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Market exit and minimax regret

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Abstract

We study an overcrowded duopoly market where the only strategic variable is the exit time. We suppose that the surviving firm gets a positive monopoly profit and we focus on the classic context with complete information and identical firms. The only symmetric Nash equilibrium of this war of attrition is a mixed-strategy equilibrium that leads to a null expected payoff, i.e. the payoff a firm gets when it immediately exits the market. This result is not persuasive, both from an economic and from a strategic viewpoint. We argue that the minimax regret approach, that builds upon two opposite regrets - exiting the market too late and exiting the market too early - is more convincing. The minimax regret behavior, quite different from the mixed-strategy Nash equilibrium behavior, allows both firms to get a positive expected payoff.

Keywords: war of attrition, minimax regret, Nash equilibrium, maximin payoff, mixed strategy, duopoly.

JEL Classification: C72, D4

1. Introduction

We study an overcrowded duopoly where both firms decide if and when they exit the market, knowing that the surviving firm gets a positive monopoly profit. As is known since Maynard Smith (1974), the only symmetric Nash equilibrium of this war of attrition is a full support mixed-strategy equilibrium where each firm gets a null equilibrium payoff, i.e. the payoff obtained by immediately leaving the market. This result is not convincing, neither from an economic viewpoint, nor from a strategic viewpoint: why should a firm play a mixed and risky strategy, just to earn what it can earn by immediately leaving the market? Most authors bypass this result by introducing some incomplete information, for example on the duopoly profits. By doing so, they get a pure-strategy symmetric Nash equilibrium with positive payoffs (see for example Fudenberg and Tirole (1986)), that is also more easy to test experimentally (Hörisch and Kirchkamp (2010), Oprea & al. (2013)).

In this paper, we stick to the complete information context, but we construct the mixed strategies in a new way. We focus on the minimax regret approach introduced in game theory by Hayashi (2008), Renou and Schlag (2010) and Halpern and Pass (2012). The minimax regret approach is well known in single agent decision problems with a strong uncertainty (see Savage (1951) and Niehans (1948)). In these contexts economic agents may opt for a strategy that will minimize their regret, which is the difference between the payoff linked to their decision and the payoff they would obtain with the best decision in the realized state of the world (state of Nature). In a game, players are not confronted to Nature – at least not only-, but to other players, so they may face a strong *strategic* uncertainty, in that it may be difficult to anticipate what the

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other players will play. As a matter of fact, a Nash equilibrium is just a mathematical fixed point: a player's strategy is a best reply to the other players' (Nash equilibrium) strategies. But what happens if the others do not play their Nash equilibrium strategies? What happens when there are many different Nash equilibria, when they seem risky or are difficult to calculate? Wars of attrition are contexts where it is difficult to anticipate the opponent's strategy, namely because many strategies may be best responses. In the game studied, leaving the market at any time t is a best response when the opponent exits the market early, before time t. And it is better to leave the market immediately (at time 0), when the opponent decides to never exit the market. So there is a strong strategic uncertainty. Moreover, in this "should I stay or should I go" problem, each firm may suffer from two types of regrets: it is worth staying in the market, even by losing money, if the opponent exits the market fast (so a firm may regret to leave the market too early), but it is better to leave the market immediately if the opponent exits the market very late (so a firm may regret to leave the market too late, in that the potential monopoly payoffs will not cover the too many duopoly losses). The minimax regret criterion, that looks for a strategy that minimizes the maximal regret it may lead to, adapts well to contexts with multiple types of regrets, in that it will balance the possible regrets. The minimax regret behavior reveals to be completely different from the mixed-strategy symmetric Nash equilibrium behavior. Moreover it ensures a positive expected profit to both firms, contrary to the Nash equilibrium.

In section 2 we introduce the game studied, recall the symmetric mixed-strategy Nash equilibrium in the game in continuous time, and give this equilibrium in the game in discrete time. In section 3 we turn to the minimax regret behavior, both in the discrete and in the continuous setting. In section 4 we discuss the philosophy of the mixed-strategy Nash equilibrium and the minimax regret criterion. In section 5 we show that the minimax regret behavior, contrary to the mixed-strategy Nash equilibrium, always leads to a positive expected payoff. Section 6 concludes on the meaning of mixed strategies. It also opens the discussion on the necessity to put limits on the maximal exit time.

2. Exit time game and Nash equilibrium

The game goes as follows: two firms compete in a duopoly model, in discrete or continuous time. The market is overcrowded and each firm gets the *negative duopoly profit D* as long as both firms stay in the market. If one firm decides to exit the market, then the other firm gets the *positive monopoly profit M* forever. Time is discounted at the rate r, with r>0. In many countries, a firm cannot earn indefinitely negative profits, so the duopoly market cannot survive more than T periods (an amount of time T in the continuous setting).

In the discrete-time game, each firm decides in which period t she leaves, definitively, the duopoly market (if the opponent has not yet left). Leaving at time 0 means immediately leaving the market, in which case the firm earns a null payoff. Leaving in period t, if the opponent stays longer in the market, leads to the negative payoff D in each period, from period 1 to period t, i.e.: $D+\frac{D}{1+r}+\cdots+\frac{D}{(1+r)^{t-1}}=\sum_{i=0}^{t-1} D/(1+r)^i$.

If the firm leaves in period t and the opponent leaves in period u, with t>u, then the firm's payoff becomes: $\sum_{i=0}^{u-1} D/(1+r)^i+\frac{M}{(1+r)^u}\frac{(1+r)}{r}$ because she gets the positive monopoly payoff forever, from period u+1 onwards.

For the model to be interesting, we suppose that there exists a positive period t*, such that $\sum_{i=0}^{t*-2}\frac{D}{(1+r)^i}+\frac{M}{(1+r)^{t*-1}}.\frac{1+r}{r}>0$ and $\sum_{i=0}^{t*-1}\frac{D}{(1+r)^i}+\frac{M}{(1+r)^{t*}}.\frac{1+r}{r}<0$, so that it is worth staying in the duopoly market one additional period if the opponent decides to leave in period t*-1, but it is better to exit the market at time 0 if the opponent leaves in period t* or later.

$$\sum_{i=0}^{t*-1}\frac{D}{(1+r)^i}+\frac{M}{(1+r)^{t*}}.\frac{1+r}{r}=\frac{D\left(1-\frac{1}{(1+r)^{t*}}\right)(1+r)}{r}+\frac{M}{(1+r)^{t*}}\frac{1+r}{r}.$$

Hence $\sum_{i=0}^{t*-1}\frac{D}{(1+r)^i}+\frac{M}{(1+r)^{t*}}.\frac{1+r}{r}<0 \Leftrightarrow$ t* is the first period checking t* $\geq \frac{ln\left(\frac{M-D}{-D}\right)}{ln(1+r)}$. For the model to make sense, we assume $\frac{ln\left(\frac{M-D}{-D}\right)}{ln(1+r)}>1$, hence –D<M/r, which means that it is worth enduring one period the duopoly loss when getting the monopoly profit thereafter forever.

For example, if T=6, -D=M=1 and r=0.25, we get the following normal form game matrix 1:

		Firm 2						
		0	1	2	3	4	5	6
	0	(0 , 0)	(0 , 5)	(0 , 5)	(0 , 5)	(0 , 5)	(0 , 5)	(0 , 5)
	1	(5 , 0)	(-1, -1)	(-1 , 3)	(-1 , 3)	(-1 , 3)	(-1 , 3)	(-1 , 3)
	2	(5 , 0)	(3 , -1)	(-1.8,-1.8)	(-1.8, 1.4)	(-1.8, 1.4)	(-1.8, 1.4)	(-1.8, 1.4)
Firm1	3	(5 , 0)	(3 , -1)	(1.4, -1.8)	(-2.44,-2.44)	(-2.44, 0.12)	(-2.44, 0.12)	(-2.44, 0.12)
	4	(5 , 0)	(3 , -1)	(1.4, -1.8)	(0.12, -2.44)	(-2.952,-2.952)	(-2.952,-0.904)	(-2.952,-0.904)
	5	(5 , 0)	(3 , -1)	(1.4, -1.8)	(0.12, -2.44)	(-0.904,-2.952)	(-3.3616,-3.3616)	(-3.3616,-1.7232)
	6	(5 , 0)	(3 , -1)	(1.4, -1.8)	(0.12, -2.44)	(-0.904,-2.952)	(-1.7232,-3.3616)	(-3.68928,-3.68928)

Matrix 1: exit game for T=6, M= -D=1 r=0.25

In this example, t*=4. It is profitable to stay in the market if the opponent leaves before or in period 3, in that the firm achieves the positive payoff 0.12 (thanks to the monopoly profit from period 4 onwards), but it is better to exit the market at time 0, when the opponent leaves in period t*=4 or later (in which case the best payoff when leaving later is -0.904 even with the monopoly profit from period 5 onwards).

In the continuous-time game, each firm decides at which time t she exits, definitively, the duopoly market (if the opponent has not yet left), with t from 0 to T. Leaving at time 0 means immediately leaving the market, in which case the payoff is 0. Leaving at time t, when the opponent stays longer in the market, leads to a payoff D from 0 to t, hence to the actualized payoff: $\int_0^t De^{-rs}ds$

But when the opponent leaves the market sooner, at time u<t, then the payoff becomes: $\int_0^u De^{-rs}ds+\int_u^\infty Me^{-rs}ds$, because the firm gets the monopoly payoff from time u to +∞.

In the continuous-time game, it is profitable to stay in the market if the opponent exits the market at time t* or sooner, but it is better to leave the market at time 0 if the opponent leaves the market at time t* or later, for t* checking: $\int_0^{t*} De^{-rs}ds+\int_{t*}^\infty Me^{-rs}ds=0$. So we get t*=ln$(-\frac{M-D}{D})/r$. t*=2.77 in the numerical example.

In both the discrete-time game and the continuous-time game, there is only one symmetric Nash equilibrium and it is in mixed strategies.

Proposition 1 (close to Maynard Smith (1974))
In the continuous-time game, the unique mixed-strategy symmetric Nash equilibrium is given by:

- *The support of the equilibrium is [0, T],*
- *The cumulative probability distribution on [0,T[is given by:* $F(t) = 1 - e^{\frac{Dr}{M}t}$
- *T is a mass point played with probability g(T)=* $e^{\frac{Dr}{M}T}$

Proof: see Appendix A

It derives from this proposition that the probabilities decrease from 0 to T^-. The density function f(t)= $-\frac{Dr}{M}e^{\frac{Dr}{M}t}$ decreases at the rate Dr/M and has a mass point, T. This function derives from the differential equation $D\big(1 - F(t)\big) + \frac{M}{r}f(t) = 0$, which, by definition of a mixed-strategy Nash equilibrium, says that switching from t to t+dt, so leaving at t or at t+dt, does not change the obtained payoff. As a matter of fact, staying in the market up to the time t+dt instead of up to the time t, doesn't change the firm's payoff in front of an opponent leaving before t. It lowers her payoff by D each time she faces an opponent leaving after t (hence she gets Ddt with probability 1-F(t) because she stays in "dt" time longer), but it increases her payoff in front of an opponent leaving at time t, in that she gets the endless monopoly payoff M/r (hence she gets M/r with probability f(t)dt). Given that a firm earns a null payoff when exiting the market at time 0, this equation ensures that each firm always gets a null payoff in the mixed-strategy Nash equilibrium.

In the discrete-time game we get:

Proposition 2
We note p_t the probability of exiting the market in period t, t from 0 to T. In the discrete-time game, the only mixed-strategy symmetric Nash equilibrium is given by:

$p_0 = -\frac{rD}{M+r(M-D)}$

$p_t = (\frac{M(1+r)}{M+r(M-D)})^t p_0$ *for t from 1 to T-1*

$p_T = (\frac{M(1+r)}{M+r(M-D)})^T$

Proof: see Appendix B

We get again decreasing probabilities from 0 to T-1 (given that $\frac{M(1+r)}{M+r(M-D)} < 1$) and an extra probability p_T, larger than p_{T-1}. The probabilities decrease at the rate $(p_{t+1} - p_t)/p_t = \frac{rD}{M+r(M-D)}$ from p_0 to p_{T-1}, so the equilibrium distribution behaves similarly in the discrete-time game and in the continuous-time game (which is not always the case in a war of attrition, see Umbhauer (2017)).

These probabilities, in our numerical example, are given and illustrated in figure 1.

Both in the discrete-time game and in the continuous-time game, the equilibrium payoff is 0 by construction, given that all the strategies in the equilibrium support lead to the same expected payoff, and given that leaving the market at time 0, which leads to a null payoff, is played with positive probability.

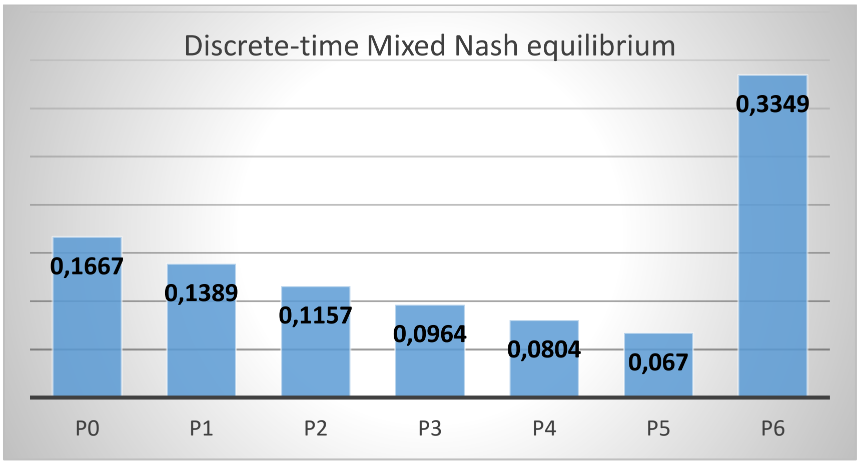

Figure 1: mixed-strategy Nash equilibrium for T=6, M= -D=1, r=0.25

This is not appealing from an economic and from a strategic viewpoint. Why should a firm play in a mixed-strategy way, risking to get a negative payoff, just to earn an expected payoff equal to 0, whereas it could be sure to not lose any money by immediately exiting the market? In other words, the mixed-strategy Nash equilibrium does not better than the maximin payoff strategy which simply consists in exiting the market at time 0 *(see Appendix C)*. Moreover, in the asymmetric pure-strategy Nash equilibria, one firm gets 0 and the other firm gets the monopoly payoff forever; so one may reasonably expect a possibility for both firms to earn a positive payoff.

3. Exit time game and minimax regret behavior

Minimax regret is a concept that is well known in decision theory (see Savage (1951)). It has been introduced in game theory, namely by Hayashi (2008), Renou and Schlag (2010) and Halpern and Pass (2012). In many games, players find it difficult to anticipate the others' behavior for different reasons. They may be unsure about others' rationality, which prevents them from best replying. But, even if they expect to be in front of rational players, they may be unsure of their way of playing, namely because many strategies are rationalizable. And this is especially true in a war of attrition. Any strategy may be a best response: in our context, it is optimal for a firm to exit at (any) time t>0, if she expects that the other firm leaves the market fast and before t, so that the positive monopoly profits cover the duopoly losses. And it is optimal to exit the market at time 0 if the opponent exits after t*. It follows from this fact that it is difficult to anticipate the opponent's behavior and, consequently, to choose a best reply. It ensues that a player, rather than trying to best reply to an unknown behavior, may prefer playing a strategy that does not generate too much regret. As a matter of fact, leaving at time t, either leads to no regret, in that exiting at time t is the best response to the other firm's strategy, or it leads to a regret, which is the difference between the payoff the firm could get by best replying to the opponent's strategy, and the payoff she obtains by leaving at time t. The minimax regret philosophy consists in trying to minimize the maximal regret a player may endure due to the chosen strategy.

This philosophy adapts well to contexts with a strong strategic uncertainty (like a war of attrition with many rationalizable strategies) and to contexts that lead to different kinds of regrets. Such contexts are numerous. In a Bertrand duopoly for example, by playing a price lower than the competitor's one, a firm may regret to not have played a slightly higher price that would have allowed the firm to serve the entire market with a higher profit. But by playing a price larger than the competitor's one, a firm simply regrets to have lost her part of the market. This is also true in the market exit game. A firm may regret exiting the market before the opponent, when the latter chose to leave before t*, but may also regret staying in the market too long, namely when the opponent took the same decision. In these multiple regrets contexts, minimax regret behavior balances the different possible regrets in order to minimize them.

We first illustrate the concept in the discrete numerical example before switching to the general game. To do so, we compute, for each strategy, the regret it leads to in front of each opponent's strategy, this regret being the difference between the best-reply payoff and the payoff linked to the chosen strategy. For example, if firm 1 chooses to leave in period 2 whereas player 2 leaves in period 3, player 1 gets 2 times the negative duopoly profit (-1.8), whereas she can get 0.12 (three times the negative duopoly profit plus the endless monopoly payoff from period 4 onwards) with the best decision, which consists in exiting the market in period 4, 5 or 6. So her regret when firm 2 leaves in period 3 and she leaves in period 2 is 0.12-(-1.8)=1.92. Matrix 2 is firm 1's matrix of regrets:

regrets	0	1	2	Firm 2 3	4	5	6
0	**5**	3	1.4	0.12	0	0	0
1	0	**4**	2.4	1.12	1	1	1
2	0	0	**3.2**	*1.92*	1.8	1.8	1.8
Firm1 3	0	0	0	**2.56**	2.44	2.44	2.44
4	0	0	0	0	**2.952**	**2.952**	**2.952**
5	0	0	0	0	0.904	**3.3616**	**3.3616**
6	0	0	0	0	0.904	1.7232	**3.68928**

Matrix 2: regret matrix for firm 1 in the exit game for T=6, M=-D=1, r=0.25

For each strategy, we compute the maximal regret (the regret in bold in matrix 2). For example, when firm 1 chooses to exit the market in period 2, her regret is 1.92 if the opponent chooses to leave in period 3, but her maximal regret is 3.2; this regret is linked to firm 2's decision to leave the market in period 2. If so, firm 1 earns 2 times the negative duopoly payoff by leaving in period 2 (-1.8) whereas she could obtain 2 times the duopoly payoff plus the endless monopoly payoff from period 3 onwards (1.4), by leaving at least one period later (hence the regret is 1.4-(-1.8)= 3.2).

So, if we only work with pure strategies, the strategy that minimizes the maximal regret consists in exiting the market in period 3 (regret=2.56), i.e. t*-1. This result generalizes as follows.

Proposition 3
When t≤T, in the discrete-time game, the pure minimax regret strategy consists in leaving the market in period t*-1 if* $-\sum_{i=0}^{t*-1}\frac{D}{(1+r)^i} > \frac{M}{(1+r)^{t*-1}}\frac{(1+r)}{r}$, *in period t* if not. In the continuous-time game, the pure minimax regret strategy consists in leaving the market at time t*.*
When t>T, the pure minimax regret strategy consists in leaving the market in period T, both in the discrete-time game and in the continuous-time game.*
Proof see Appendix D

In the numerical example, $-\sum_{i=0}^{t*-1}\frac{D}{(1+r)^i} = 2.952$ and $\frac{M}{(1+r)^{t*-1}}\frac{(1+r)}{r} = 2.56$, so the pure minimax regret strategy consists in leaving in period t*-1= 3. The minimax regret is at the junction between the two types of regrets: the regret a firm gets when leaving too early (so she regrets the monopoly payoff $\frac{M}{(1+r)^{t*-1}}\frac{(1+r)}{r}$ she could get by staying in the market an additional period) and the regret a firm has because she has not left at time 0 (she could avoid the duopoly losses $-\sum_{i=0}^{t*-1}\frac{D}{(1+r)^i}$).

$\sum_{i=0}^{t*-1}\frac{D}{(1+r)^i}$ and $-\frac{M}{(1+r)^{t*-1}}\frac{(1+r)}{r}$ may be large negative numbers. It is possible to lower the minimax regret by switching to mixed strategies. The idea is to construct a mixed strategy that minimizes the regret regardless of the exit time chosen by the opponent. Insofar we follow Renou and Schlag (2010). We call p_t firm 1's probability of exiting in period t. The probabilities are chosen in order to minimize the maximal regret, y, regardless of the exit time chosen by firm 2. This amounts to solving the optimization program:

$$\min_{y\, p_0 p_1 p_2 p_3 p_4 p_5 p_6} y$$

$$\begin{gathered}
\text{u.c. } 5p_0 \leq y \\
3p_0 + 4p_1 \leq y \\
1.4p_0 + 2.4p_1 + 3.2p_2 \leq y \\
0.12p_0 + 1.12p_1 + 1.92p_2 + 2.56p_3 \leq y \\
p_1 + 1.8p_2 + 2.44p_3 + 2.952p_4 + 0.904p_5 + 0.904p_6 \leq y \\
p_1 + 1.8p_2 + 2.44p_3 + 2.952p_4 + 3.3616p_5 + 1.7232p_6 \leq y \\
p_1 + 1.8p_2 + 2.44p_3 + 2.952p_4 + 3.3616p_5 + 3.68928p_6 \leq y \\
p_0 + p_1 + p_2 + p_3 + p_4 + p_5 + p_6 = 1 \\
0 \leq p_t \qquad \text{t from 0 to 6}
\end{gathered}$$

This program has a unique solution,

$$\left(y = \frac{205}{144}, p_0 = \frac{41}{144}, p_1 = \frac{41}{288}, p_2 = \frac{41}{192}, p_3 = \frac{41}{128}, p_4 = \frac{5}{128}, p_5 = 0, p_6 = 0\right)$$

The minimax regret distribution is represented in figure 2.

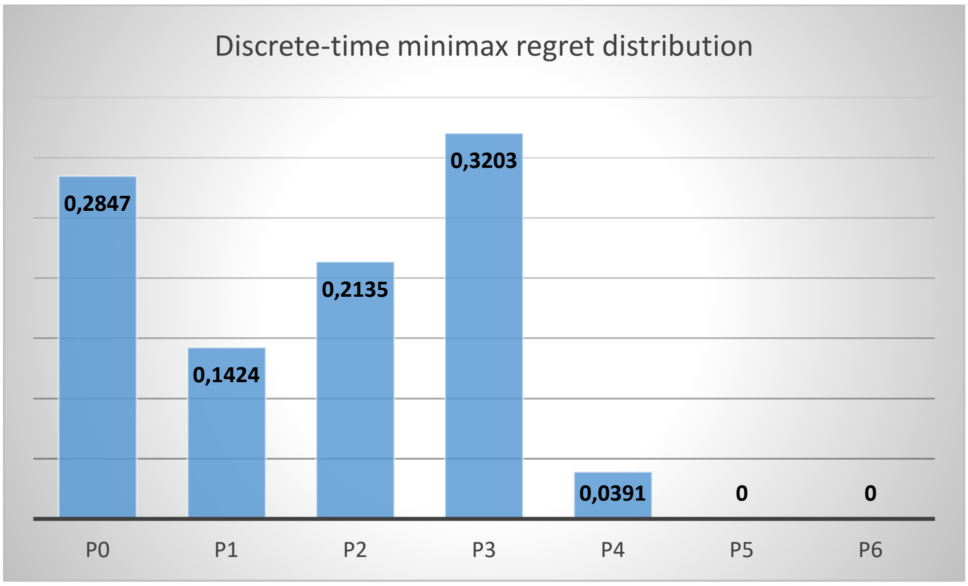

Figure 2: minimax regret distribution for T=6, M=-D=1, r= 0.25

We observe that the probabilities are increasing from period 1 to period 3 (t*-1), with a constant rate $(p_{t+1}-p_t)/p_t$=0.5, that leaving in period 4=t* is played with a smaller probability, and that leaving immediately (time 0) is played with a probability that is different from the other probabilities. The minimax regret is y=1.4236 which means that a firm, regardless of the exit time chosen by the opponent, will never get less than the best-reply payoff minus 1.4236. Given that at the optimum all inequations equalize in y, firm 1, by playing this strategy, always gets the best possible payoff minus 1.4236. So, if firm 2 exits in period 0, respectively in period 1, 2, 3, 4, 5 or 6, firm 1 gets 3.5764, respectively 1.5764, -0.0236, -1,3036 and -1.4236. Hence, if the opponent also plays the minimax regret strategy, her mean payoff is 0.7645>0. We can add that her payoff is positive each time her opponent leaves at time 0 or in period 1 (42.7% of the time), and positive or almost positive (-0.0236 is close to 0), when her opponent leaves at time 0, in period 1 or in period 2 (64.1% of the time).

We now give the minimax regret distribution in the general case:

Proposition 4

In the discrete-time game, the mixed minimax regret strategy consists in leaving the market with positive probability p_t in any period t from 0 to min(t,T), but not later. The probabilities check the following equations:*

$$p_0 = -\left(\frac{D}{M}\right).\frac{(1+r)^{\min(t*,T)}-1}{\left(\frac{M+r(M-D)}{M}\right)^{\min(t*,T)}}$$

$$p_1 = \frac{r(M-D)}{M}p_O$$

$$p_t = \left(\frac{M+r(M-D)}{M}\right)^{t-1}\frac{r(M-D)}{M}p_O \quad \textit{for t from 2 to min (t*-1,T-1)}$$

$$p_{\min(t*,T)} = \frac{p_0\left(\frac{M+r(M-D)}{M}\right)^{\min(t*,T)-1}\left((M-D)\left(\frac{1}{1+r}\right)^{\min(t*,T)-1}+D\right)}{-D\left(1-\left(\frac{1}{1+r}\right)^{\min(t*,T)}\right)}$$

$p_t = 0$ *for t from t*+1 to T when t*<T*

The minimax regret is equal to $\frac{p_0 M(1+r)}{r}$

Proof see Appendix E

We observe that p_0 and $p_{\min(t*,T)}$ do not behave like the other probabilities, and that the probabilities from p_1 to $p_{\min(t*-1,\, T-1)}$ are increasing with the constant increasing rate $\frac{r(M-D)}{M}$. If t*≤T, p_{t*} is a kind of "adjustment" probability and is very small. As a matter of fact $D+(M-D)\left(\frac{1}{1+r}\right)^{t*-1}$ is equal to $\frac{\left(D+\frac{D}{1+r}+\cdots+\frac{D}{(1+r)^{t*-2}}+\frac{M}{(1+r)^{t*-1}}\cdot\frac{1+r}{r}\right)r}{r+1}$, the profit a firm earns by staying in the market longer than the opponent, who stays in up to period t*-1, multiplied by r/(r+1). This profit becomes very small, goes to 0, when the interval between two periods becomes small. By contrast, if t*>T, then $D+(M-D)\left(\frac{1}{1+r}\right)^{T-1}$ is not necessarily small and p_T can be a large probability.

We now turn to the game in continuous time:

Proposition 5

$$t* = \frac{\ln\left(\frac{M-D}{-D}\right)}{r}$$

In the continuous-time game, if t ≤ T, the mixed minimax regret strategy is given by:*

- *The support of the strategy is [0, t*],*
- *The cumulative probability distribution on [0,t*] is given by:* $F(t) = (-\frac{D}{M-D})^{\frac{M-D}{M}}.e^{\frac{(M-D)rt}{M}}$
- *0 is a mass point played with probability* $g(0) = (-\frac{D}{M-D})^{\frac{M-D}{M}}$
- *The minimax regret is equal to* $\frac{g(0)M}{r}$.

If t > T, the mixed minimax regret strategy is given by:*

- *The support of the strategy is [0, T],*
- *The cumulative probability distribution on [0,T] is given by:* $F(t) = g(0)e^{\frac{(M-D)rt}{M}}$
- *0 is a mass point played with probability* g(0)=$(e^{rT}-1)(\frac{-D}{M})e^{-(M-D)rT/M}$ *>0 and T is a mass point played with probability* $g(T) = 1 + \frac{e^{rT}D}{M} - \frac{D}{M}$.
- *g(0) is increasing in T and equal to* $(-\frac{D}{M-D})^{\frac{M-D}{M}}$ *when T=t*. Symetrically, g(T) is decreasing in T and equal to 0 when T = t*.*
- *The minimax regret is equal to* $\frac{g(0)M}{r}$
  Proof see Appendix F

If our discrete-time example (with –D=M=1 and r=0.25) were studied in a continuous setting, we would obtain g(0)=0.25 and a minimax regret of 1.
In the game in continuous time, the probabilities increase at the rate r(M-D)/M, like in the discrete- time game, from 0+ to either t* (when t*≤T) or T- (if t*>T).
If t*≤T, $p_0=-(\frac{D}{M}).\frac{(1+r)^{t*}-1}{(\frac{M+r(M-D)}{M})^{t*}}$ in the discrete-time game becomes close to g(0) in the continuous-time game as soon as r →0 in the discrete-time game. As a matter of fact, if r →0, then $t^* \rightarrow \frac{\ln\left(-\frac{M-D}{D}\right)}{\ln(1+r)}$, hence $p_0 \rightarrow 1/e^{t*\ln(1+\frac{(M-D)r}{M})} \rightarrow 1/e^{\ln\left(-\frac{M-D}{D}\right)\ln\left(1+\frac{(M-D)r}{M}\right)/\ln(1+r)}$; the latter expression tends towards $1/e^{\ln\left(-\frac{M-D}{D}\right)(\frac{(M-D)r}{M})/r} = \left(-\frac{M-D}{D}\right)^{\frac{(M-D)}{M}}$=g(0) when r →0.
When T<t*, then T becomes an additional mass point. Given that g(0) is increasing in T and equal to $(-\frac{D}{M-D})^{\frac{M-D}{M}}$ only for T=t*, the firm leaves the market at time 0 with a smaller probability. For example, in our numerical example, when T=2<t*= 2.77, -D=M=1, r= 0.25, g(0)= 0.239 <0.25 and g(2)= 0.351. The minimax regret, equal to 0.955, is lower than 1, the minimax regret obtained for T>t*.

4. Minimax regret, mixed-strategy Nash equilibrium and maximin payoff: different behavior philosophies

We now comment the differences between the mixed-strategy Nash equilibrium distribution and the minimax regret one. It follows from figures 1 and 2 and from propositions 1 to 5 that the structure of both distributions is completely different.
For sake of clarity, we restrict attention in this section to the game in continuous time with t*<T. First, in the mixed-strategy Nash equilibrium, T is a mass point, whereas T is not even in the support of the minimax regret distribution. Second the probabilities decrease from 0 to T- in the mixed-strategy Nash distribution, at the rate Dr/M, whereas the probabilities increase from

0+ to t* in the minimax regret distribution, at the rate r(M-D)/M. Why do we have such a strong divergence in behavior?

The divergence follows from the difference in the philosophy of both concepts.

We recall that the mixed-strategy Nash equilibrium distribution follows from the differential equation $D\big(1-F(t)\big)+\frac{M}{r}f(t)=0$, which says that for firm 1, switching from exiting at t to exiting at t+dt, does not change her payoff *when firm 2 behaves according to the probability distribution f(.);* this is what is expected, by definition, in a mixed-strategy Nash equilibrium. Given that a player is completely indifferent between all the pure strategies in the support of his equilibrium distribution, his probabilities have no meaning for himself: so player 2's distribution has no meaning for himself (the only function of his distribution is to ensure firm 1's indifference between all her pure strategies). This is well illustrated in the numerical example: player 2 leaves the market in period 6 with the large probability 0.3349 not because this strategy is interesting for himself, but because it ensures that firm 1 is indifferent between the other strategies and leaving in period 6: as a matter of fact, leaving in period 6 appears as a very efficient strategy, except if the opponent also chooses this strategy with a strong probability. So paradoxically, a firm stays in the market up to period 6 with a large probability only to avoid that this strategy becomes the (only) best reply for the opponent.

The Nash distributions are logical (best-reply fixed-point logic) but they do not necessarily convince from a behavioral viewpoint, at least in a context where a player cannot anticipate what the other players may play.

According to the minimax regret philosophy, a player builds a strategy without trying (or without being able) to anticipate the strategies played by the others. She builds a probability distribution on *her* pure strategies so as to never suffer from a too large regret, regardless of the strategies chosen by the others. It follows that the probabilities she calculates are useful for herself. As a matter of fact, the differential equation that leads to the minimax regret distribution is $\frac{M}{r}f(t)+F(t)(D-M)=0$. This equation, quite different from the one obtained for the Nash equilibrium, says that firm 1, by *behaving herself according to the distribution f(.),* has the same regret when *firm 2* switches from leaving in period t to leaving in period t+dt. As a matter of fact, what changes for firm 1 due to this switch? Nothing changes if she stays in for a time longer than t+dt (she has no regret). For each of her strategies from leaving at time 0 to leaving at time t, she does not suffer from the duopoly loss at time t+dt (i.e. during an interval of time dt), and she does not regret the monopoly profit at time t (i.e. during an interval of time dt), hence her regret decreases by the amount (M-D)F(t)dt. But, when she leaves in period t+dt, she has no regret when player 2 plays t, but regrets the monopoly profit forever (M/r) when he leaves at t+dt. Hence she has the additional regret (M/r)f(t)dt. Given that the total regret should not change, we get $\frac{M}{r}f(t)+F(t)(D-M)=0$. What matters is that the probability distribution f(.) is firm 1's distribution. When *she* plays according to this distribution, she is sure to always get the same regret g(0)M/r, regardless of the other firm's exit time. She always gets (at least) the best-reply payoff to this exit time minus the regret g(0)M/r.

The minimax regret philosophy shares with the maximin payoff philosophy the fact that each player builds a probability function for herself, in order to protect herself from the others' behavior. But the link between both philosophies stops here. With the maximin payoff philosophy, a firm builds a probability distribution on her actions that ensures her a payoff that she tries to maximize, regardless of what is played by the other. So the maximin payoff philosophy is very pessimistic: in some way, a player, when choosing a strategy, only focuses on the worst thing that may happen with this strategy. This explains that, each time a firm chooses to stay in the market up to the time t, she fears that the opponent stays in longer, so that she will lose money. So the best thing a firm can do is to leave the market immediately, which is the only way to be sure to not get a negative payoff. This explains that the maximin payoff strategy leads to a null payoff (which is also the mixed-strategy Nash equilibrium expected payoff). The minimax regret philosophy is less pessimistic: a player, when choosing a strategy, focuses on the fact that it may not be the best response to the opponent's behavior: given the unknown way of playing of the opponent, another strategy may be the best reply and therefore may lead to a better payoff. So the philosophy is "we could have done better" which is much less pessimistic than the maximin payoff philosophy.

This also explains that the minimax regret philosophy is much less "one side" focalized than the maximin payoff philosophy. The minimax regret strategy takes into account that firm 2 may exit the market at time T (or at t>t*) but it also takes into account that firm 2 may exit the market much earlier, which generates a different regret. By aiming to minimize all types of regrets, the minimax regret philosophy takes into account that leaving at time 0 is indeed the best reply if the opponent leaves at T (or at t>t*), hence that a firm regrets to not leave at time 0 if the other firm stays in the market for a long time, but it takes also into account that, if the opponent leaves the market early, then the firm regrets to not have stayed in the market longer. So it leads to a more nuanced strategy that copes with all possible regrets. This way of doing is especially fruitful, in that it leads to a positive expected payoff, when the other firm also plays the minimax regret strategy (see proposition 6 below).
Despite this main difference, there is a technical link worth to mention between the minimax regret strategy and the maximin payoff strategy. If we transform the regret matrix into a two-player zero-sum game, where firm 1's payoffs are the opposite of the regrets, and player 2 is an artificial player who gets the regrets, and if we look for firm 1's maximin payoff strategy in this new game, then we get her minimax regret strategy in the original game. This follows from the fact that it is equivalent to maximize a function h(.) and to minimize the function –h(.), under a same set of constraints. Moreover, given that, in two-player zero-sum games, the maximin payoff behavior is a mixed-strategy Nash equilibrium, we get a link between the minimax regret behavior and the mixed-strategy Nash equilibrium (in another game). This link has been observed by Renou and Schlag (2010).

5. Minimax regret: a positive payoff for both firms

Proposition 6
If the opponent plays the minimax regret strategy, then, for each firm, the minimax regret strategy leads to a positive expected payoff, E(g). E(g) is the sum of the best -reply payoffs to each exit time t, with t from 0+ to min(t,T) in the continuous setting, respectively from 1 to min(t*,T) in the discrete setting, weighted by the probabilities of exiting at time t.*
In the continuous model, for $t^* \leq T$, E(g) is equal to $\frac{M}{r}\left(1-\frac{g(0)(M-2D)}{-D}\right)$.
In the continuous model, for $t^* > T$, E(g) is equal to
$\frac{g(0)(M-2D)}{D}.\frac{M}{r}+\frac{g(0)}{Dr}(D^2e^{\frac{(M-D)rT}{M}}-(M-D)^2e^{-\frac{DrT}{M}})$.
Proof see Appendix G

The expected payoff is positive by construction. A firm, for each opponent's exit time, gets the best-reply payoff minus the regret, equal to $p_0M(1+r)/r$ in the discrete version, to g(0)M/r in the continuous version. So the expected sum of regrets is this regret. Yet the best-reply payoff, when the opponent exits the market at time 0, weighted by the probability to do so, is $p_0M(1+r)/r$ in the discrete game, g(0)M/r in the continuous game. It follows that E(g) is the sum of best-reply payoffs to the opponent's exit times t from 1 to min (t*,T) in the discrete-time game, from 0+ to min(t*,T) in the continuous-time game, weighted by the probability of exiting the market at time t.

In the game in continuous time, for t*<T (or T infinite), we make the following comments.

We set x = -D/M. So we get: $g(0) = \left(-\frac{D}{M-D}\right)^{\frac{M-D}{M}} = \left(\frac{x}{1+x}\right)^{1+x}$, $t^* = \frac{\ln\left(\frac{M-D}{-D}\right)}{r} = \frac{\ln\left(\frac{1+x}{x}\right)}{r}$ and

$$E(g) = \frac{M}{r}\left(1-\frac{g(0)(M-2D)}{-D}\right) = \frac{M}{r}\left(1-\left(\frac{x}{1+x}\right)^{x}-\left(\frac{x}{1+x}\right)^{1+x}\right)$$

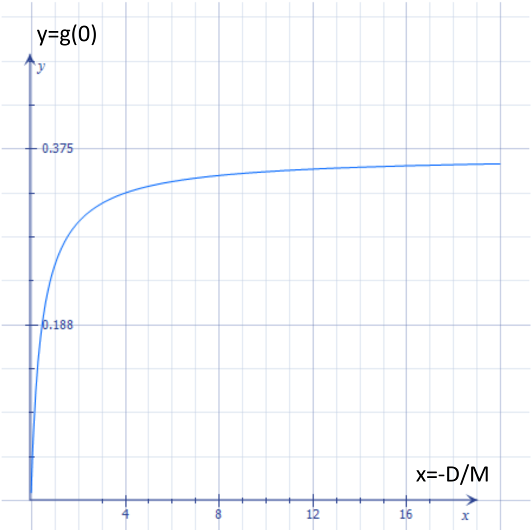

Figure 3: g(0)

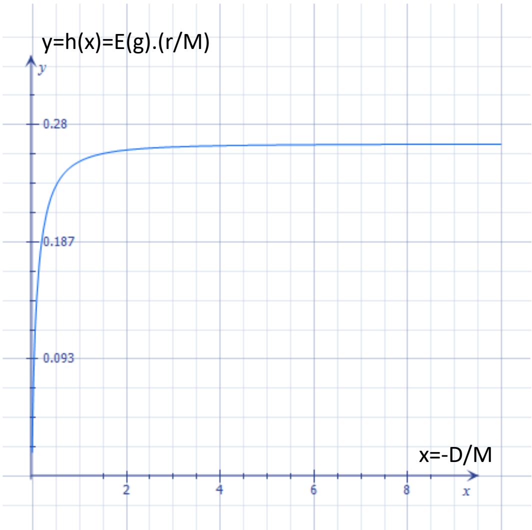

Figure 4:E(g)r/M

We first comment g(0), depicted in figure 3. Clearly, for a fixed value M, when D→0, then g(0)→0, given that a firm never loses money by staying in the market. The more D becomes negative, hence the more -D/M is large, the more we expect the firm to exit the market early, so g(0) is increasing, as expected. Yet g(0) has an upper bound. We show in appendix G that even if D is very negative for a fixed M, i.e. if -D/M →+∞, g(0) only tends towards e^{-1}, i.e. 0.368, a rather surprising result. This result derives from the fact that a firm, by leaving at time 0, still regrets the large amount M/r she could get by staying in the market when the opponent also leaves the market at time 0. That is why leaving immediately the market with probability 1 is not acceptable from a regret viewpoint. So even in the worst scenario, minimax regret strategies do not converge to the maximin payoff strategy (which assigns probability 1 to leaving the market at time 0).

We now comment the expected minimax regret payoff E(g). Function h(x)=E(g)r/M is represented in figure 4 . It follows that:

- For a fixed value x= -D/M, E(g) is increasing in M, which is not astonishing, and clearly shows that the minimax regret criterion, contrary to the Nash equilibrium concept, allows both firms to earn a nice expected payoff.
- E(g) is decreasing in r, which is quite logical in that the firms benefit better from the infinite monopoly payoff when the actualization rate is low. When r tends towards 0, the expected payoff tends towards $+\infty$.
- For fixed values M and r, E(g) is increasing in x= -D/M. It goes to 0 when x goes to 0 because, on the one hand, no firm leaves the market for D=0 (there is no regret when staying in the market), but, on the other hand, the firms cannot make any positive profit because they earn the null duopoly profit forever. E(g) tends towards $(1\text{-}2e^{-1})M/r$ = 0.264M/r when -D/M tends towards $+\infty$. Hence, once again, even with very bad duopoly payoffs, the minimax regret expected payoff is positive. Moreover it is growing in –D/M, so the expected payoff is larger when the duopoly payoffs are very negative. This surprising result stems from two facts. When D is very low (-D large), a firm is induced to exit the market faster (g(0) grows). So it does not earn the bad duopoly payoffs very long. Moreover, the opponent also leaves the market earlier, so, with a large probability she will be a monopoly sooner, and earn the monopoly payoff for a longer time. Putting things together induces a larger payoff. Observe that the payoff function fast converges to 0.264M/r, given that E(g) > 0.25M/r for x > 1.

E(g), the *expected* payoff is positive, but the payoff (difference between the best-reply payoff and the minimax regret strategy's payoff) is positive when the opponent leaves the market fast enough and it is negative when he leaves the market later. So we introduce t**, the opponent's last exit time that ensures a positive payoff. t** is defined by: $\int_0^{t^{**}} De^{-rs}ds + \int_{t^{**}}^{\infty} Me^{-rs}ds = \frac{g(0)M}{r}$.

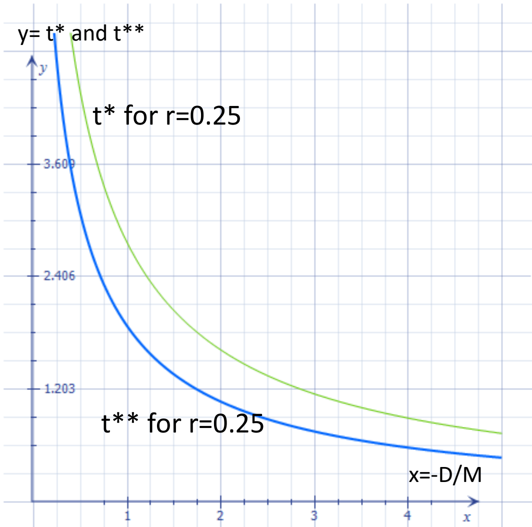

Figure 5 : t and t***

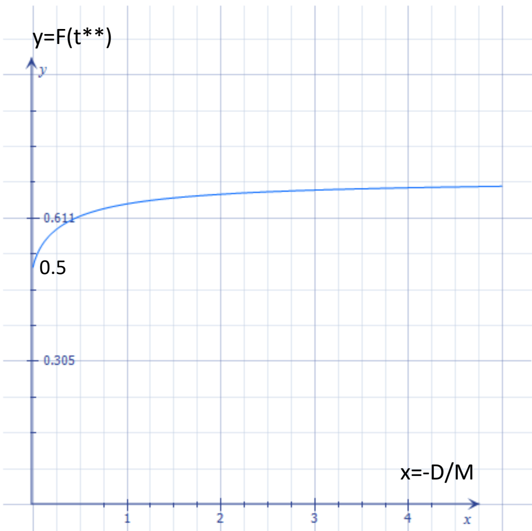

*Figure 6 : F(t**)*

We get t** = $-\ln\left(\frac{\left(\frac{x}{1+x}\right)^{1+x}}{1+x} + \frac{x}{1+x}\right)/r$ with x = -D/M. And the opponent exits the market before t** with probability F(t**) = $\left(\frac{\left(\frac{x}{1+x}\right)^{x}}{1+x} + 1\right)^{-(1+x)}$ if he also plays the minimax regret strategy.

The evolutions of t* and t** are given in figure 5 for r=0.25. F(t**) is given in Figure 6. It is nice to observe that the payoff is positive more than half the time regardless of x, and that F(t**) fast becomes larger than 0.6. F(t**) is increasing in x, going from 0.5 to $e^{-e^{-1}}$(≈0.70) when x tends towards +∞ , and it is larger than 0.64 for x larger than 1.

6. Conclusion, limit exit time and mixed strategies

The main result of the paper is: whereas the mixed-strategy Nash equilibrium leads to a risky null expected payoff in the duopoly exit time game, minimax regrets allow both firms to earn a potentially large positive expected payoff.

We open the discussion on the limit time T and on the notion of mixed strategy.

Is it interesting, from an economic viewpoint, to include a limit time T? Very often, in many countries, it is forbidden to stay in a market when losing money. As regards the mixed-strategy Nash equilibrium concept, the question does not matter, in that both firms have a null expected payoff regardless of T (and whether T be finite or infinite). This is not the case with the minimax regret criterion.

We observe, on the one hand, that T>t* has no impact on the firms' behavior in that they exit the market at the latest at t* <T. On the other hand, for T<t*, the more T is low, the more the expected payoff, E(g), is low. As a matter of fact, for T<t*, E(g) is increasing in T, and equal to the expected payoff obtained for T>t* only when T=t* . For example, for T=2 and –D=M=1, we get E(g)= 0.837 < 1 (=the expected payoff for T>t*)

We illustrate this fact in figure 7, for –D=M=1, r=0.25 (hence t*= 2.77 and E(g)=1 for T>t*).

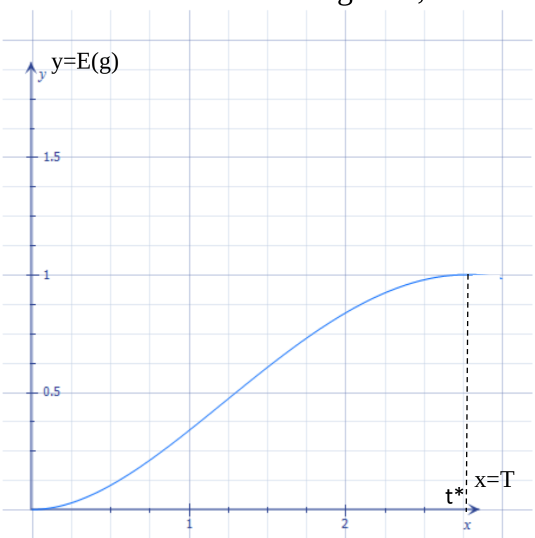

Figure 7: E(g) for T<t,–D=M=1, r=0.25*

It follows from this observation that the minimax regret criterion suggests to not introduce a limit exit time T (or to introduce a large enough one, so that T>t*).

We now open the discussion on the notion of mixed strategy, by first focusing on the meaning of a mixed strategy. We recall that in a mixed-strategy Nash equilibrium, the probabilities of a player only make sense for the other players in that they stabilize their behavior; by contrast, the minimax regret probabilities of a player make sense for herself, in that they help the player to balance her regrets. In this sense the minimax regret criterion belongs to the behavioral approaches of mixed strategies, in that it gives meaning to the mixed strategies. Other approaches, for example Best Reply Matching (see Kosfeld & al. (2002)) also aim to give more behavioral meaning to mixed strategies. According to the Best Reply Matching philosophy, a strategy is played as often as it is a best reply: so, if player 1's action A is a best reply to player 2's action B and player 2 plays B with probability p, then player 1 plays A with the same probability p. Our point of view it that thinking mixed strategies differently (not in the mixed-strategy Nash equilibrium way) can bring new insights to old economic topics.

We finally focus on a technical point. In some way, even when we look for a mixed minimax regret behavior, we do not completely depart from a pure strategy study. As a matter of fact we study a firm's regret for any opponent's exit time, that is to say for any opponent's pure strategy. This is not a problem from a conceptual point of view: a player who is playing a mixed strategy is supposed to play the pure strategy selected by a random device, which selects each of the pure strategies in accordance with the mixed strategy distribution. So, in the end, the player plays a pure strategy, which makes Renou and Schlag's approach, we adopt in this paper, meaningful. Yet Halpern and Pass (2012) observe that conforming to a true mixed behavior (of the opponent) may lead to another way of doing. We show why in our numerical example. When firm 2 leaves in period 3, firm 1's regrets are given in column 3 (matrix 2) because the best reply is to leave in period 4. When firm 2 exits in period 4, firm 1's regrets are given in column 4 (matrix 2) because the best reply is to leave at time 0. Yet when firm 2 leaves in period 3 with probability 0.9 and in period 4 with probability 0.1, the best reply is to leave in period 5 or 6, because 0.12x0.9+(-0.904)x0.1= 0.0176 >0. It follows a new column of regrets, we compare to column 3 and to column 4, given in matrix 3.

		3	Firm 2 4	Mixed (0.9 on period 3, 0.1 on period 4)
	0	0.12	0	0.0176
	1	1.12	1	1.0176
	2	1.92	1.8	1.8176
Firm 1	3	2.56	2.44	2.4576
	4	0	2.952	*0.2048*
	5	0	0.904	0
	6	0	0.904	0

Matrix 3: firm 1's regrets when player 2 exits in period 3, in period 4, and in period 3 and period 4 respectively with probability 0.9 and 0.1

For example, firm 1's regret when leaving in period 4, 0.2048, when firm 2 exits in period 3 with probability 0.9 and exits in period 4 with probability 0.1, is the difference between 0.12x0.9-2.952x0.1 and 0.0176. Clearly, in this example, the mixed column does not change the minimax regret, because, if column 3 and column 4 lead to the regret y, the mixed column leads to a regret strictly lower than y (because the real numbers in this column are lower than 0.93 (numbers in column 3) + 0.07 (numbers in column 4)). Given the structure of the payoffs in matrix 2, we can reasonably conjecture that Halpern and Pass' (2012) approach would not change the study, but it would complicate it.

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Appendix A

We look for a symmetric mixed-strategy Nash equilibrium with a mass point on T.
Suppose that firm 2's strategy is given by the density function f(t) and the cumulative probability function F(t) defined on [0, T].
By going out at the end of period d, firm 1 gets:
$\int_d^{T^-}(\int_0^d De^{-rs}ds)f(t)dt + (\int_0^d De^{-rs}ds)g(T) + \int_0^d(\int_0^t De^{-rs}ds + \int_t^{\infty} Me^{-rs}ds)f(t)dt$
This payoff is constant over [0, T], so the derivative in d has to be equal to 0, from 0 to T-.
It follows: $-(\int_0^d De^{-rs}ds)f(d) + \int_d^{T^-} De^{-rd}f(t)dt + De^{-rd}g(T) + (\int_0^d De^{-rs}ds + \int_d^{\infty} Me^{-rs}ds)f(d) = 0$
Developing this equation leads to $D - DF(d) + \frac{M}{r}f(d) = 0$.
Solving this differential equation leads to $F(t) = 1 - e^{\frac{Dr}{M}t}$ with f(t)= $-\frac{Dr}{M}e^{\frac{Dr}{M}t}$ and $g(T) = 1 - F(T^-) = e^{\frac{Dr}{M}T}$.
The mean payoff obtained by leaving in period d is the payoff obtained by leaving at time 0, i.e. 0. It can be checked that the same payoff is obtained by leaving in period T.
This payoff is:
$(\int_0^T De^{-rs}ds)g(T) + \int_0^{T^-}(\int_0^t De^{-rs}ds + \int_t^{\infty} Me^{-rs}ds)f(t)dt$
$= \frac{D-De^{-rT}}{r}e^{\frac{Dr}{M}T} + \int_0^{T^-}\left(\frac{D-De^{-rt}}{r} + \frac{M}{r}e^{-rt}\right)\left(-\frac{Dr}{M}\right)e^{\frac{Dr}{M}t}dt$
$= \frac{D-De^{-rT}}{r}e^{\frac{Dr}{M}T} + \int_0^{T^-}\left(\frac{(M-D)}{r}\right)\left(-\frac{Dr}{M}\right)e^{\frac{(D-M)r}{M}t}dt + \int_0^{T^-}\left(-\frac{D^2}{M}\right)e^{\frac{Dr}{M}t}dt$
$= \frac{De^{\frac{Dr}{M}T} - De^{\frac{(D-M)r}{M}T}}{r} + \frac{D}{r}\left(e^{\frac{(D-M)r}{M}T} - 1\right) - \frac{D}{r}(e^{\frac{Dr}{M}T} - 1) = 0$

f(t) is decreasing in t given that f'(t) = $-(\frac{Dr}{M})^2 e^{\frac{Dr}{M}t}$<0.

Appendix B

The only symmetric Nash equilibrium is a full support mixed Nash equilibrium. First, it is not possible to only exit the market in (some) periods up to k (with k<T) because, if so, leaving in period k+1 leads to a higher payoff than leaving in period k. As a matter of fact, it leads to the same payoff than leaving in period k in front of an opponent leaving before period k, and it leads to a larger payoff in front of an opponent leaving in period k (by ensuring a monopoly payoff in all periods later than k). Second, it is not possible to have one hole in the support, for example between the periods k and r, with r>k+1 (which means that the firm does not leave the market in the periods from k+1 to r-1). As a matter of fact, exiting in period k+1 is a better reply than exiting in period r because it leads to the same payoff, respectively to a larger payoff, in front of an opponent leaving from time 0 to period k, respectively in front of an opponent leaving in period r or later. Finally, only exiting the market in periods from k (>0) to M cannot be an equilibrium, because leaving in period k always leads to a negative payoff, lower than the null payoff obtained when leaving at time 0. So we have a full support Nash equilibrium.
The discrete-time Nash equilibrium is similar to the continuous-time one. We construct it step by step. Exiting the market at time 0 and in period 1 give rise to the same payoff if and only if: $0 = \frac{M(1+r)}{r}p_0 + D(1-p_0)$. So $p_0 = -\frac{rD}{M+r(M-D)}$

Leaving in period T and leaving in period T-1 give rise to the same payoff, except if the opponent also leaves in period T or T-1, so we need :

$p_{T-1}\sum_{t=0}^{T-2}\frac{D}{(1+r)^t} + p_T\sum_{t=0}^{T-2}\frac{D}{(1+r)^t} = p_{T-1}(\sum_{t=0}^{T-2}\frac{D}{(1+r)^t} + \frac{\frac{M}{(1+r)^{T-1}}(1+r)}{r}) + p_T\sum_{t=0}^{T-1}\frac{D}{(1+r)^t}$

Hence $p_T = \frac{M(1+r)}{-rD}p_{T-1}$ and $p_{T-1} + p_T = \frac{M+r(M-D)}{-rD}p_{T-1}$

Leaving in period T-1 and leaving in period T-2 give rise to the same payoff, except if the opponent leaves in period T, T-1, or T-2, so we need:

$p_{T-2}\sum_{t=0}^{T-3}\frac{D}{(1+r)^t} + p_{T-1}\sum_{t=0}^{T-3}\frac{D}{(1+r)^t} + p_T\sum_{t=0}^{T-3}\frac{D}{(1+r)^t} = p_{T-2}\left(\sum_{t=0}^{T-3}\frac{D}{(1+r)^t} + \frac{\frac{M}{(1+r)^{T-2}}(1+r)}{r}\right) +$
$p_{T-1}\sum_{t=0}^{T-2}\frac{D}{(1+r)^t} + p_T\sum_{t=0}^{T-2}\frac{D}{(1+r)^t}$

Hence $p_{T-2}\left(\frac{\frac{M}{(1+r)^{T-2}}(1+r)}{r}\right) + p_{T-1}\left(\frac{D}{(1+r)^{T-2}}\right) + p_T\left(\frac{D}{(1+r)^{T-2}}\right) = 0$

So we get $p_{T-1} = \frac{M(1+r)}{M+r(M-D)}p_{T-2}$ and $p_{T-2} + p_{T-1} + p_T = \frac{M+r(M-D)}{-rD}p_{T-2}$

More generally we assume that $\sum_{i=t}^{T} p_i = \frac{M+r(M-D)}{-rD}p_t$

Leaving in period t-1 leads to the same payoff than leaving in period t if and only if:

$\sum_{i=t-1}^{T}(p_i\sum_{j=0}^{t-2}\frac{D}{(1+r)^j}) = p_{t-1}(\sum_{j=0}^{t-2}\frac{D}{(1+r)^j} + \frac{\frac{M}{(1+r)^{t-1}}(1+r)}{r}) + \sum_{i=t}^{T}(p_i\sum_{j=0}^{t-1}\frac{D}{(1+r)^j})$

Hence $p_{t-1}\left(\frac{\frac{M}{(1+r)^{t-1}}(1+r)}{r}\right) + \left(\frac{D}{(1+r)^{t-1}}\right)\sum_{i=t}^{T}p_i = 0$. It follows from the assumption:

$p_{t-1}\left(\frac{M(1+r)}{r}\right) + D(\frac{M+r(M-D)}{-rD})p_t = 0$

So we get $p_t = \frac{M(1+r)}{M+r(M-D)}p_{t-1}$ and $\sum_{i=t-1}^{T}p_i = \frac{M+r(M-D)}{-rD}p_{t-1}$

The same relation holds till t = 2, hence $p_2 = \frac{M(1+r)}{M+r(M-D)}p_1$ and $p_t = \left(\frac{M(1+r)}{M+r(M-D)}\right)^{t-1} p_1$ for t from 2 to T-1.
Exiting in period 1 and exiting in period 2 lead to the same payoff if and only if:
$D(1-p_0) = p_1\left(D + \frac{\frac{M}{(1+r)}(1+r)}{r}\right) + (1-p_0-p_1)\left(D + \frac{D}{1+r}\right)$
Given $p_0 = -\frac{rD}{M+r(M-D)}$, we get $p_1 = -\frac{rDM(1+r)}{(M+r(M-D))^2} = \frac{M(1+r)}{M+r(M-D)}p_0$ and $p_t = (\frac{M(1+r)}{M+r(M-D)})^t p_0$ for t from 1 to T-1.
Hence $p_t = -(\frac{M(1+r)}{M+r(M-D)})^t \frac{rD}{M+r(M-D)}$ for t from 1 to T-1 and $p_T = \frac{M(1+r)}{-rD}p_{T-1} = (\frac{M(1+r)}{M+r(M-D)})^T$.
We get a geometric sequence from p_0 to p_{T-1} and a different probability for p_T. It can be checked that the sum of all the probabilities is equal to 1.
When there is no limit period T, that is to say when T → +∞, the Nash equilibrium is built in the same way, and $p_t = -(\frac{M(1+r)}{M+r(M-D)})^t \frac{rD}{(M+r(M-D))}$ for t from 0 to +∞.
The expected payoff is 0.

Appendix C

To establish the maximin payoff y, we solve the optimization problem:
$\max_{p_0 \dots p_T y} y$
u.c.
$\sum_{i=0}^{T} p_i P(i,j) \geq y$ j from 0 to T
$\sum_{i=0}^{T} p_i = 1$
$p_i \geq 0$ i from 0 to T

where p_i is the probability firm 1 assigns to the strategy "leaving in period i" and P(i,j) is firm 1's payoff when she leaves in period i and the opponent leaves in period j.
If i > j' > j, then P(i,j) > P(i,j') because firm 1 gets the monopoly payoff faster when the opponent exits in period j than when he leaves in period j'.
If i = j' > j, then P(i,j) > P(i,j'), because firm 1, when meeting an opponent exiting in period j', never gets the monopoly payoff and gets the duopoly payoff longer than when she meets an opponent leaving in period j.
If j'> j ≥ i, then P(i,j) = P(i,j') because firm 1 gets the duopoly payoff during i periods when her opponent exits the market later than herself.
So, when the last constraint $\sum_0^T p_i P(i,T) \geq y$ is satisfied, all the other constraints are satisfied too. And given that P(i,T) is lower than P(0,T) for any i from 1 to T, because it is better to leave immediately when the opponent leaves in period T, $\sum_0^T p_i P(i,T)$, hence y, is maximal for p_0 = 1 and p_i = 0 for i from 1 to T. It follows that leaving at time 0 and getting a null payoff is the maximum of this program.

Appendix D

We first assume t* ≤ T and consider the game in discrete time.
For any strategy that consists in leaving in period t, with t ≤ t*-1, the maximal regret is observed when the other firm exits in period t. If so, the firm gets t times the negative duopoly payoff instead of t times the negative duopoly payoff plus the positive monopoly payoff forever, from period t+1 onwards, by leaving one period later. Hence the maximal regret is the monopoly

payoff obtained from period t+1 onwards (and the lowest regret of this kind is obtained for t = t*-1). If the opponent leaves earlier, the firm has no regret; if he leaves in period d, with d higher than t but lower than t*, the best decision consists in going out in period d+1, and it leads to earning more times the duopoly payoff and to earning less often the monopoly payoff, so it leads to a lower regret than if the opponent exits in period t. Finally if the opponent leaves in period d, with d≥ t*, the best reply is to leave immediately, hence the regret is t times the opposite of the duopoly payoff, $-\sum_{i=0}^{t-1} D/(1+r)^i$, which is lower than the monopoly payoff obtained from period t+1 onwards. Hence the maximal regret is the monopoly payoff obtained from period t+1 onwards and the lowest regret of this kind is obtained for t=t*-1.
For any strategy that consists in leaving in period t, with t larger than or equal to t*, the regret is equal to t times the opposite of the duopoly payoff if the opponent leaves in period t or later (and the lowest regret of this type is obtained for t=t*). If the opponent leaves in period d, with d lower than t but larger than t*-1, the firm gets less often the duopoly payoff and gets the monopoly payoff from period d+1 on, so her regret is lower. If the opponent leaves before period t*, the firm has no regret. Hence the maximal regret, when leaving in period t later than t*-1, is t times the opposite of the duopoly payoff and the lowest regret of this type is obtained for t=t*.
So the pure minimax regret strategy consists in leaving in period t*-1 or in period t*, depending on whether $-\sum_{i=0}^{t*-1} \frac{D}{(1+r)^i}$ is larger than $\frac{M}{(1+r)^{t*-1}} \frac{(1+r)}{r}$ or not.
The previous reasoning easily adapts to the continuous setting. In this setting, we have $-\int_0^{t*} De^{-rt}dt = \int_{t*}^{+\infty} Me^{-rt}dt$ and the pure minimax regret strategy simply consists in leaving at time t*.

We now assume t*>T.
In the discrete-time game, as above, leaving in period t, with t≤T-1, leads to the maximal regret when the opponent leaves in period t. This regret is the monopoly payoff obtained from period t+1 onwards (and the lowest regret of this kind is obtained for t=T-1, because T<t*). The maximal regret, when leaving in period T, is obtained when the opponent also leaves in period T, and it is equal to $-\sum_{i=0}^{T-1} \frac{D}{(1+r)^i}$. Given that $-\sum_{i=0}^{T-1} \frac{D}{(1+r)^i} < \frac{M}{(1+r)^{T-1}} \frac{(1+r)}{r}$ because T<t*, the pure minimax regret strategy consists in leaving in period T.
The same result holds for the game in continuous time, given that $-\int_0^{T} De^{-rt}dt < \int_{T}^{+\infty} Me^{-rt}dt$ for T<t*.

Appendix E

Study with t*≤T.

We call column t, the column of firm 1's regrets when firm 2 leaves in period t. The regrets in each column have a regular pattern.
Consider a period $\tilde{t}$ with $\tilde{t} \leq t*-1$, and suppose that the opponent (firm 2) exits the market in period $\tilde{t}$. By leaving in period t <$\tilde{t}$, firm 1 gets the negative payoff $\sum_{i=0}^{t-1} \frac{D}{(1+r)^i}$ and the best payoff she could obtain is $\sum_{i=0}^{\tilde{t}-1} \frac{D}{(1+r)^i} + \frac{M}{(1+r)^{\tilde{t}}} . \frac{1+r}{r}$. So her regret is $\sum_{i=t}^{\tilde{t}-1} \frac{D}{(1+r)^i} + \frac{M}{(1+r)^{\tilde{t}}} . \frac{1+r}{r}$.
Given that D is negative, this regret is increasing in t (<$\tilde{t}$) and is maximum for t=$\tilde{t}$, where it is equal to $\frac{M}{(1+r)^{\tilde{t}}} . \frac{1+r}{r}$.

By comparing two adjacent columns, where firm 2 leaves in period $\tilde{t}-1$ or $\tilde{t}$, player 1's regret when leaving in period t, with t < $\tilde{t}-1$, is $\sum_{i=t}^{\tilde{t}-2}\frac{D}{(1+r)^i}+\frac{M}{(1+r)^{\tilde{t}-1}}.\frac{1+r}{r}$ when firm 2 exits in period $\tilde{t}$-1, and $\sum_{i=t}^{\tilde{t}-1}\frac{D}{(1+r)^i}+\frac{M}{(1+r)^{\tilde{t}}}.\frac{1+r}{r}$ when firm 2 exits in period $\tilde{t}$. So her regret is larger when firm 2 leaves in period $\tilde{t}-1$; she gets an additional regret $\frac{M-D}{(1+r)^{\tilde{t}-1}}$, because it is more regrettable to leave earlier than the opponent when he leaves the market faster. A similar observation holds for $t=\tilde{t}-1$.
In contrast, when firm 2 leaves in period $\tilde{t}$, with $\tilde{t}$≥t* the best thing firm 1 can do is to leave immediately, so, when she exits in period t ≤ $\tilde{t}$, her regret is $-\sum_{i=0}^{t-1}\frac{D}{(1+r)^i}$. This regret is growing in t, and constant in $\tilde{t}$ with $\tilde{t}$≥t* .When firm 1 exits after period $\tilde{t}$, her regret is $-\sum_{i=0}^{\tilde{t}-1}\frac{D}{(1+r)^i}-\frac{M}{(1+r)^{\tilde{t}}}.\frac{1+r}{r}$. This regret is growing in $\tilde{t}$ given that firm 1 earns the negative duopoly profit longer when $\tilde{t}$ is larger and earns the positive monopoly profit later.
Finally the regrets in column t*-1 are larger than the regrets in column t*, when firm 1 leaves in period t, with t≤t*-1. If so, when the opponent leaves in period t*-1, firm 1's regret is $\sum_{i=t}^{t*-2}\frac{D}{(1+r)^i}+\frac{M}{(1+r)^{t*-1}}.\frac{1+r}{r}$. We have $\sum_{i=0}^{t*-2}\frac{D}{(1+r)^i}+\frac{M}{(1+r)^{t*-1}}.\frac{1+r}{r}>0$ so $\sum_{i=t}^{t*-2}\frac{D}{(1+r)^i}+\frac{M}{(1+r)^{t*-1}}.\frac{1+r}{r}>-\sum_{i=0}^{t-1}\frac{D}{(1+r)^i}$. So this regret is larger than the regret with an opponent leaving in period t*. In contrast, firm 1 has no regret when she leaves in period t* and firm 2 leaves in period t*-1, whereas her regret is $-\sum_{i=0}^{t*-1}\frac{D}{(1+r)^i}$ when firm 2 leaves in period t* too.

We write p_t the probability firm 1 assigns to the strategy "leaving in period t", Regret($\tilde{t}$) firm 1's expected regret when her opponent leaves in period $\tilde{t}$, and X_{it} firm 1's regret when she exits in period i and firm 2 exits in period t. The properties of the regrets ensure that the program

$\min\limits_{p_0 \dots p_T y} y$
u.c. Regret(t) ≤ y for t from 0 to T
$\sum_{t=0}^{T}p_t=1$
p_t ≥ 0 for t from 0 to T

has a unique solution that checks p_t >0 for t from 0 to t* and p_t =0 for t>t*. This property ensures that Regret(t) = Regret(t*) for t > t*.
To prove this result, we assume that, at the optimum, Regret(t)=y for t from 0 to t*. The positivity of the optimal p_t, t from 0 to t*, results from the fact that the regret, when leaving in period t (before the opponent), is lower when the opponent leaves the market later. As a matter of fact, for two columns $\tilde{t}$ and $\tilde{t}+1$, with $\tilde{t}+1<$ t*, we get: Regret ($\tilde{t}$) = $p_oX_{0\tilde{t}}+\dots+p_{\tilde{t}}X_{\tilde{t}\tilde{t}}$ and Regret ($\tilde{t}+1$) =$p_oX_{0\tilde{t}+1}+\dots+p_{\tilde{t}}X_{\tilde{t}\tilde{t}+1}+p_{\tilde{t}+1}X_{\tilde{t}+1,\tilde{t}+1}$. Suppose that p_i>0 for i from 0 to $\tilde{t}$. We get:
$p_oX_{0\tilde{t}}+\dots+p_{\tilde{t}}X_{\tilde{t}\tilde{t}}$ = y = $p_oX_{0\tilde{t}+1}+\dots+p_{\tilde{t}}X_{\tilde{t}\tilde{t}+1}+p_{\tilde{t}+1}X_{\tilde{t}+1,\tilde{t}+1}$ $\Rightarrow p_{\tilde{t}+1}>0$ because $X_{i\tilde{t}}$>$X_{i\tilde{t}+1}$for any i from 0 to $\tilde{t}$. And the same is true for p_{t*} given the structure of the regrets in the two columns t*-1 and t*. So when p_0 is positive, p_t is positive too, for any t from 1 to t*.
The way the probabilities are determined ensures unicity. The first column leads to $p_0X_{00}=y$, so $p_0=\frac{y}{X_{00}}$, then the second column leads to $p_0X_{01}+p_1X_{11}=y$ so $p_1=\frac{y-p_0X_{01}}{X_{11}}$ and so on, up to p_{t*}.

The Karush Kuhn Tucker (KKT) function is $y + \sum_{t=0}^{T} \lambda_t (Regret(t) - y) - \sum_{t=0}^{T} \mu_t p_t + \lambda(\sum_{t=0}^{T} p_t - 1)$. For the obtained values of p_t, t from 0 to T, to be optimal, the multipliers λ_i for i from 0 to T have to be positive or null, the multipliers μ_t have to be null for t from 0 to t* and positive or null for t from t*+1 to T. These conditions are satisfied thanks to the structure of the regrets.

As a matter of fact, the derivative of the KKT function in p_0 leads to $\sum_{i=0}^{t*-1} \lambda_i X_{0i} + \lambda = 0$. More generally the derivative in p_t leads to $\sum_{i=t}^{T} \lambda_i X_{ti} + \lambda = 0$ for any t from 1 to t* and $\sum_{i=t*}^{T} \lambda_i X_{ti} + \lambda - \mu_t = 0$ for any t from t*+1 to T.

We first look at the equations for t and k larger than t*. First X_{tk} does not depend on t for t>k, and $X_{tk} < X_{tk+1}$ for t > k+1. So the two last equations $\sum_{i=t*}^{T} \lambda_i X_{T-1i} + \lambda - \mu_{T-1} = 0$ and $\sum_{i=t*}^{T} \lambda_i X_{Ti} + \lambda - \mu_T = 0$ lead to $\lambda_{T-1} X_{T-1T-1} + \lambda_T X_{T-1T} - \mu_{T-1} = \lambda_{T-1} X_{TT-1} + \lambda_T X_{TT} - \mu_T$, given $X_{T-1t} = X_{Tt}$ for t from t* to T-2.

We set $\mu_T = \mu_{T-1} = 0$ and we suppose $\lambda_T > 0$. Given $X_{T-1T-1} > X_{TT-1}$ and $X_{T-1T} < X_{TT}$, it follows $\lambda_{T-1} > 0$. We get $\lambda_{T-1} = \alpha_{T-1} \lambda_T$ with α_{t-1}>0

We now compare the adjacent equations for T-1 and T-2. We have $X_{T-2t} = X_{T-1t}$ for any t from t* to T-3 and we set $\mu_{T-2} = 0$. So we get: $\lambda_{T-2} X_{T-2T-2} + \lambda_{T-1} X_{T-2T-1} + \lambda_T X_{T-2T} = \lambda_{T-2} X_{T-1\,T-2} + \lambda_{T-1} X_{T-1T-1} + \lambda_T X_{T-1T}$.

$X_{T-1T-1} = X_{T-1T}$ and $X_{T-2T-2} = X_{T-2T-1} = X_{T-2T} < X_{T-1T-1} = X_{T-1T}$

imply $\lambda_{T-2} X_{T-2T-2} + (\lambda_{T-1} + \lambda_T) X_{T-2T-1} = \lambda_{T-2} X_{T-1T-2} + (\lambda_{T-1} + \lambda_T) X_{T-1T-1}$.

Given $X_{T-1\,T-2} < X_{T-2\,T-2}$ and $X_{T-2T-1} < X_{T-1T-1}$, the positivity of $(\lambda_{T-1} + \lambda_T)$ implies the positivity of λ_{T-2}. It follows $\lambda_{T-2} = \alpha_{T-2} \lambda_T$, with $\alpha_{T-2} > 0$. And so on, down to t*.

For t*-1 and t*, we have the equations: $\sum_{i=t*-1}^{T} \lambda_i X_{t*-1i} + \lambda = 0$ and $\sum_{i=t*}^{T} \lambda_i X_{t*i} + \lambda = 0$ i.e. $\lambda_{t*-1} X_{t*-1t*-1} + (\lambda_{t*} + \cdots + \lambda_T) X_{t*-1t*} + \lambda = 0$ and $(\lambda_{t*} + \cdots + \lambda_T) X_{t*t*} + \lambda = 0$. Given the positivity of λ_t for t from t* to T, and given $X_{t*-1t*} < X_{t*t*}$, we get $\lambda_{t*-1} > 0$, so $\lambda_{t*-1} = \alpha_{t*-1} \lambda_T > 0$, with $\alpha_{t*-1} > 0$.

We now turn to t*-2 and t*-1. The two adjacent equations become: $\lambda_{t*-2} X_{t*-2t*-2} + \lambda_{t*-1} X_{t*-2t*-1} + (\lambda_{t*} + \cdots + \lambda_T) X_{t*-2t*} + \lambda = 0$ and $\lambda_{t*-1} X_{t*-1t*-1} + (\lambda_{t*} + \cdots + \lambda_T) X_{t*-1t*} + \lambda = 0$
Given that $X_{t*-2t*-1}$ and X_{t*-2t*} are respectively lower than $X_{t*-1t*-1}$ and X_{t*-1t*}, it follows λ_{t*-2}>0, and $\lambda_{t*-2} = \alpha_{t*-2} \lambda_T$, with $\alpha_{t*-2} > 0$. Proceeding in the same way downwards ensures the positivity of each value λ_t down to t = 0, as soon as λ_T > 0.

The derivative of the KKT function in y leads to 1-$\sum_{i=0}^{T} \lambda_i = 0$, hence $\lambda_T = 1/(1 + \sum_{t=0}^{T-1} \alpha_t)$, which is positive, and the derivative in p_0 gives the value λ.

Moreover, given λ_t>0 for t from 0 to T, we get, due to the exclusion relations, that all the constraints have to be checked with equality, which is our starting assumption.

So, given the convexity of the problem, y=$\frac{p_0 M(1+r)}{r}$ will be the solution of the minimization program (global minimum), and the obtained values of p_t, t from 0 to T, will be the optimal probabilities. To get these probabilities, we equalize the regrets in the first t columns, t going from 0 to t*.

$X_{00} = \frac{M(1+r)}{r}$ so Regret(0) = $\frac{p_0 M(1+r)}{r} = y$
Regret(1) = $p_0 \left(D + \frac{M}{1+r} . \frac{1+r}{r}\right) + p_1 (\frac{M}{1+r} . \frac{1+r}{r})$

The equality of both regrets leads to $p_1 = \frac{r(M-D)}{M} p_O$

Regret(2) = $p_0\left(D + \frac{D}{1+r} + \frac{M}{(1+r)^2}.\frac{1+r}{r}\right) + p_1\left(\frac{D}{1+r} + \frac{M}{(1+r)^2}.\frac{1+r}{r}\right) + p_2\frac{M}{(1+r)^2}.\frac{1+r}{r}$

Regret(1) = Regret(2) $\Longrightarrow (p_0 + p_1)(M-D)r = p_2 M$

Given $Mp_1 = r(M-D)p_O$, the previous equation becomes: $p_2 = \frac{M+r(M-D)}{M}.p_1$

More generally, suppose $p_t = (\frac{M+r(M-D)}{M})^{t-1} p_1$ and $\sum_{i=0}^{t-1} p_i\,(M-D)r = p_t M$ for t<t*-1.

We get: Regret(t) = $\sum_{i=0}^{t-1} p_i\left(\frac{M(1+r)}{(1+r)^t r} + \sum_{j=i}^{t-1}\frac{D}{(1+r)^j}\right) + p_t\frac{M}{(1+r)^t}.\frac{1+r}{r}$

Regret(t+1) = $\sum_{i=0}^{t-1} p_i\left(\frac{M(1+r)}{(1+r)^{t+1} r} + \sum_{j=i}^{t-1}\frac{D}{(1+r)^j} + \frac{D}{(1+r)^t}\right) + p_t(\frac{D}{(1+r)^t} + \frac{M}{(1+r)^{t+1}}.\frac{1+r}{r}) + p_{t+1}\frac{M}{(1+r)^{t+1}}.\frac{1+r}{r}$

Regret(t) = Regret(t+1) $\Longrightarrow r\sum_{i=0}^{t} p_i\,(M-D) = p_{t+1}M$, so $p_t M + p_t(M-D)r = p_{t+1}M$ and $p_{t+1} = \frac{M+r(M-D)}{M}.p_t = (\frac{M+r(M-D)}{M})^t p_1$

Finally we get Regret(t*) = $-p_1 D - p_2\left(D + \frac{D}{1+r}\right) - \dots - p_{t*-1}\left(D + \frac{D}{1+r} + \dots . + \frac{D}{(1+r)^{t*-2}}\right) - p_{t*}\left(D + \frac{D}{1+r} + \dots . + \frac{D}{(1+r)^{t*-1}}\right)$

$= -D(p_1 + p_2 + \dots + p_{t*-1}) - \frac{D}{1+r}.(p_2 + \dots + p_{t*-1}) - \dots - \frac{D}{(1+r)^{t*-2}}.p_{t*-1} - p_{t*}\left(D + \frac{D}{1+r} + \dots . + \frac{D}{(1+r)^{t*-1}}\right)$

$= -\frac{Dp_1(1-B^{t*-1})}{(1-B)} - \frac{DBp_1(1-B^{t*-2})}{(1+r).(1-B)} - \frac{DB^{t*-2}p_1(1-B)}{(1+r)^{t*-2}.(1-B)} - p_{t*}D\left(\frac{1-\left(\frac{1}{1+r}\right)^{t*}}{1-\frac{1}{1+r}}\right)$ with $B = \frac{M+r(M-D)}{M}$

$= -\frac{Dp_1}{1-B}.((1-B^{t*-1}) + \frac{B(1-B^{t*-2})}{(1+r)} + \dots + \frac{B^{t*-2}(1-B)}{(1+r)^{t*-2}}) - p_{t*}D\left(\frac{1-\left(\frac{1}{1+r}\right)^{t*}}{1-\frac{1}{1+r}}\right)$

$= -\frac{Dp_1}{1-B}.(\frac{1-\left(\frac{B}{1+r}\right)^{t*-1}}{1-\frac{B}{1+r}} - B^{t*-1}.\frac{1-\left(\frac{1}{1+r}\right)^{t*-1}}{1-\frac{1}{1+r}}) - p_{t*}D\left(\frac{1-\left(\frac{1}{1+r}\right)^{t*}}{1-\frac{1}{1+r}}\right)$

We equalize Regret(t*) with Regret(0) and we get:

$\frac{p_0 M(1+r)}{r} = Dp_0(\frac{1-\left(\frac{B}{1+r}\right)^{t*-1}}{1-\frac{B}{1+r}} - B^{t*-1}.\frac{1-\left(\frac{1}{1+r}\right)^{t*-1}}{1-\frac{1}{1+r}}) - p_{t*}D\left(\frac{1-\left(\frac{1}{1+r}\right)^{t*}}{1-\frac{1}{1+r}}\right)$ because $-\frac{Dp_1}{1-B} = Dp_0$

We get $\frac{p_0 M(1+r)}{r}.(\left(\frac{B}{1+r}\right)^{t*-1} + \frac{D}{M}.B^{t*-1}\left(1-\left(\frac{1}{1+r}\right)^{t*-1}\right)) + p_{t*}D\left(\frac{1-\left(\frac{1}{1+r}\right)^{t*}}{1-\frac{1}{1+r}}\right) = 0$

It follows $-p_{t*}D\left(1-\left(\frac{1}{1+r}\right)^{t*}\right) = p_0 B^{t*-1}\left((M-D)\left(\frac{1}{1+r}\right)^{t*-1} + D\right)$

And we get $p_{t*} = \frac{p_0 B^{t*-1}\left((M-D)\left(\frac{1}{1+r}\right)^{t*-1} + D\right)}{-D\left(1-\left(\frac{1}{1+r}\right)^{t*}\right)}$

which is positive because $D + (M-D)\left(\frac{1}{1+r}\right)^{t*-1} = \frac{\left(D + \frac{D}{1+r} + \dots . + \frac{D}{(1+r)^{t*-2}} + \frac{M}{(1+r)^{t*-1}}.\frac{1+r}{r}\right)r}{r+1}$ = a multiple of the positive payoff a firm gets by leaving in period t* when the opponent leaves in period t*-1. This payoff, given the definition of t*, is usually quite small.

Summing to 1 the probabilities leads to:

$$p_0 + p_1(1 + B + \cdots + B^{t*-2}) + \frac{p_0 B^{t*-1}\left((M-D)\left(\frac{1}{1+r}\right)^{t*-1} + D\right)}{-D\left(1-\left(\frac{1}{1+r}\right)^{t*}\right)} = 1$$

which leads to $p_0 = -D.\frac{(1+r)^{t*}-1}{(M(1+r)-Dr)\left(\frac{M+r(M-D)}{M}\right)^{t*-1}} = -(\frac{D}{M}).\frac{(1+r)^{t*}-1}{(\frac{M+r(M-D)}{M})^{t*}}$

All the other probabilities follow.

Study with t*>T

Nothing changes as regards the equations, by replacing t* by T. The only important change is that $D + (M-D)\left(\frac{1}{1+r}\right)^{T-1}$, which is equal to $\frac{\left(D+\frac{D}{1+r}+\cdots+\frac{D}{(1+r)^{T-2}}+\frac{M}{(1+r)^{T-1}}.\frac{1+r}{r}\right)r}{r+1}$, is not necessarily small. So p_T may be large.

Appendix F

Study with t*≤T

For the same reasons than in appendix E, to get the minimax regret density function f(t), we equalize the regrets in the columns from 0 to t*, where t* = $\ln(-\frac{M-D}{D})/r$.

We look for a density function f(t) with a mass point on 0, g(0) being the probability assigned to 0, and we get $g(0) + \int_{0+}^{t*} f(t) = 1$.

F(t) is the cumulative distribution function, with F(0)= g(0) and F(t*)=1.

In the continuous-time game, the regrets in column t* can be calculated in two ways: either by comparing the payoffs with the null payoff obtained by leaving the market at time t*, $\int_0^{t*} De^{-rt}dt + \int_{t*}^{\infty} Me^{-rt}dt = 0$, or by comparing the payoffs with the null payoff obtained by leaving at time 0.

We equalize the regrets and we get:

Regret(0) = g(0) $\int_0^{\infty} Me^{-rt}dt = g(0)M/r$

By exiting the market at time s (<t*) when the opponent exits at time d, with d>s, firm 1 gets $\int_0^s De^{-rt}dt$, whereas she could obtain $\int_0^d De^{-rt}dt + \int_d^{\infty} Me^{-rt}dt$, so her regret is $\int_s^d De^{-rt}dt + \int_d^{\infty} Me^{-rt}dt$

We get: Regret(d) = g(0)($\int_0^d De^{-rt}dt + \int_d^{\infty} Me^{-rt}dt) + \int_{0+}^{d}(\int_s^d De^{-rt}dt + \int_d^{\infty} Me^{-rt}dt)f(s)ds$

Regret(d) is constant over [0, t*], so we need Regret'(d)=0, for d from 0+ to t*. We get:

$g(0)\left(De^{-rd} - Me^{-rd}\right) + (\int_d^{\infty} Me^{-rt}dt)f(d) + \int_{0+}^{d}(\ De^{-rd} - Me^{-rd})f(s)ds$

= $g(0)(De^{-rd} - Me^{-rd}) + \frac{Me^{-rd}}{r}f(d) + (F(d) - g(0))(De^{-rd} - Me^{-rd})$

= $\frac{Me^{-rd}}{r}f(d) + F(d)(De^{-rd} - Me^{-rd})$.

We need $\frac{M}{r}f(d) + F(d)\ (D-M) = 0$, with F(0)=g(0) and F(t*)=1.

This differential equation leads to $F(t) = g(0)e^{\frac{(M-D)rt}{M}}$, $f(t) = \frac{g(0)(M-D)r}{M}e^{\frac{(M-D)rt}{M}}$ and

$g(0) = (-\frac{D}{M-D})^{\frac{M-D}{M}}$

We check that Regret(0) = Regret(d) for d from 0+ to t*

Regret(0) = g(0)M/r

Regret(d) = g(0)($\int_0^d De^{-rt}dt + \int_d^\infty Me^{-rt}dt$) + $\int_{0+}^d (\int_s^d De^{-rt}dt + \int_d^\infty Me^{-rt}dt) f(s)ds$

$=\frac{g(0)(D+(M-D)e^{-rd})}{r} + g(0)\int_0^d (\frac{M-D}{r}e^{-rd} + \frac{D}{r}e^{-rs}).\frac{M-D}{M}re^{\frac{(M-D)rs}{M}}ds = \frac{g(0)M}{r}$

after development.

Study with t*>T

When t*>T, the distribution has two mass points 0 and T, played with probability g(0) and g(T), and F(T) = $F(T-) + g(T)$.

We have: Regret(T) = $\int_{0+}^{T-} (\int_0^s -De^{-rt}dt) f(s)ds + g(T)\int_0^T -De^{-rt}dt$.

Regret(s) is constant over [0, T], so we get again the differential equation $\frac{M}{r}f(d) + F(d)(D-M) = 0$, with F(0)=g(0) and F(T)=1, which leads to $F(t) = g(0)e^{\frac{(M-D)rt}{M}}$ hence $f(t) = \frac{g(0)(M-D)r}{M}e^{\frac{(M-D)rt}{M}}$.

Given F(T) = F(T-)+g(T), we get g(0) and g(T) by requiring Regret(0) = Regret(T). We get:

$\frac{g(0)M}{r} = \int_{0+}^{T-} (\int_0^s -De^{-rt}dt) f(s)ds + g(T)\int_0^T -De^{-rt}dt$

$\int_{0+}^{T-} (\int_0^s -De^{-rt}dt) f(s)ds + g(T)\int_0^T -De^{-rt}dt = \int_{0+}^{T-} \frac{(De^{-rs}-D)}{r} f(s)ds + \frac{g(T)(De^{-rT}-D)}{r}$

$= -\frac{D}{r}\left(1 - g(0) - g(T)\right) + \frac{D}{r}\int_{0+}^{T-} \frac{g(0)(M-D)r}{M}e^{\frac{-Drs}{M}}ds + \frac{g(T)(De^{-rT}-D)}{r}$

$= -\frac{D}{r}\left(1 - g(0)\right) + \frac{g(0)D(M-D)}{M}\int_{0+}^{T-} e^{\frac{-Drs}{M}}ds + \frac{g(T)(De^{-rT})}{r}$

$= -\frac{D}{r} + \frac{g(0)M}{r} - \frac{g(0)(M-D)e^{-\frac{DrT}{M}}}{r} + \frac{g(T)(De^{-rT})}{r}$

$\frac{g(0)M}{r} = -\frac{D}{r} + \frac{g(0)M}{r} - \frac{g(0)(M-D)e^{-\frac{DrT}{M}}}{r} + \frac{g(T)(De^{-rT})}{r} \Longrightarrow$

$g(T) = e^{rT} - \frac{g(0)(M-D)}{-D}e^{\frac{(M-D)rT}{M}}$.

F(T-)+g(T) = 1 hence $g(0)e^{\frac{(M-D)rT}{M}} + e^{rT} - \frac{g(0)(M-D)}{-D}e^{\frac{(M-D)rT}{M}} = 1$.

We get g(0)=$(e^{rT}-1)(\frac{-D}{M})e^{-(M-D)rT/M}$ > 0 and $g(T) = 1 + \frac{e^{rT}D}{M} - \frac{D}{M}$.

g(T) is decreasing in T and it is easily checked that for 0< T< t* we get 1> g(T) >0, and that for T=t*=$\frac{\ln\left(\frac{M-D}{-D}\right)}{r}$ we get g(T)=0 and g(0)= $(-\frac{D}{M-D})^{\frac{M-D}{M}}$.

g(0) is increasing in T:

$g'_T(0) = re^{rT}\left(\frac{-D}{M}\right)e^{-\frac{(M-D)rT}{M}} + (e^{rT}-1)(\frac{-D}{M})(-\frac{(M-D)r}{M})e^{-(M-D)rT/M}$

$= \frac{re^{rT}\left(\frac{-D}{M}\right)e^{-\frac{(M-D)rT}{M}}D}{M} + \frac{r\left(\frac{-D}{M}\right)(M-D)}{M}e^{-\frac{(M-D)rT}{M}} = \frac{r\left(\frac{-D}{M}\right)}{M}e^{-\frac{(M-D)rT}{M}}(De^{rT} + M - D) > 0$ because T<t*.

Given that g(0) is equal to $(-\frac{D}{M-D})^{\frac{M-D}{M}}$ for T = t*, g(0) is always lower than the probability assigned to 0 in the model with T > t*. Hence, when firms are constrained to leave before t*, they less often leave at time 0.

When T goes to 0, g(0) goes to 0 and g(T) goes to 1. So, when T becomes small, firms are more and more incited to exit the market as late as possible (as does the pure minimax regret strategy).

Appendix G

Game in discrete time

$E(g) = \sum_{t=0}^{\min(t*,T)}(best - reply\ payoff\ to\ t)p_t - \sum_{t=0}^{\min(t*,T)}(p_0\mathrm{M}(1+\mathrm{r})/\mathrm{r})p_t = \frac{p_0\mathrm{M}(1+\mathrm{r})}{\mathrm{r}} + \sum_{t=1}^{\min(t*,T)}(best - reply\ payoff\ to\ t)p_t - \frac{p_0\mathrm{M}(1+\mathrm{r})}{\mathrm{r}} = \sum_{t=1}^{\min(t*,T)}(best - reply\ payoff\ to\ t)p_t > 0$ (because the best-reply payoff is larger than 0 for t <min (t*,T) and equal to 0 for t=min (t*, T)).

Game in continuous time with t*≤T

E(g)=$\left(\frac{M}{r} - \frac{M}{r}g(0)\right)g(0) + \int_{0+}^{t*}(\int_0^t De^{-rs}ds + \int_t^\infty Me^{-rs}ds - \frac{g(0)M}{r})f(t)dt = \frac{M}{r}g(0) + \int_{0+}^{t*}(\int_0^t De^{-rs}ds + \int_t^\infty Me^{-rs}ds)f(t)dt - \frac{M}{r}g(0) = \int_{0+}^{t*}(\int_0^t De^{-rs}ds + \int_t^\infty Me^{-rs}ds)f(t)dt > 0$

$g(0) = \left(-\frac{D}{M-D}\right)^{\frac{M-D}{M}} = \left(\frac{x}{1+x}\right)^{1+x}$ and t* = $\frac{\ln\left(\frac{M-D}{-D}\right)}{r} = \frac{\ln\left(\frac{1+x}{x}\right)}{r}$ with x=-D/M

Developing $\int_{0+}^{t*}(\int_0^t De^{-rs}ds + \int_t^\infty Me^{-rs}ds)f(t)dt > 0$ leads to

E(g)= $\int_{0+}^{t*}\frac{D+(M-D)e^{-rt}}{r}f(s)ds = g(0)(\frac{D}{r}\left(e^{\frac{(M-D)rt*}{M}} - 1\right) + \frac{(M-D)^2}{Dr}\left(1 - e^{-\frac{Drt*}{M}}\right))$

$= g(0)(\frac{(M-2D)}{D}.\frac{M}{r} + \frac{D^2e^{\frac{(M-D)rt*}{M}} - (M-D)^2e^{-\frac{Drt*}{M}}}{Dr})$

$= g(0)(\frac{(M-2D)}{D}.\frac{M}{r} + \frac{D^2(\frac{M-D}{-D})^{\frac{(M-D)}{M}} - (M-D)^2(\frac{M-D}{-D})^{-\frac{D}{M}}}{Dr})$

$= \frac{\mathrm{M}}{\mathrm{r}}\left(1 - \frac{\mathrm{g(0)(M-2D)}}{\mathrm{-D}}\right) = \frac{M}{r}(1 - \left(\frac{x}{1+x}\right)^x - \left(\frac{x}{1+x}\right)^{1+x})$ with x = -D/M

Because $\int_0^t De^{-rs}ds + \int_t^\infty Me^{-rs}ds$ is decreasing in t, we calculate t**, the last opponent's exit time that leads to a positive payoff i.e.: $\int_0^{t**} De^{-rs}ds + \int_{t**}^\infty Me^{-rs}ds = \frac{g(0)M}{r}$.

We get $(M - D)e^{-rt**} + D = g(0)M$ i.e. $e^{-rt**} = \frac{g(0)}{1-\frac{D}{M}} - \frac{\frac{D}{M}}{1-\frac{D}{M}}$

It follows: t**=$-\ln(\frac{\left(\frac{x}{1+x}\right)^{1+x}}{1+x} + \frac{x}{1+x})/r$ with x = -D/M

Given that the opponent leaves before t** with probability F(t**), a firm gets a positive payoff with probability F(t**) = $g(0)e^{(1+x)rt**} = g(0)\left(\frac{g(0)}{1+x} + \frac{x}{1+x}\right)^{-(1+x)} = \left(\frac{x}{1+x}\right)^{1+x}\left(\frac{\left(\frac{x}{1+x}\right)^{1+x}}{1+x} + \frac{x}{1+x}\right)^{-(1+x)} = \left(\frac{\left(\frac{x}{1+x}\right)^{x}}{1+x} + 1\right)^{-(1+x)}$

We now study g(0), t*, t**, F(t**) and E(g).

g(0) = 0 for x = 0 and g(x) → e^{-1} = 0.368 when x → +∞ because $\left(\frac{x}{1+x}\right)^{1+x} = e^{(1+x)\ln(1-\frac{1}{1+x})}$, which tends towards e^{-1}. The same result holds when –D/M=1/r and r → 0.

$t* = \frac{\ln\left(\frac{1+x}{x}\right)}{r} \to +\infty$ when r → 0 and/or x → 0. t*→ 0 when x → +∞ for a given r because t*→ 1/(xr), but t* → 1 if x = 1/r and r → 0.
t** → +∞ when r → 0 and/or x → 0 . t** → 0 when x → +∞ for a given r, but t** → 1-e^{-1} = 0.6321 when x = 1/r and r → 0.

As a matter of fact $-\ln(\left(\frac{x}{1+x}\right)\left(\frac{\left(\frac{x}{1+x}\right)^x}{1+x}+1\right))/r \;= -\frac{\ln\left(\frac{x}{1+x}\right)+\ln\left(1+\frac{\left(\frac{x}{1+x}\right)^x}{1+x}\right)}{r} \to -\frac{-\frac{1}{1+x}+\frac{e^{-1}}{1+x}}{r} =$ $\frac{-1}{1+r}(-1+e^{-1}) = 1-e^{-1}$
F(t**) → 0.5 when x → 0 because:
$\left(\frac{x}{1+x}\right)^{1+x}\left(\frac{\left(\frac{x}{1+x}\right)^{1+x}}{1+x}+\frac{x}{1+x}\right)^{-(1+x)} \to \frac{x}{1+x}\left(\frac{\frac{x}{1+x}}{1+x}+\frac{x}{1+x}\right)^{-1} \to \left(\frac{1}{1+x}+1\right)^{-1} \to 0.5$

F(t**) → $e^{-e^{-1}} = 0.692$ when x →+∞ or x = 1/r and r → 0, because $\left(\frac{\left(\frac{x}{1+x}\right)^x}{1+x}+1\right)^{-(1+x)}$ $= e^{-(1+x)\ln\left(\frac{\left(\frac{x}{1+x}\right)^x}{1+x}+1\right)} = e^{-(1+x)\ln\left(\frac{e^{x\ln(\frac{x}{1+x})}}{1+x}+1\right)} \longrightarrow e^{-(1+x)\ln\left(\frac{e^{-x/(1+x)}}{1+x}+1\right)} \longrightarrow e^{-(1+x)\frac{e^{-x/(1+x)}}{1+x}} \to e^{-e^{-1}}$.
E(g) is increasing in M and decreasing in r for a fixed value -D/M. E(g) → 0 when x → 0, E(g) → (1-2e^{-1})M/r when x → +∞, E(g)→ +∞ for x = 1/r and r → 0. This results follow from: E(g) = $\frac{M}{r}\left(1-\left(\frac{x}{1+x}\right)^x-\left(\frac{x}{1+x}\right)^{1+x}\right) = \frac{M}{r}\left(1-e^{x\ln\left(\frac{x}{1+x}\right)}-e^{(1+x)\ln(\frac{x}{1+x})}\right) \to$ $\frac{M}{r}\left(1-e^{x(-\frac{1}{1+x})}-e^{(1+x)(\frac{-1}{1+x})}\right) \to$ (1-2e^{-1})M/r when x → +∞.

Game in continuous time with t*>T

E(g) = $\left(\frac{M}{r}-\frac{M}{r}g(0)\right)g(0)+\int_{0+}^{T-}(\int_0^t De^{-rs}ds+\int_t^{\infty}Me^{-rs}ds-\frac{g(0)M}{r})f(t)dt+g(T)(0-g(0)M/r) = \frac{M}{r}g(0)+\int_{0+}^{T-}(\int_0^t De^{-rs}ds+\int_t^{\infty}Me^{-rs}ds)f(t)dt-\frac{M}{r}g(0) =$

$\int_{0+}^{T-}(\int_0^t De^{-rs}ds+\int_t^{\infty}Me^{-rs}ds)f(t)dt > 0$

Developing this expression leads to:
E(g) = $g(0)(\frac{D}{r}\left(e^{\frac{(M-D)rT}{M}}-1\right)+\frac{(M-D)^2}{Dr}\left(1-e^{-\frac{DrT}{M}}\right))$
$= \frac{g(0)(M-2D)}{D}.\frac{M}{r}+\frac{g(0)}{Dr}(D^2e^{\frac{(M-D)rT}{M}}-(M-D)^2e^{-\frac{DrT}{M}}) = \frac{M}{r}\left(1-\frac{g(0)(M-2D)}{-D}\right)$ for T=t*

because $\frac{g(0)}{Dr}(D^2e^{\frac{(M-D)rT}{M}}-(M-D)^2e^{-\frac{DrT}{M}}) = \frac{M}{r}$ for T=t* (the calculi are the same than that for T>t*).

We show that E(g) is increasing in T over]0+, t*]. So the firm's expected payoff, when it is constrained to exit before t*, is lower than the expected payoff obtained when the firm is free to leave at a time later than t* (or at anytime).
We have E'(g)=
$\frac{g\prime(0)(M-2D)}{D}.\frac{M}{r}+\frac{g'^{(0)}}{Dr}\left(D^2e^{\frac{(M-D)rT}{M}}-(M-D)^2e^{-\frac{DrT}{M}}\right)+\frac{g(0)}{Dr}(\frac{D^2(M-D)r}{M}e^{\frac{(M-D)rT}{M}}-\frac{(M-D)^2(-Dr)}{M}e^{-\frac{DrT}{M}})$.

$\frac{g\prime(0)(M-2D)}{D}\cdot\frac{M}{r}+\frac{g'(0)}{Dr}\left(D^2e^{\frac{(M-D)rT}{M}}-(M-D)^2e^{-\frac{DrT}{M}}\right)>0$ because g’(0)>0 and E(g)>0. So it remains to show that $\frac{g(0)}{Dr}(\frac{D^2(M-D)r}{M}e^{\frac{(M-D)rT}{M}}-\frac{(M-D)^2(-Dr)}{M}e^{-\frac{DrT}{M}})$ is larger than 0.

We have $\frac{g(0)}{Dr}\left(\frac{D^2(M-D)r}{M}e^{\frac{(M-D)rT}{M}}-\frac{(M-D)^2(-Dr)}{M}e^{-\frac{DrT}{M}}\right)=\frac{g(0)}{Dr}\left(\frac{Dr}{M}\right)(M-D)e^{-\frac{DrT}{M}}(De^{rT}+M-D)>0$ because T<t*.