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Approval voting and Shapley ranking

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Abstract

Approval voting allows electors to list any number of candidates and their scores are obtained by summing the votes cast in their favor. Equal-and-even cumulative voting instead follows the *One-person-one-vote* principle by endowing electors with a single vote that they may evenly distribute among several candidates. It corresponds to satisfaction approval voting introduced by Brams and Kilgour (2014) as an extension of approval voting to a multiwinner election. It also corresponds to the concept of Shapley ranking introduced by Ginsburgh and Zang (2012) as the Shapley value of a cooperative game with transferable utility. In the present paper, we provide an axiomatic foundation of Shapley ranking and analyze the properties of the resulting social welfare function.

Keywords: approval voting, equal-and-even cumulative voting, ranking game, Shapley value
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> "Which candidate ought to be elected in a single-member constituency *if all we take into account* is the order in which each of the electors ranks the candidates? ... At the very outset of the argument, we try to move from the *is* to the *ought* and to jump the unbridgeable chasm between the universe of science and that of morals." (Duncan Black, 1958, p. 55)

Introduction

Approval voting is a method that was formally studied in the 1970's by Weber (1977) and Brams and Fishburn (1978).[1] Given a set of candidates, electors have the possibility to list any number of candidates whom they consider as "good for the job." The method simply consists in assigning to each candidate a score equal to the number of electors who have listed that candidate.[2] The winners are those with the largest scores. Beyond being a voting method, rational collective preferences are derived from approval voting like in Borda count and other scoring methods.

Approval voting has its supporters, starting with Brams and Fishburn, but also its opponents like Saari and van Newenhizen (1988).[3] Approval voting has actually not been much implemented except in some scientific societies such as the *American Mathematical Society* or the *Institute of Electrical and Electronics Engineers*. There are a few other exceptions such as the election of the UN Secretary General.[4] Several experiments have been conducted, in particular by Baujard and Igersheim (2010) following the 2002 and 2007 French presidential elections.

Under approval voting, there is a priori no limit to the number of candidates an elector is allowed to list and there is no direct "cost" in listing additional candidates. If an elector adds a candidate to her ballot, it has no impact on the scores of the other candidates. Furthermore, electors who list several candidates carry more weight. In this sense, approval voting violates the *One-person-one-vote principle* often emphasized by the advocates of political equality. This principle is satisfied by *equal-and-even cumulative voting* (also called *block approval voting*) where an elector's vote is divided evenly among the candidates she retains. For instance, if an elector retains three candidates, each one gets 1/3 instead of 1. Hence, an elector's vote weights less the larger the number of candidates in her ballot and electors have an incentive to limit the size of their ballots: Adding a candidate reduces the chance that those already present will be elected. If the objective of an elector is to see elected one of the candidates that she places high in her preferences, she will tend to submit a limited number of candidates among which she is relatively indifferent. Cumulative voting is typically used in multiplewinner elections where

[1] See also Brams and Fishburn (1983/2007, 2005), Weber (1995), Brams (2008) and the *Handbook of approval voting* edited by Laslier and Remzi Sanver (2010).
[2] Convention: we use "she" for voters and "he" for candidates.
[3] See the ensuing discussion in the issue of *Public Choice* where their paper was published.
[4] See Brams and Fishburn (2005). Recently, the electoral system in the city of Fargo (North Dakota) was changed from plurality voting to approval voting. See www.electionscience.org

electors can spread a fixed number of votes – usually equal to the number of seats to be filled – over one or more candidates.[5]

In what follows, we make a distinction between voting, ranking and ordering. Voting is the procedure by which electors submit ballots. Ranking aggregates the electors' choices by assigning a score to each candidate.[6] And a ranking leads to an ordering which is the ordinal relation on the set of candidates induced by the ranking.

Electors have preferences over candidates. An elector whose preferences are complete orders the candidates from the most preferred to the least preferred and draws a line somewhere to partition the set of candidates into two sublists, *as if* her preferences were dichotomous. We only observe that the candidates above the line are strictly preferred to those below it and two electors with identical preferences may well draw the line at different levels. If an elector has incomplete preferences, the candidates whom she cannot rank would just not be listed.[7] Even if approval voting is compatible with incomplete preferences, we keep the assumption that preferences are complete, leaving open the possibility of dichotomous preferences where electors are indifferent within and outside their ballots. Under the assumption of dichotomous preferences, Mongin and Maniquet (2015) show that approval voting induces a non-dictatorial social welfare function that satisfies the Pareto criterion and Independence of irrelevant alternatives. This result does not contradict Arrow's impossibility theorem because the domain of individual preferences is restricted.[8]

The assumption of dichotomous preferences is however far too strong. The candidates listed by an elector are in some sense relatively "close" to each other, but assuming indifference within and outside ballots is not plausible. This is even less plausible for the candidates that an elector does not list. In the present paper, we only assume that electors strictly prefer the candidates they list to those they do not list, an assumption that is an integral part of the definition of a ballot. Notice that, under dichotomous preferences, approval voting is equivalent to the Borda method that, for each elector, gives 1 to the candidates she has listed and 0 to the others. The absence of information on electors' preferences is a fundamental difficulty when electors are required to name candidates without ordering them. It is so for approval voting as well as for any other method that limits the number of candidates an elector can list, including plurality voting.

Equal-and-even cumulative voting corresponds to the concept of *Shapley ranking* defined by Ginsburgh and Zang (2012) as the Shapley value of a transferable utility game derived from

[5] For an overall analysis of multiwinner elections based on approval balloting, see Brams et al. (2018).
[6] Ranking is cardinal and, in some contexts such as wine competitions, rankings matter.
[7] Alcantud and Laruelle (2014) study and characterize a voting rule that allows voters to divide candidates into three classes, approved, disapproved and indifferent, thereby allowing for incomplete preferences.
[8] The reference is Arrow 1951's famous book. The 1963 edition reproduces the first edition and adds a chapter reviewing the developments in social choice theory since 1951.

approval ballots. It also corresponds to the concept of *satisfaction approval voting* introduced by Brams and Kilgour (2014). Even if they limit this aggregation method to multiwinner elections, nothing actually prevents from applying it to single-winner elections. In the present paper, we provide an axiomatic foundation to equal-and-even cumulative voting based on the One-person-one-vote principle. We then move from ranking to ordering and look at the properties of the induced social welfare function.

The paper is organized as follows. Approval and equal-and-even cumulative voting are introduced in Section 1 using the concept of ballot profile that specifies, for each subset of candidates, the number of electors who support it. Ranking games associated to ballot profiles are introduced in Section 2. Their Shapley values are shown to coincide with the ranking derived from equal-and-even cumulative voting. The resulting "Shapley ranking" is then axiomatized in terms of ballot profiles by using Shapley's axioms where Efficiency is translated into the One-man-one-vote principle. Section 3 looks at the properties of the orderings derived from approval and Shapley rankings, in a social choice perspective. The last section is devoted to concluding remarks.

1. Approval, fractional, plurality and majority voting

Consider a set N of n candidates with $n \geq 2.$[9] There are at least two electors. They have preferences over candidates: $i \succ_h j$ reads "elector h prefers candidate i to candidate j" and $i \sim_h j$ reads "elector h is indifferent between candidates i and j." The weak preference relation $\succsim_h$ represents the preferences of elector h. Preferences are assumed to be complete and rational: $\succsim_h$ is a complete, transitive and reflexive binary relation (a complete preorder) over N. A preference profile specifies a preference ordering for each elector.

1.1 Approval voting

Under approval voting, electors are asked to list the candidates they approve. We denote by $N_h \subset N$ the *approval set* or *ballot* of elector h, the set of candidates submitted by h. We assume that $N_h \neq \varnothing$ for all h but do not exclude $N_h = N$: $0 < n_h \leq n.$ The choice of elector h can be identified to an n-tuple $q_h \in \{0,1\}^n$ with $q_{ih} = 1$ if and only if $i \in N_h.$ If M denotes the set of electors, a ballot profile can be arranged in a $n \times m$ matrix whose rows are attributed to candidates and columns to electors. Alternatively, a ballot profile can be written as a mapping π that associates to each subset of candidates, the number of electors whose approval set *coincides* with that subset: $\pi(S) = \left|\{h \in M \mid N_h = S\}\right|.$ There is a one-to-one relation between ballot profiles and the matrix representation, the number of electors being obtained by summing the $\pi(S).$ The set of all ballot profiles on a set N of candidates is given by:

$$\Pi(N) = \left\{\pi \in \mathbb{N}^{2^n - 1} \mid \pi \neq (0,...,0), 0 \leq \pi(S) \leq n \text{ for all } S \subset N, S \neq \varnothing \right\}$$

[9] *Notation*: The cardinality of a finite set A is denoted $|A|$. Upper-case letters are used to denote finite sets and subsets, and the corresponding lower-case letters are used to denote the number of their elements: $n = |N|$, $s = |S|$,...

where $\mathbb{N} = \{0,1,2,\dots\}$ denotes the set of nonnegative integers. Notice that the profile (0,0,...,0) has been excluded because we only retain the electors that have submitted a nonempty ballot.

Example 1 Consider the 4-candidate situation described by the following ballot profile: $\pi(1) = \pi(1,2) = \pi(2,3) = \pi(2,3,4) = 1$, $\pi(3,4) = 2$, and $\pi(S) = 0$ for all other subset S.[10] Hence, there are 6 electors and their ballots are $N_1 = \{1\}$, $N_2 = \{1,2\}$, $N_3 = \{2,3\}$, $N_4 = N_5 = \{3,4\}$ and $N_6 = \{2,3,4\}$. The associated matrix is given by:

$$Q = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 1 & 0 & 0 & 1 \\ \hline 0 & 0 & 1 & 1 & 1 & 1 \\ \hline 0 & 0 & 0 & 1 & 1 & 1 \\ \hline \end{array}$$

The *approval score* of candidate *i* is the number of electors who have listed that candidate:

$$AR_i(N,\pi) = \sum_{S:i\in S} \pi(S) \quad (i = 1,\dots, n). \tag{1}$$

It is obtained by summing along each row of the representative matrix *Q*. In Example 1, the approval scores are given by (2, 3, 4, 3). It leads to the following ordering: $3 \succ 2 \sim 4 \succ 1$.

1.2 Equal-and-even cumulative voting

Under equal-and-even cumulative voting, each of the candidates listed by elector *h* receives a fraction $1/n_h$ where n_h is the size of elector *h*'s ballot. The scores are obtained by summing the fractions allocated to each candidate:

$$SR_i(N,\pi) = \sum_{S:i\in S} \frac{1}{s} \pi(S) \quad (i = 1,\dots, n). \tag{2}$$

This coincides with the concept of *Shapley ranking* introduced by Ginsburgh and Zang (2012) and with the concept of *satisfaction approval voting* introduced by Brams and Kilgour (2014).

	1	**2**	**3**	**4**	**5**	6	*AR*	*SR*
1	1	1	0	0	0	0	2	1.50
2	0	1	1	0	0	1	3	1.33
3	0	0	1	1	1	1	4	1.83
4	0	0	0	1	1	1	3	1.33

Table 1: The scores in Example 1

Table 1 shows the scores obtained in Example 1. The ordering resulting from Shapley ranking is $3 \succ 1 \succ 2 \succ 4$. It places candidate 1 on top. Not surprisingly, it differs from the ordering $3 \succ 2 \sim 4 \succ 1$ that results from approval ranking. Both orderings obviously coincide in the case

[10] Braces are omitted in the absence of ambiguity.

of two candidates. Example 1 shows that they may not coincide beyond two candidates. The following example illustrates a situation where the two orderings coincide.

Example 2 Consider a 3-candidate situation involving five electors whose approval sets are $N_1 = \{1\}$, $N_2 = \{1,2\}$, $N_3 = \{2,3\}$, $N_4 = \{1,3\}$ and $N_5 = \{1,2,3\}$. The scores are given in Table 2. In both cases, candidate 1 comes first while the other two obtain the same score: $1 \succ 2 \sim 3$.

	1	**2**	**3**	**4**	**5**	*AR*	*SR*
1	1	1	0	1	1	4	2.33
2	0	1	1	0	1	3	1.33
3	0	0	1	1	1	3	1.33

Table 2: The scores in Example 2

Notice that by normalizing the scores given by (2), we obtain the probabilities that a particular candidate is elected under the *random dictatorship* procedure.[11] Each of the *m* electors is asked to identify a subset of candidates, knowing that an elector will first be chosen at random and that the winner will be chosen at random in her approval set. The resulting probabilities are then proportional to the scores:

$$\text{Prob}[\, i \text{ is elected}\,] = \frac{1}{m} \sum_{h:\, i \in N_h} \frac{1}{n_h} = \frac{1}{m} SR_i(N, \pi).$$

In Example 1 (see Table 1), the probabilities are given by (0.25, 0.22, 0.30, 0.22). In Example 3 (see Table 2), they are given by (0.47, 0.27, 0.27).

1.3 Plurality and majority voting

A number of voting rules in which electors are allowed to submit only one candidate are special cases of approval voting. This is the case of plurality and majority voting. These methods are well defined only in the absence of indifference in individual preferences. Each elector has then a unique most preferred candidate and candidates are ordered according to their approval scores given by (1) or equivalently by (2). In plurality voting, the winners are the candidates with the largest approval score. In majority voting, the winner is the candidate with an approval score exceeding half the number of electors. The latter is therefore not a decisive method. The following example shows that a candidate who appears first in a majority of electors' preferences may be defeated under approval voting and equal-and-even cumulative voting. It illustrates how voting by approval reveals some information on electors' intensities of preferences.

Example 3 Consider a situation with four candidates and five electors whose preferences are given by $1 \succ_1 3 \succ_1 2 \succ_1 4$, $1 \succ_2 2 \succ_2 3 \succ_2 4$, $1 \succ_3 2 \succ_3 4 \succ_3 3$, $2 \succ_4 3 \succ_4 4 \succ_4 1$ and $2 \succ_5 4 \succ_5 1 \succ_5 3$.

[11] This is the terminology used by Bogolmania et al. (2005).

The first candidate has a majority: he comes on top of 3 out of 5 orderings.[12] Now assume that the electors submit the following ballots: $N_1 = \{1,3\}$, $N_2 = \{1,2\}$, $N_3 = \{1,2,4\}$, $N_4 = \{2,3\}$ and $N_5 = \{2,4\}$. Table 3 shows the resulting scores. Candidate 2 gets the largest score in both cases.

	1	**2**	**3**	**4**	**5**	*AR*	*SR*
1	1	1	1	0	0	3	1.33
2	0	1	1	1	1	4	1.83
3	1	0	0	1	0	2	1
4	0	0	1	0	1	2	0.83

Table 3: The scores in Example 3

When some electors are indifferent between candidates, plurality and majority voting are not well defined because some electors may have several most preferred candidates. If this is the case for an elector, she has to make a selection and we could assume that the candidate that she submits is drawn at random among her top candidates. Denoting by N_h the subset of most preferred candidates of elector *h*, each one is assigned a probability equal to $1/n_h$ and the score of a candidate is then given by the sum of the probabilities that his name be submitted. In this case, plurality voting and equal-and-even cumulative voting give rise to the same result.

2. Shapley ranking

2.1 Ranking games

We first recall the definition of a ranking games introduced by Ginsburgh and Zang (2012). In our voting context, the players are the candidates. Given a ballot profile (N,π), we define the *transferable utility game* (N,w) whose characteristic function associates to each subset *S* of candidates, the number of electors whose approval set is *included* in *S*:

$$w(S) = \sum_{T \subset S} \pi(T) \text{ for all } S \subset N. \tag{3}$$

$w(S)$ is the number of electors who are *exclusively* supporting some candidates in *S*. In particular, $w(i)$ is the number of electors who have only listed candidate *i* and $w(N)$ is the total number of electors. A solution of a ranking game provides a ranking of candidates by specifying for each of them a score equal to a fraction of the total number of electors. The characteristic function associated to Example 1 is given by:

$w(1)=1,\ w(2)=0,\ w(3)=0,\ w(4)=0,$

$w(1,2)=2,\ w(1,3)=1,\ w(1,4)=1,\ w(2,3)=1,\ w(2,4)=0,\ w(3,4)=1,$

$w(1,2,3)=3,\ w(1,2,4)=2,\ w(1,3,4)=2,\ w(2,3,4)=3,$

$w(1,2,3,4)=5.$

[12] That candidate is therefore also the unique Condorcet winner (see Section 3).

We denote by $G(N)$ the set of all characteristic functions on a given set N of players. It can be identified to the vector space $\mathbb{R}^{2^n-1}$. In proving the uniqueness of his value, Shapley (1953) shows that the $2^n - 1$ *unanimity games* defined for all $T \subset N$ by

$$u_T(S) = \begin{cases} 1 & \text{if } T \subset S \\ 0 & \text{otherwise.} \end{cases}$$

form a basis of the vector space $G(N)$: For any characteristic function v, there exists a *unique* collection $(\alpha_T \mid T \subset N, T \neq \varnothing)$ of $2^n - 1$ real numbers such that:

$$v(S) = \sum_{T \subset N} \alpha_T(N,v) u_T(S) = \sum_{T \subset S} \alpha_T(N,v) \text{ for all } S \subset N. \quad (4)$$

The coefficients α_T are known as the *Harsanyi dividends* (dividends for short).[13] The following proposition follows immediately from (3) and (4).

Proposition 1 The dividends of the ranking game (N,w) associated to the ballot profile (N,π) coincide with the ballot profile: $\alpha_S(N,w) = \pi(S)$ for all $S \subset N$.

Hence, because (4) is invertible, there is a *one-to-one* relation between a ranking game and its ballot profile. The subset $RG(N) \subset G(N)$ of all ranking games on a set N generated by the set of ballot profiles $\Pi(N)$ forms a remarkable class of games. The characteristic functions defining ranking games are *monotonic* (increasing), convex (thereby superadditive) and take values in $\mathbb{N}$. They are *positive* in the sense that their dividends are all non-negative.[14] Furthermore, the set $RG(N)$ is *closed under addition*: starting from any two ballot profiles (N,π') and (N,π'') on a common set of candidates, and their associated ranking games (N,v') and (N,v''), the ranking game $(N,v'+v'')$ is associated to the ballot profile $(N,\pi'+\pi'')$.[15]

2.2 The Shapley value of a ranking game

Ginsburgh and Zang (2012) prove that the Shapley value of a ranking game coincides with equal-and-even cumulative voting as defined by (2), hence the terms Shapley ranking.

Proposition 2 The Shapley ranking associated to a ballot profile (N,π) is the Shapley value of the associated ranking games (N,w): $SR_i(N,\pi) = SV_i(N,w)$, $i = 1,...,n$.

Proof. Following Harsanyi (1959), the Shapley value of a game (N,v) is given by the uniform distribution of its dividends:

$$SV_i(N,v) = \sum_{T: i \in T} \frac{1}{t} \alpha_T(N,v) \quad (i = 1,..., n).$$

The identity then follows from Proposition 1. ♦

[13] See Harsanyi (1959).
[14] Positive games form a particular subclass of convex games on which the set of asymmetric values obtained by considering all distributions of dividends (the "Harsanyi set") coincides with the set of weighted Shapley values and the core. See Dehez (2017) for details.
[15] It is implicitly assumed that the sets of voters are disjoint.

2.3 Axiomatization of Shapley ranking

For a given set N, we denote by $P(N)$ the set of *permutations* of N. For a given subset $S \subset N$, pS denotes the image of S under the permutation $p \in P(N)$. For a given set function v on N, the function pv is defined by $pv(pS) = v(S)$ for all $S \subset N$. The following axioms are the translations of Shapley's axioms in terms of ballot profiles. They apply to the ranking rules $\varphi : \Pi(N) \to \mathbb{R}^n$ defined on the set of ballot profiles.

One-person-one-vote (Efficiency) The scores add-up to the number of electors:

$$\sum_{i \in N} \varphi_i(N, \pi) = \sum_{S \subset N} \pi(S).$$

Neutrality (Anonymity) If candidates' names are permuted, scores are permuted accordingly:

For all $p \in P(N)$ and $i \in N$, $\varphi_{pi}(N, p\pi) = \varphi_i(N, \pi).$

Null candidate (Null player) Candidates appearing in no ballot get a zero score:

$\pi(S) = 0$ for all $S \subset N$ such that $i \in S \;\Rightarrow\; \varphi_i(N, \pi) = 0.$

AdditivityThe score associated to a sum of ballot profiles on a common set of candidates is equal the sum of the scores associated to each ballot profile:

For any two ballot profiles π' and π'' on N, $\varphi(N, \pi' + \pi'') = \varphi(N, \pi') + \varphi(N, \pi'').$

Additivity makes sense here because the set $\Pi(N)$ of all possible ballot profiles is closed under addition. These four axioms are natural requirements and characterize Shapley ranking.

Proposition 3 Shapley ranking is the unique ranking rule defined on $\Pi(N)$ that satisfies One-person-one-vote, Neutrality, Null candidate and Additivity.

Proof . Shapley ranking obviously satisfies the four axioms. Now, consider a ranking rule φ satisfying all four axioms. Any ballot profile π on N can be decomposed as a sum of elementary ballot profiles $\pi = \sum_{T \subset N} \pi_T$ where

$$\begin{aligned} \pi_T(S) &= \pi(T) \quad \text{if } S = T, \\ &= 0 \qquad\;\; \text{if } S \neq T. \end{aligned}$$

By the Null candidate axiom, we have:

$\varphi(N, \pi_T) = 0$ for all $i \notin T$.

Combining the One-person-one-vote and the Neutrality axioms, we have:

$$\varphi(N, \pi_T) = \frac{\pi(T)}{t} \text{ for all } i \in T.$$

We then obtain (3) by using Additivity. ♦

It is easily seen that approval ranking obtains by replacing the One-person-one-vote axiom by an axiom specifying that the scores *add-up to the number of votes* (One-person-many-votes).

3. From ranking to ordering

Approval and Shapley orderings are derived from approval and Shapley rankings. They generally differ, as shown in Example 1. They coincide in the two extreme voting situations where either $n_h = 1$ for all h or $n_h = n$ for all h. In the first situation, analogous to plurality voting, $AR_i = SR_i$ for all i. In the second situation, $AR_i = m$ and $SR_i = m/n$ for all i where m is the number of electors.

Referring to the underlying individual preferences, approval and Shapley orderings are two social welfare functions that assign collective preferences to individual preferences. What are their properties and how do they compare to each other? Not surprisingly, we will see that little can be said outside the case of dichotomous preferences.

3.1 Individual preferences

Consider the following three possible assumptions on individual preferences:

A1 $i \in N_h$ and $j \notin N_h$ implies $i \succ_h j$.

A2 $i, j \in N_h$ implies $i \sim_h j$.

A3 $i, j \in N \setminus N_h$ implies $i \sim_h j$.

A1 is part of the definition of approval voting. The other two assumptions are less natural and far too restrictive, especially A3. The three assumptions together characterize *dichotomous preferences*.

3.2 From individual to collective preferences

The validity of four properties will be considered, *Pareto*, *Independence of irrelevant alternatives*, *Condorcet* and *Monotonicity*, on the basis of the approval sets $(N_1, ..., N_m)$ submitted by electors.

The Pareto principle requires that unanimity should be reflected in collective preferences. Here is a formulation that allows electors to be indifferent between candidates:

Pareto principle If candidate i is preferred to candidate j by all electors, then j cannot be collectively preferred to i.

Under dichotomous preferences, candidate i is preferred to candidate j by all electors if and only if $i \in N_h$ and $j \notin N_h$ for all $h \in M$. Clearly, approval and Shapley ranking both satisfy the Pareto principle under dichotomous preferences. It remains true assuming only A1.

Proposition 4 Under assumption A1, both approval and Shapley orderings satisfy the Pareto principle.

Proof. Consider two candidates i and j such that $i \succ_h j$ for all $h \in M$. For each elector h, there are three cases that define a partition in three subsets of the set of electors:

(a) $i \in N_h$ and $j \notin N_h \;\rightarrow\; h \in M_1$,

(b) $i, j \in N_h \;\rightarrow\; h \in M_2$,

(c) $i, j \notin N_h \;\rightarrow\; h \in M \setminus (M_1 \cup M_2)$.

The difference in approval scores is then equal to $m_1 \geq 0$. Nothing excludes a situation where M_1 is empty. The difference in Shapley scores is non-negative as well. Indeed, referring to the representative matrix Q that describes electors' ballots, we have:

$$SR_i - SR_j = \sum_{h \in M_1} \frac{q_{hi}}{b_h}$$

where $b_h = \sum_{l \in N} q_{hl} > 0$ for all $h \in M$ by assumption. ♦

The axiom of independence of irrelevant alternatives (IIA for short) was introduced by Arrow (1951). It may be considered as a natural requirement although it is generally not satisfied in the absence of restrictions on individual preferences. Consider two preference profiles with a common set of electors, on a common set of candidates, and two candidates *i* and *j*.

IIA If electors have the same preferences regarding *i* and *j* in both profiles, the collective preferences regarding *i* and *j* derived from the two profiles must be identical.

Arrow's impossibility theorem states that, without restrictions on preferences (axiom of *Unrestricted domain*), dictatorship is the only social welfare function that satisfies the Pareto principle and IIA. It is easy to show that, under dichotomous *preferences*, approval voting satisfies IIA.[16] Indeed, consider two ballot profiles $(N_1', ..., N_m')$ and $(N_1'', ..., N_m'')$, and the associated representative matrices Q' and Q''. If electors have the same preferences regarding *i* and *j*, the rows *i* and *j* of the matrices Q' and Q'' are identical and therefore $AR_i' = AR_i''$ and $AR_j' = AR_j''$. Shapley ranking instead does not satisfy IIA, whether preferences are dichotomous or not, because of the strong interdependence that characterizes it. This is confirmed by the following example.

Example 4 Consider two ballot profiles on a set of three candidates and five electors, represented by the matrices Q' and Q''. Assume that the electors have the same preferences regarding both candidates.

$$Q' = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 0 & 0 \\ \hline \end{array} \qquad Q'' = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 & 1 \\ \hline 0 & 0 & 1 & 1 & 1 \\ \hline \end{array}$$

In Q' candidate 2 has a higher Shapley score than candidate 1. The order is reversed in Q''.

[16] This is acknowledged by Brams and Fishburn (2007, p.137) and confirmed by Maniquet and Mongin (2015).

The Condorcet principle is often considered to be desirable property. A candidate is a *Condorcet winner* if he never loses in duels (Condorcet, 1785). There may be no Condorcet winner and, if such a candidate exists, one could argue that he should be elected.

Condorcet principle The set of Condorcet winners, if non-empty, should be on top of the collective preferences.

Few aggregation methods satisfies this principle. If individual preferences were known (like in Borda count), one could first check whether there is a Condorcet winner and eventually elect him. If preferences are assumed to be dichotomous, the result of a duel between two candidates only depends on their approval scores. Hence, a candidate is a Condorcet winner if he has the highest approval score.[17] Shapley ranking instead does not satisfy the Condorcet principle, as shown in Example 1 where candidates 2 and 3 are Condorcet winners but none of them is elected under Shapley ranking. Example 3 confirms that, outside dichotomous preferences, approval and Shapley rankings *both* fail to satisfy the Condorcet principle. However, both rankings are exempt of cycles.[18]

What happens to collective preferences when the preferences of a single elector change? This is the object of the following axiom.

Monotonicity Consider a preference profile such that candidate *i* is *collectively* preferred to *j*. If an elector who prefers *j* to *i* changes his mind in favor of candidate *i*, candidate *i* must remain collectively preferred to candidate *j*.

Proposition 5 Under assumption A1 both approval ordering and Shapley ordering satisfy Monotonicity.

Proof. Assume that $i \succ j$ while $j \succ_k i$ for some $k \in M$. There are three possible cases:

(a) $j \in N_k$ and $i \notin N_k$,

(b) $i, j \in N_k$,

(c) $i, j \notin N_k$.

Assume that elector *k* changes his mind and now prefer *i* to *j*. We denote by N'_k his modified approval set. In case (a), we have three possible cases:

(a1) $N'_k = N_k \setminus j$,

(a2) $N'_k = N_k \cup i$,

(a3) $N'_k = (N_k \cup i) \setminus j$.

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[17] This is Theorem 3.1 in Brams and Fishburn (2007, p. 38).

[18] Cycles are possible if more than two grades are used giving rise to what Brams and Potthoff (2015) call the "paradox of grading systems" that is comparable to the Condorcet paradox.

In case (b), there are two possible cases:

(b1) $N'_k = N_k \setminus j$,

(b2) $N'_k = N_k$.

In case (c), there are also two possible cases:

(c1) $N'_k = N_k \cup i$,

(c2) $N'_k = N_k$.

Consider first approval voting. Initially, we have $AR_i > AR_j$. In cases (b2) and (c2), AR_i and AR_j remain unchanged. In cases (a1) and (b1), AR_i is unaffected while AR_j decreases by 1. In cases (a2) and (c1), AR_j is unaffected while AR_i increases by 1. In case (a3), AR_i increases by 1 and AR_j decreases by 1. Hence, $AR'_i > AR'_j$. Consider now Shapley ranking. Initially, we have $SR_i > SR_j$. In cases (b2) and (c2), SR_i and SR_j remain unchanged. In cases (a1), SR_i is unaffected while SR_j decreases by $1/n_k$. In case (a2), SR_j decreases by $1/n_k(1+n_k)$ while SR_i increases by $1/(1+n_k)$. In case (b1), SR_j decreases by $1/n_k$ while SR_i increases by $1/n_k(n_k-1)$. In case (c1), SR_j is unaffected while SR_i increases by $1/(1+n_k)$. In case (a3), SR_i increases by $1/n_k$ and SR_j decreases by $1/n_k$. Hence, $SR'_i > SR'_j$. ♦

4. Concluding remarks

Approval voting has its advantages and drawbacks like any other preference aggregation methods, although most of its advantages cannot be formalized. The same applies to equal-and-even cumulative voting. However, equal-and-even cumulative voting may be preferable to approval voting because under the former, electors have an incentive to limit the number of candidates they decide to retain. Furthermore, if an elector retains several candidates, it is likely that the candidates retained will be "close" to each other in terms of preferences, in which case assumption A2 becomes more plausible. It would be interesting to conduct experiments within approval balloting in order to evaluate the effect of fractioning votes.

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