at his turn of play, so that player i is again called on to play. And at the end, the opponent will lead her to a remaining distance lower than or equal to f and she wins the game. By complementary, this reasoning ensures that she loses the game in front of a strictly stronger opponent (if he is called on to play), whatever strategy she plays. So this strategy is a best response to the strategy of the opponent whether the opponent is weaker than her (in which case she wins) or stronger than her (in which case she loses regardless of the played strategy if the opponent is called on to play). ■

Yet of course, this strategy is a bad one if both players are of same strength. In this case, it may lead a player i, even though he has the opportunity to win by playing as in proposition 4, to be in front of a remaining distance equal to $f+1$, in which case he loses the game.

This fact gives rise to two remarks:

One the one hand, in real life, players are not really able to control their strength, so that it is difficult for them to drive the nail exactly a given number of millimeters into the support. Moreover people are unable to measure precisely the remaining distance when they are called on to play (because usually they have no ruler on hand). So proposition 5 is interesting in that it requires much less measures.

But on the other hand, in the "Fort Boyard" TV show where the hammer-nail game is played, it is quite natural to expect that the two players are of same (at least similar) strength, given that it would not be a good show to confront a very weak player to a very strong one. So proposition 5 is unfortunately useless and only proposition 4 has to be taken into account.

6. Concluding remarks: what about dexterity?

In this short paper, we get a rather easy result, even in an incomplete information context: the equilibrium strategies do not depend on the opponent's strength, and hence do not require any information on the opponent. That is rather uncommon. Yet this may be due, at least partly, to the fact that our notion of strength does not care about dexterity. In the whole paper, we supposed that being of strength f means that one is able to drive the nail 1, 2, up to f millimeters into the support. Yet it may be that some persons, namely very strong ones, are unable to hammer smoothly, so that, with one swing of the hammer, they drive the nail at least b millimeters into the support, with $b > 1$. In other terms, they are unable to dose their strength, that is to say they are of low dexterity. So the player's dexterity is a decreasing function of b. This may of course have an impact on the equilibrium strategies.

Therefore, in this concluding section, we just study what happens if both players are of same strength f but differ in dexterity: one player is able, with one swing of the hammer, to drive the nail from 1 to f millimeters into the support -we will say that he is of high dexterity- , whereas the other player is of lower dexterity in that he is able, with one swing of the hammer, to drive the nail from b to f millimeters into the support, with $f > b > 1$. In that case, the player with high dexterity most often wins the game except if he starts the game with $D = 1 + f$, or if the less gifted player starts the game with $D \leq f$ or with $f + b + 1 \leq D \leq 2f + 1$.

Proposition 6
We call f, $f \geq 3$, the strength of both players and b ($1<b<f$) the minimal number of millimeters that can be played by the player with lower dexterity. A subgame perfect Nash equilibrium way of playing is given by:
At each turn of play, the player with high dexterity plays as follows:

- *If the remaining distance d is lower than or equal to f, she plays d.*
- *If $d > f$, she plays in order to bring the opponent in front of a remaining distance that is a multiple of $f + 1$. If this is not possible (because d is a multiple of $f + 1$), then she plays f.*

At each turn of play, the player with lower dexterity ($b > 1$), plays as follows:

- *If the remaining distance d is lower than or equal to f, he plays d.*
- *If $2f + 1 \geq d \geq f + b + 1$, he plays from f to b so that the player with high dexterity is in front of a remaining distance equal to $f + 1$.*
- *If $d \notin [0, f]$ and $d \notin [f + b + 1, 2f + 1]$, he can play any number from b to f.*

Proof of proposition 6:

Dexterity matters only if the remaining distance is larger than b. If $d < b$ and a player is of lower dexterity, then, even if he hammers too strongly, that is not a problem, because he for sure fully drives the nail into the support.
So a player's behavior in front of a remaining distance lower than or equal to f is obvious.
It is also obvious that the player with high dexterity cannot win the game if she starts the game with $D = (f + 1)$ (obvious), if the opponent (the player with lower dexterity) starts the game with $D \leq f$ (obvious), or if he starts the game with D between $f + b + 1$ and $2f + 1$ (in which case he optimally plays from b to f, so that the player with high dexterity is in front of a remaining distance $f + 1$ and can only lose the game).
So we focus on the other configurations. Suppose that the player with high dexterity is in front of a remaining distance d strictly larger than $f + 1$. Either $d = (f + 1) + r$, with $0 < r < f + 1$, in which case she plays r, so that the opponent cannot finish the game and she will win in the next round. Or $d = k(f + 1) + r$, where r is the remainder of the division of d by $f + 1$ and k is an integer strictly larger than 1. This gives rise to two possibilities:
Case 1: r is different from 0.
In that case, she plays r, so the opponent is in front of a remaining distance that is a multiple of $f + 1$. So regardless of the number he chooses, from b to f, she completes to $f + 1$ at her next turn of play, and the same process goes on, till she wins the game.
Case 2: r is equal to 0.
Hence, by playing f, the opponent is in front of the remaining distance $d = (k - 1)(f + 1) + 1$. If $k > 2$, then d is strictly larger than $2f + 1$, and if $k = 2$, then $d = f + 2 < f + b + 1$.
If $k = 2$, given that the opponent can choose any number from $b \geq 2$ to f, the player with high dexterity is in front of a remaining distance going from 2 to $f + 2 - b$ and so she wins the game.
If $k > 2$, given that the opponent can choose any number from $b \geq 2$ to f, the player with high dexterity, at her next turn of play, is in front of a remaining distance going from $d = (k -$

$2)(f+1)+2$ to $d = (k-2)(f+1)+f+2-b$. So she plays a number going from 2 to $f+2-b$, and the opponent is in front of a remaining distance that is a multiple of $(f+1)$ (with $k-2>0$) and so the player with high dexterity wins the game. ∎

In proposition 6, the player with high dexterity, when she faces a distance d that is a multiple of $f+1$, plays f because, by so doing, she impedes the player with lower dexterity from bringing her back to a distance that is a multiple of $f+1$, because he is unable to play 1. As a matter of facts, if she would play 1, as in proposition 4, instead of *f*, the player with lower dexterity could play *f* and bring her back to a multiple of $f+1$. In other terms, in proposition 4, the more gifted (stronger) player exploits the fact that the opponent cannot play large numbers, whereas in proposition 6, the more gifted (with higher dexterity) player exploits the fact that the opponent cannot play low numbers. Hence it easy to conjecture that a game with incomplete information, both on dexterity and on strength, will perhaps be more complicated to play.

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