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Pollution effects on disease transmission and economic stability*

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Abstract

In this article, we embed a model of disease spread into a Ramsey model. A stock of pollution, viewed as a productive externality, affects both the disease transmission and the consumption demand. An eco-friendly government levies a proportional Pigouvian tax on production to depollute. We show the coexistence of two steady states in the long run: a disease-free and an endemic steady state. At the endemic steady state, a higher green-tax rate always reduces the pollution level. In the short run, we show the existence of limit cycles (through a Hopf bifurcation) as well as more complex dynamics of codimension two (a Gavrilov-Guckenheimer bifurcation). We complete the study with a numerical illustration of these bifurcations and a new facet of the Green Paradox: a higher tax rate can allow more scope for cycles by lowering the critical aversion to pollution and, thus, contribute to destabilize the economy and promote intergenerational inequalities.

Keywords: SIS model, Ramsey model, pollution, transcritical bifurcation, Hopf bifurcation, Gavrilov-Guckenheimer bifurcation.

JEL Classification: C61, E32, O44.

1 Introduction

Bosi and Desmarchelier (2016a) study the role of pollution in disease transmission and the economic consequences in a growth model à la Ramsey.[1] In their model, a pollution externality coming from production activities promote the

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[1] In the spirit of Goenka et al. (2014), Bosi and Desmarchelier (2016a) have embedded a SIS model (Susceptible-Infected-Susceptible) into a Ramsey model. In epidemiology, the SIS model remains very popular. The reader interested in a complete survey of this literature is refereed to Hethcote (2009).

transmission of an infectious disease. The role of pollution on disease transmission was yet considered in the epidemiological literature.[2] In the long run, as in Goenka et al. (2014), Bosi and Desmarchelier (2016a) recover a robust feature of the SIS model: a disease-free steady state coexists with an endemic one. In the short run, under dominant income effects, they find that a rise in the harmfulness of production can give birth to two successive limit cycles around the endemic steady state, the one stable (through a supercritical Hopf bifurcation), the other unstable (through a subcritical Hopf).

There is a literature on the destabilizing impact of pollution. Two main mechanisms of pollution effect on macroeconomic volatility have been identified with either finite or infinite-lived agents: (1) when pollution increases the consumption demand[3] or (2) it is strongly inertial[4]. When these mechanisms work together richer dynamics take place as shown by Bosi and Desmarchelier (2016b): limit cycles arise through a Hopf bifurcation in a market economy à la Ramsey. Despite the mathematical complexity, the economic interpretation for cycles remains somewhat intuitive: a positive pollution effect on consumption demand and a large pollution inertia play the role of a repulsive and a restoring force respectively, giving rise to fluctuations.

Interestingly, Bosi and Desmarchelier (2016a) consider pollution as a flow and assume that pollution does not enter the utility function. Since their framework embeds a SIS model, the comparison with the existing literature is difficult. To overcome this difficulty, we propose a new unified framework where the stock of pollution is an externality affecting both the consumption demand and the disease transmission. In addition, we introduce a proportional tax at the firm level to finance depollution expenditures according to a balanced budget rule. We try to understand whether the pollution effect on disease transmission can change significantly the qualitative impact of a green tax on the steady state.

This unified approach leads to intersting results either in the short or in the long run. In the long run, as in Bosi and Desmarchelier (2016a), we recover the main feature of the SIS model: a disease-free steady state coexists with an endemic one. In addition, we show that a higher green-tax rate always improve the environmental quality by reducing the pollution level. In this sense, we find a usual property of the competitive Ramsey model with pollution. In the short run, around the endemic steady state, we recover one of the main result of the literature: a Hopf bifurcation can occur if and only if pollution increases the consumption demand and exhibits a strong inertia. Moreover, we recover the complex dynamics pointed out in Bosi and Desmarchelier (2016a): a change in the environmental impact of production generates the collision of the two steady states through a transcritical bifurcation and the possible occurrence of two successive limit cycles with opposite stability properties. Differently from Bosi and Desmarchelier (2016a), we do not require the income effect to be too large in order to observe the occurrence of these Hopf bifurcations. Surprisingly, the whole is greater than the sum of the parts: the unified model leads to richer dynamics

[2] See Caren (1981) among the others.

[3] See Heal (1982), Itaya (2008), Fernandez et al. (2012) and Zhang (1999) among others.

[4] See Zhang (1999) and Seegmuller and Verchère (2004) among others.

absent from the existing literature on environmental Ramsey economies: when the two steady states coalesce, the endemic one can be surrounded by a persistent cycle through a Gavrilov-Guckenheimer bifurcation. Following Kuznetsov (1998), this bifurcation can give rise to three different phenomena: (1) an invariant torus, (2) a preserved limit cycle, (3) a blown-up limit cycle. In our paper, a computer simulation shows that the limit cycle is preserved.

In an environmental context, the existence of limit cycles entails not only a kind of macroeconomic volatility but also intergenerational inequalities: some generations enjoy ecological quality while others suffer from a highly polluted environment. From the policy maker's perspective, we study whether the green tax can shelter the economy from unpleasant fluctuations. Numerical simulations point out an ambiguous role of the green tax. However, we show that, when the tax pressure is low, a higher rate can create more scope for limit cycles by reducing the critical pollution aversion in the preferences. In other words, an increase in the green-tax rate to finance depollution, can promote macroeconomic volatility and, thus, intergenerational inequalities. This unpleasant effect represents an extension of the Green Paradox literature. In a seminal work, Sinn (2008) showed how a higher green tax can accelerate the global warming and named this effect Green Paradox. In our case, the fiscal effect on macroeconomic volatility can be interpreted as a new facet of the Green Paradox (1) because of the intergenerational inequalities and (2) because, in some part of the limit cycle, the pollution level increases.

The paper is organized as follows: in section 2, we present the model; sections 3 and 4 focus on the equilibrium and the steady state respectively; the local dynamics are studied in section 5, while the section 6 provides a numerical illustration. Section 7 concludes.

2 The model

2.1 Disease

Epidemiologists use the SIS model to study the spread of endemic diseases. Population (N) is divided in two classes: susceptible (S) and infective (I) with $S + I = N$. The proportion of susceptible and infective are given by $s = S/N$ and $i = I/N$. $\beta > 0$ denotes the average number of adequate contacts (sufficient to transmit the disease) of an infective per unit of time and S/N the probability to face a susceptible during a contact. Thus, $\beta S/N$ is the average number of adequate contacts with susceptibles of one infective per unit of time, while the number of new infectives per unit of time is given by $\beta IS/N$. An infective is seek during a period of time after which he recovers and becomes a new susceptible ($\gamma = -\dot{I}/I$ is the recovery rate in absence of new contamination, a sort of exponential decay rate from infection). Indeed, the SIS model postulates that the infection does not confer immunity. In the following, for the sake of simplicity, we will omit the time argument t.

The evolution of S and I over time is simply given by:

$$\dot{S} = -\beta \frac{I}{N} S + \gamma I \tag{1}$$

$$\dot{I} = \beta \frac{I}{N} S - \gamma I \tag{2}$$

In an oversimplified world with no births, no deaths, no migrations, the population remains constant over time. Therefore, $N = S + I$ gives $\dot{S} + \dot{I} = 0$ and equation (1) becomes:

$$\dot{s} = (1 - s)(\gamma - \beta s) \tag{3}$$

As in Goenka *et al.* (2014), we assume that the labor force (L) consists only of healthy people: $L = S$. Since $l = L/N \leq 1$, l inherits the dynamics of s:

$$\dot{l} = (1 - l)(\gamma - \beta l) \tag{4}$$

We can see that (4) exhibits two steady state: $l = 1$ and $l = \gamma/\beta$ with $\gamma < \beta$. The first one is called *disease-free* because the disease disappears while the other is called *endemic* because the disease persists. As seen above, some medical evidence highlights the negative effects of pollution (P) on the immune system (Caren (1981), Bauer *et al.* (2012)) and supports the theoretical assumption of β and γ and as increasing and decreasing functions of P respectively.

Assumption 1 *The function $\beta(P) : \mathbb{R}_+ \to \mathbb{R}_+$, is C^2 with $\beta'(P) > 0$, $\lim_{P \to 0} \beta(P) = 0$ and $\lim_{P \to +\infty} \beta(P) = +\infty$. $\gamma(P) : \mathbb{R}_+ \to \mathbb{R}_+$ is also C^2 with $\gamma'(P) < 0$, $\lim_{P \to 0} \gamma(P) = +\infty$ and $\lim_{P \to +\infty} \gamma(P) = 0$.*

The following first-order elasticities capture the main informations about the role of pollution:

$$\varepsilon_\beta(P) \equiv \frac{P\beta'(P)}{\beta(P)} > 0 \text{ and } \varepsilon_\gamma(P) \equiv \frac{P\gamma'(P)}{\gamma(P)} < 0$$

We introduce a measure of the pollution impact on the disease transmission:

$$d(P) \equiv \varepsilon_\beta(P) - \varepsilon_\gamma(P) > 0$$

2.2 Preferences

The household earns a capital income rh and a labor income ω, where r and h denote respectively the real interest rate and the individual wealth at time t. Income is consumed and saved/invested according to the budget constraint:

$$\dot{h} \leq (r - \delta) h + \bar{\omega} - c \tag{5}$$

In this model, healthy people work while sick people don't. However, for simplicity, we assume a perfect social security, that is a full unemployment insurance in the case of illness. Healthy and sick agents earn the same labor income $\bar{\omega}$. L healthy people supply one unit of labor at a wage w. Under a balanced-budget rule for social security, we obtain $\bar{\omega}N = wL$. Therefore, $\bar{\omega} = wl$.

Gross investments include the capital depreciation at the rate δ. For simplicity, the population of consumers-workers is normalized to unity: $N = 1$. Such a normalization implies $L = Nl = l$, $K = Nh = h$ and $h = K/N = kl$.

Assumption 2 *Preferences are rationalized by a non-separable utility function $u(c, P)$. First and second-order restrictions hold on the sign of derivatives: $u_c > 0$, $u_P < 0$ and $u_{cc} < 0$, jointly with the limit conditions:* $\lim_{c \to 0^+} u_c = \infty$ *and* $\lim_{c \to +\infty} u_c = 0$.

Assumption 2 does not impose any restriction on the sign of the cross-derivative $u_{cP} \lesseqgtr 0$. Following Michel and Rotillon (1995), the household's preferences exhibit a *distaste effect* (*compensation effect*) when pollution decreases (increases) the marginal utility of consumption. If the household wishes to consume in a pleasant environment, a higher pollution level lowers its consumption demand ($u_{cP} < 0$) giving rise to a *distaste effect* (Michel and Rotillon, 1995). Conversely, the household may decide to increase its consumption demand to compensate the utility loss due to a higher pollution level ($u_{cP} > 0$): in this case, a *compensation effect* arises (Michel and Rotillon, 1995).

We introduce the first and second-order elasticities:

$$
\begin{aligned}
(\varepsilon_c, \varepsilon_P) &\equiv \left(\frac{cu_c}{u}, \frac{Pu_P}{u} \right) \\
\begin{bmatrix} \varepsilon_{cc} & \varepsilon_{cP} \\ \varepsilon_{Pc} & \varepsilon_{PP} \end{bmatrix} &\equiv \begin{bmatrix} \frac{cu_{cc}}{u_c} & \frac{Pu_{cP}}{u_c} \\ \frac{cu_{Pc}}{u_P} & \frac{Pu_{PP}}{u_P} \end{bmatrix}
\end{aligned}
\tag{6}
$$

According to Assumption 2, $\varepsilon_c > 0$ and $\varepsilon_P < 0$. $-1/\varepsilon_{cc}$ represents the intertemporal elasticity of substitution in consumption while ε_{cP} captures the effect of pollution on the marginal utility of consumption. Typically, if $\varepsilon_{cP} > 0$ (< 0), pollution and consumption are complement (substitute) for households.

The illness lowers labor supply and the individual income in turn. The agent maximizes the intertemporal utility function

$$
\int_0^\infty e^{-\theta t} u(c, P) \, dt \tag{7}
$$

under the budget constraint (5), where $\theta > 0$ is the rate of time preference.

Proposition 1 *The first-order conditions of the consumer's program are given by a static relation*

$$
\mu = u_c(c, P) \tag{8}
$$

a dynamic Euler equation and the budget constraint (1), now binding:

$$
\dot{\mu} = \mu(\theta + \delta - r) \tag{9}
$$

$$
\dot{h} = (r - \delta) h + wl - c \tag{10}
$$

jointly with the transversality condition $\lim_{t\to\infty} e^{-\theta t}\mu(t)\, h(t) = 0$. μ *denotes the multiplier associated to the budget constraint.*

Applying the Implicit Function Theorem to the static relation $\mu = u_c(c, P)$, we obtain the consumption function $c \equiv c(\mu, P)$ with elasticities

$$\frac{\mu}{c}\frac{dc}{d\mu} = \frac{1}{\varepsilon_{cc}} < 0 \text{ and } \frac{P}{c}\frac{dc}{dP} = -\frac{\varepsilon_{cP}}{\varepsilon_{cc}} \tag{11}$$

2.3 Firms

The firm chooses the amount of capital and labor to maximize the profit taking as given the real interest rate r as well as the wage rate w. In addition, the government levies a proportional tax $\tau \in (0,1)$ on polluting production $F(k_j, l_j)$ of firm j to finance depollution expenditures.

Assumption 3 *The production function* $F : \mathbb{R}^2_+ \to \mathbb{R}_+$ *is* C^2*, homogeneous of degree one, strictly increasing and concave. Inada conditions hold.*

The profit maximization $\max_{K_j,N_j} [F(K_j, L_j) - rK_j - wL_j - \tau F(K_j, L_j)]$ entails the following first-order conditions:

$$r = (1-\tau) f'(k_j) \text{ and } w = (1-\tau)[f(k_j) - k_j f'(k_j)]$$

where $k_j \equiv K_j/L_j$ is the capital intensity and $f(k_j) \equiv F(k_j, 1)$ the average productivity of the firm j.

All the firms share the same technology and address the same demand for capital.

Proposition 2 *Let* $k \equiv K/L$ *with* $K \equiv \sum_{j=1}^{J} K_j$ *and* $L \equiv \sum_{j=1}^{J} L_j$. *In aggregate terms,* $Y = F(K, L)$ *and profit maximization yields*

$$r = (1-\tau)\rho(k) \text{ and } w = (1-\tau)\omega(k) \tag{12}$$

with $\rho(k) \equiv f'(k)$ *and* $\omega(k) \equiv f(k) - kf'(k)$.

We introduce the capital share in total disposable income and the elasticity of capital-labor substitution:

$$\alpha(k) \equiv \frac{rk}{(1-\tau) f(k)} = \frac{kf'(k)}{f(k)} \text{ and } \sigma(k) \equiv \frac{f'(k)}{kf''(k)}\left[\frac{kf'(k)}{f(k)} - 1\right] = \alpha(k)\frac{\omega(k)}{k\omega'(k)}$$

In addition, we determine the elasticities of factor prices:

$$\frac{k\rho'(k)}{\rho(k)} = -\frac{1-\alpha(k)}{\sigma(k)} \text{ and } \frac{k\omega'(k)}{\omega(k)} = \frac{\alpha(k)}{\sigma(k)}$$

2.4 Government

The government uses all the tax revenues to finance depollution expenditures (G) according to a balanced budget rule:

$$G = \tau F(K, L) \tag{13}$$

2.5 Pollution

The aggregate stock of pollution P is a pure externality coming from production. The government takes care of depollution through the abatement expenditures G. The pollution accumulation follows a linear process:

$$\dot{P} = -aP + bY - mG \tag{14}$$

$a \geq 0$, $b \geq 0$ and $m \geq 0$ capture respectively the natural rate of pollution absorption, the environmental impact of production and the pollution abatement efficiency. Because $N = 1$, process (14) becomes in intensive terms: $\dot{P} = -aP + (b - m\tau)\, lf(k)$. The linear process (14) is widely used in the literature.[5]

3 Equilibrium

At the equilibrium all markets clear (good, capital and labor). Noticing that

$$\frac{\dot{k}}{k} = \frac{\dot{h}}{h} - \frac{\dot{l}}{l}$$

and considering (4), (12), (14) and Proposition 1, we obtain a four-dimensional dynamic system.

$$\begin{aligned}
\dot{\mu} &= f_1(\mu, k, l, P) = \mu\left[\theta + \delta - (1-\tau)\,\rho(k)\right] && (15)\\
\dot{k} &= f_2(\mu, k, l, P) = \left[(1-\tau)\,\rho(k) - \delta\right] k + (1-\tau)\,\omega(k) - \frac{c(\mu, P)}{l} - z(l, P)\,k && (16)\\
\dot{l} &= f_3(\mu, k, l, P) = lz(l, P) && (17)\\
\dot{P} &= f_4(\mu, k, l, P) = -aP + (b - m\tau)\, lf(k) && (18)
\end{aligned}$$

with

$$z(l, P) \equiv \frac{1-l}{l}\left[\gamma(P) - \beta(P)\, l\right]$$

In this system, three variables are predeterminate (k, l and P), while μ is a jump variable. Equations (15) and (16) represent the Ramsey model, equation (17) captures the SIS dynamics, while equation (18) adds the environmental layer. Our framework bridges two strand of literature: environmental economics and epidemiology.

4 Steady state

At the steady state, all variables remain constant, that is $\dot{\mu} = \dot{k} = \dot{l} = \dot{P} = 0$. Equation (15) gives the Modified Golden Rule (MGR). Assumption 3 entails a unique capital level (k^*) at the steady state:

$$k^* = \rho^{-1}\left(\frac{\theta + \delta}{1 - \tau}\right) \tag{19}$$

[5] See Heal (1982) or Seegmuller and Verchère (2004) among others.

Equations (19) and (18) at the steady state give

$$P^* = \frac{b - m\tau}{a} l f(k^*) \equiv P(l) \tag{20}$$

In order to have a positive stock of pollution at the steady state we introduce an additional hypothesis.

Assumption 4 *$b - m\tau > 0$.*

Equation (17) gives the labor supply at the steady state. According to equation (4), there are multiple values of $l \neq 0$ satisfying equation (17) at the steady state. Nevertheless, for a given value of $l \neq 0$, we observe that equation (16) leads to a unique positive consumption level at the steady state:

$$c^* = [\theta k^* + (1 - \tau)\,\omega(k^*)]\, l > 0$$

Reconsidering Proposition 1, we find that, for a given $l \neq 0$, there is a unique positive shadow price of capital at the steady state:

$$\mu^* = u_c(c^*, P(l)) > 0 \tag{21}$$

Summing up, we see that the existence as well as the unicity/multiplicity of a steady state for this economy depends upon the number of stationary values $l \in (0, 1)$ satisfying

$$z(l, P(l)) = \frac{1 - l}{l} g(l) = 0$$

with $g(l) \equiv \gamma(P(l)) - \beta(P(l))\, l$. Assumption 1 allows us to prove the existence of a unique positive value of labor supply.

Lemma 3 *At the steady state, there is a unique $l > 0$ such that $g(l) = 0$.*

The following proposition is an immediate consequence of Lemma 3 and provides a complete picture of the steady states.

Proposition 4 *Let l^* be a solution of $g(l) = 0$.*

(1) If $l^* > 1$, there is a unique steady state given by $(\mu, k, l, P) = (\mu^*, k^*, 1, P^*)$ (disease-free steady state).

(2) If $0 < l^* < 1$, there are two steady states:

(2.1) $(\mu, k, l, P) = (\mu^*, k^*, 1, P^*)$ (disease-free steady state),

(2.2) $(\mu, k, l, P) = (\mu^*, k^*, l^*, P^*)$ (endemic steady state).

If $l^* = 1$, the endemic steady state coalesces with the disease-free steady state.

We expect the occurrence of a transcritical bifurcation: the endemic steady state collides with the disease free steady state at $l^* = 1$.

Since the main goal of our paper is to describe the pollution impact on a persistent disease, we focus on the endemic steady state. The following proposition consider the impact of the green tax rate on the endemic steady state.

Proposition 5 *The impact of τ on the endemic steady state is given by*

$$\begin{aligned}
\frac{\tau}{k^*}\frac{\partial k^*}{\partial \tau} &= -\frac{\tau}{1-\tau}\frac{\sigma}{1-\alpha} < 0 \\
\frac{\tau}{l^*}\frac{\partial l^*}{\partial \tau} &= -\varphi\frac{d}{1+d}\frac{\tau}{k^*}\frac{\partial k^*}{\partial \tau} > 0 \\
\frac{\tau}{P^*}\frac{\partial P^*}{\partial \tau} &= \varphi\frac{1}{1+d}\frac{\tau}{k^*}\frac{\partial k^*}{\partial \tau} < 0 \\
\frac{\tau}{c^*}\frac{\partial c^*}{\partial \tau} &= \left[1+\frac{1-\sigma}{\sigma}\frac{(1-\alpha)(\theta+\delta)}{\theta+(1-\alpha)\delta}\right]\frac{\tau}{k^*}\frac{\partial k^*}{\partial \tau}+\frac{\tau}{l^*}\frac{\partial l^*}{\partial \tau}
\end{aligned}$$

and

$$\frac{\tau}{W^*}\frac{\partial W^*}{\partial \tau} = \varepsilon_c\frac{\tau}{c^*}\frac{\partial c^*}{\partial \tau}+\varepsilon_P\frac{\tau}{P^*}\frac{\partial P^*}{\partial \tau} \tag{22}$$

where $\alpha = \alpha(k)$, $\sigma = \sigma(k)$, $\beta = \beta(P)$, $\gamma = \gamma(P)$, $d = d(P)$, $\varepsilon_c = \varepsilon_c(c,P)$, $\varepsilon_P = \varepsilon_P(c,P)$, $\varepsilon_{cc} = \varepsilon_{cc}(c,P)$, $\varepsilon_{cP} = \varepsilon_{cP}(c,P)$ and

$$\varphi(k) \equiv \alpha + \frac{1-\alpha}{\sigma}\frac{m-m\tau}{b-m\tau} > 0$$

are evaluated at the endemic steady state.

Let us provide some intuition. Since the green tax is levied on the production level, a higher tax rate gives incentives to lower the production level and to reduce the capital intensity in the long run. Moreover, a lower production level implies also lower pollution emissions while a higher green-tax rate means at the same time a higher depollution level. Thus, a higher green-tax rate always lowers the pollution level in the long run. According to Assumption 1, we know that pollution promotes the transmission of infectious diseases and, therefore, a higher green-tax rate lowers the pollution level and increases the labor supply at the end.

In the case of a Cobb-Douglas technology ($\sigma = 1$), we disentangle the impact on consumption:

$$\frac{\tau}{c^*}\frac{\partial c^*}{\partial \tau} = \frac{\tau}{k^*}\frac{\partial k^*}{\partial \tau}+\frac{\tau}{l^*}\frac{\partial l^*}{\partial \tau} \tag{23}$$

This impact remains ambiguous (indeed, the impacts on capital and labor supply are negative and positive respectively). To provide a clearcut interpretation, we consider an additional assumption.

Assumption 5 $b - m \geq 0$.

Assumption 5 means that the environmental impact of production (b) exceeds the depollution efficiency (m).

Proposition 6 (consumption) *In the case of a Cobb-Douglas technology, under Assumption 5, the green tax has a negative impact on consumption:*

$$\frac{\tau}{c^*}\frac{\partial c^*}{\partial \tau} < 0$$

Under Assumption 5, the negative impact on capital dominates the positive impact on labor supply. Indeed, the green tax has a moderate effect on pollution reduction because of the low depollution efficiency. This moderate effect entails a limited improvement of the immune system and an increase of labor supply too small to compensate the drop in capital intensity.

Proposition 7 (welfare) *In the case of a Cobb-Douglas technology, under Assumption 5, the green tax is welfare-improving if and only if*

$$-\frac{\varepsilon_P}{\varepsilon_c} > \frac{1+(1-\varphi)\,d}{\varphi}\,(>1) \tag{24}$$

When households overvalue the environmental quality with respect to consumption, the ratio $-\varepsilon_P/\varepsilon_c$ becomes sufficiently large to exceed the RHS in (24). In this case, the increase in the utility due to the drop in pollution dominates the decrease due to the drop in consumption (see Proposition 6) and, thus, the green tax turns out to be welfare improving.

5 Local dynamics

We focus on the endemic steady state to show how the interplay between the ecological and the epidemiological sides affects the economic equilibrium. To capture the dynamics, we linearize the dynamical system (15)-(18) around the endemic steady state. The Jacobian matrix J is given by

$$\begin{aligned} J &\equiv \begin{bmatrix} \frac{\partial f_1}{\partial \mu} & \frac{\partial f_1}{\partial k} & \frac{\partial f_1}{\partial l} & \frac{\partial f_1}{\partial P} \\ \frac{\partial f_2}{\partial \mu} & \frac{\partial f_2}{\partial k} & \frac{\partial f_2}{\partial l} & \frac{\partial f_2}{\partial P} \\ \frac{\partial f_3}{\partial \mu} & \frac{\partial f_3}{\partial k} & \frac{\partial f_3}{\partial l} & \frac{\partial f_3}{\partial P} \\ \frac{\partial f_4}{\partial \mu} & \frac{\partial f_4}{\partial k} & \frac{\partial f_4}{\partial l} & \frac{\partial f_4}{\partial P} \end{bmatrix} \\ &= \begin{bmatrix} 0 & (\theta+\delta)\,\frac{1-\alpha}{\sigma}\,\frac{\mu}{k} & 0 & 0 \\ -\frac{\kappa}{\varepsilon_{cc}}\,\frac{k}{\mu} & \theta & (\kappa+\beta-\gamma)\,\frac{k}{l} & \frac{\varepsilon_{cP}}{\varepsilon_{cc}}\,\frac{\kappa}{\pi l}+(\beta-\gamma)\,\frac{d}{\pi l} \\ 0 & 0 & \gamma-\beta & (\gamma-\beta)\,\frac{d}{\pi k} \\ 0 & \alpha a\pi l & a\pi k & -a \end{bmatrix} \end{aligned}$$

where μ, k, l, $\alpha=\alpha(k)$, $\sigma=\sigma(k)$, $\beta=\beta(P)$, $\gamma=\gamma(P)$, $d=d(P)$, $\varepsilon_{cc}=\varepsilon_{cc}(c,P)$, $\varepsilon_{cP}=\varepsilon_{cP}(c,P)$,

$$\kappa(k) \equiv \frac{\theta+[1-\alpha(k)]\,\delta}{\alpha(k)} > 0 \text{ and } \pi(k) \equiv \frac{\theta+\delta}{1-\tau}\,\frac{b-m\tau}{a\alpha(k)} > 0 \tag{25}$$

are evaluated at the endemic steady state. Notice that

$$\kappa(k^*) = \frac{c^*}{k^*l^*} = \frac{C^*}{K^*} \text{ and } \pi(k^*) = \frac{P^*}{k^*l^*} = \frac{P^*}{K^*}$$

are the aggregate consumption and pollution on aggregate capital measuring the relative size of consumption and pollution in the whole economy.

To avoid any ambiguity, we consider explicit functional forms:

$$f(k) = Ak^\alpha \text{ and } u(c, P) = \frac{(cP^{-\eta})^{1-\varepsilon}}{1-\varepsilon}$$

This implies that $\sigma = 1$ jointly with $\varepsilon_{cc} = -\varepsilon$ and $\varepsilon_{cP} = \eta(\varepsilon - 1)$.

In the spirit of Bosi and Desmarchelier (2016a), we consider the following specification of the pollution effects on health:

$$\beta \equiv B_\beta P^{\varepsilon_\beta} \text{ and } \gamma \equiv B_\gamma P^{\varepsilon_\gamma}$$

with $\varepsilon_\beta > 0$ and $\varepsilon_\gamma < 0$ (Assumption 1).

We evaluate the Jacobian matrix around the endemic steady state ($l = \gamma/\beta$). Applying the methodology introduced by Bosi and Desmarchelier (2017), we compute the sums of minors:

$$\begin{aligned}
T &= \theta - a + \gamma - \beta \\
S_2 &= (\beta - \gamma)[a + ad(1-\alpha) - \theta] - a\theta - [\alpha a\eta(1-\varepsilon) + (1-\alpha)(\theta+\delta)]\frac{\kappa}{\varepsilon} \\
S_3 &= (1-\alpha)(\theta+\delta)(a+\beta-\gamma)\frac{\kappa}{\varepsilon} + a(\beta-\gamma)\left[\theta(1+d) + \alpha\kappa\left(\eta\frac{1-\varepsilon}{\varepsilon} - d\right)\right] \\
D &= (1-\alpha)(\theta+\delta)(\gamma-\beta)(1+d)\frac{a\kappa}{\varepsilon} \qquad (26)
\end{aligned}$$

Proposition 8 *A transcritical bifurcation generically occurs if and only if $\beta = \gamma$.*

Unsurprisingly, a transcritical bifurcation occurs when $\beta = \gamma$. Indeed, in this case, $l^* = 1$ and then, the endemic steady state and the disease-free steady state coalesce. The existence of a transcritical bifurcation means that the two steady state exchange their stability properties at the bifurcation point.

When $\beta = \gamma$, $D = 0$. According to Kuznetsov (1998), this leads to three possible cases: (1) an elementary saddle-node bifurcation, (2) a transcritical bifurcation or (3) a pitchfork bifurcation. In case (1), two steady state collide and disappear. In case (2), two steady state coalesce and separate again, while exchanging their stability properties. In case (3), three steady states coalesce into one which inherits the stability properties of the laterals. According to Proposition 4, there are always two steady states, meaning that a transcritical bifurcation occurs when $\beta = \gamma$.[6]

Since Heal (1982), it is known that a limit cycle can arise (through a Hopf bifurcation) when a pollution externality affects the consumption demand. The following proposition highlights a similar mechanism.

Let

$$\begin{aligned}
b_2 &\equiv (\beta-\gamma)[a - \theta + ad(1-\alpha)] - a\theta - (1-\alpha)(\theta+\delta)\frac{\kappa}{\varepsilon} \qquad (27) \\
b_3 &\equiv \frac{(1-\alpha)(\theta+\delta)(a+\beta-\gamma)\frac{\kappa}{\varepsilon} + a(\beta-\gamma)[\theta(1+d) - \alpha\kappa d]}{\theta - a + \gamma - \beta} \qquad (28)
\end{aligned}$$

[6]In our model, there are two steady states before and after the bifurcation: $l = 1$ and $l = l^* = \gamma/\beta$. In mathematical terms, l^* coexists with $l = 1$ even when $\gamma > \beta$, but becomes meaningless in economic terms.

Proposition 9 *If $a < \theta$ (strong pollution inertia), a limit cycle generically arises through a Hopf bifurcation at*

$$\eta_H = \frac{\varepsilon}{\varepsilon - 1} \frac{1}{2a\alpha\kappa} \left(1 + \frac{\theta - a}{\gamma - \beta} \right) \left(E_- + \sqrt{E_+^2 + 4D\frac{\gamma - \beta}{\theta - a}} \right) \tag{29}$$

provided that $\eta_H > 0$, where

$$\begin{aligned} E_- &\equiv \frac{\gamma - \beta}{\theta - a} (b_3 - b_2) - b_3 \\ E_+ &\equiv \frac{\gamma - \beta}{\theta - a} (b_3 - b_2) + b_3 \end{aligned}$$

Proposition 8 shows the possibility of a transcritical bifurcation (exchange of stability properties of two steady state), while Proposition 9 shows the possibility of a limit cycle (through a Hopf bifurcation) near the endemic steady state (see the end of the section for an economic interpretation). What does it happen when the endemic steady state is surrounded by a limit cycle and collides with the disease-free steady state? In other terms, when the the conditions fot the two bifurcations hold together? The simultaneous occurrence of a transcritical bifurcation and a Hopf bifurcation gives rise to codimension-two bifurcations: either the Bogdanov-Takens or the Gavrilov-Guckenheimer bifurcation. The Bogdanov-Takens bifurcation entails the destruction of the preexisting limit cycle while the Gavrilov-Guckenheimer the persistence of a cycle. The next two propositions clarify what of these two bifurcations take place in our model.

Proposition 10 *A Bogdanov-Takens bifurcation is impossible around the endemic steady state.*

Proposition 11 *Let $\varepsilon > 1$ and $\theta > a$ jointly with $\beta = \gamma$. A Gavrilov-Guckenheimer bifurcation generically arises if and only if*

$$\eta = \eta_{GG} \equiv \frac{\theta}{\alpha(\varepsilon - 1)} \left[\frac{\varepsilon}{\kappa} + \frac{(1 - \alpha)(\theta + \delta)}{a(\theta - a)} \right]$$

Notice that $\eta_{GG} > 0$.

Proposition 12 *A double-Hopf bifurcation is impossible around the endemic steady state.*

A limit cycle arises through Hopf bifurcation only if $\eta_H > 0$. Therefore, we need to know whether the pollution level increases ($\varepsilon > 1$) or decreases ($\varepsilon < 1$) the consumption demand. However, this information is not enough to know the sign of η_H. We know that both η_H and η_{GG} belong to the same Hopf bifurcation curve in the (a, η)-space. Since $a < \theta$ (strong pollution inertia), $\eta_{GG} > 0$ if and only if $\varepsilon > 1$. A necessary (but not sufficient) condition for a Hopf bifurcation to occur for critical values close to η_{GG} is that pollution raises the consumption demand. In this case, we can interpret the existence of a limit cycle around the endemic steady state. Let the economy be at the steady state at time t and assume an exogenous rise in the pollution level entailing two effects: (1) a drop in the labor supply because the infectious disease becomes more pervasive, and (2) an increase in the consumption demand ($\varepsilon > 1$). These effects imply lower savings and a lower capital intensity. Then, the production level lowers and the pollution level as well. In other terms, a positive pollution effect on consumption demand and a negative impact on labor supply jointly generate an endogenous cycle: a higher pollution level today implies a drop in the pollution level tomorrow.

In Bosi and Desmarchelier (2016a), the pollution is a flow affecting only the labor supply (through its effects on the pervasiveness of disease). In their model, a Hopf bifurcation occurs only when ε becomes very large. In our model, conditions for a limit cycle are less demanding ($\varepsilon > 1$ rather than a very large value) because the positive impact of the stock of pollution on consumption demand contributes to promote the occurrence of endogenous cycles. In addition, we observe richer dynamics: a Gavrilov-Guckenheimer bifurcation can give rise to different complex phenomena. A simple local analysis based on the Jacobian matrix is not sufficient to disentangle these cases involving higher-order terms of the Taylor expansion. Instead, a numerical simulation based on the original nonlinear system will allow us to shed light on the type of complex dynamics at work.

6 Simulation

In the last section, we have characterized the occurrence of local bifurcations. Nevertheless, even if we know that Hopf and Gavrilov-Guckenheimer bifurcations are possible in the model, our analytical approach based on the Jacobian matrix, is uninformative about the stability of the cycles or the occurrence of more complex dynamics. The limit cycle arising through the Hopf bifurcation can be stable or unstable, while the Gavrilov-Guckenheimer bifurcation can give rise to different scenarios: (1) a torus, (2) a preserved limit cycle after a Hopf bifurcation or (3) a "blown-up" limit cycle (see Section 8.5 in Kuznetsov (1998) for more details). These complex dynamics depend on the higher-order terms of the Taylor expansion and can be simulated using MATCONT, considering the original non-linear system instead of its first-order approximation.[7]

Local bifurcations are characterized in Propositions 8, 9 and 11, and occur at some critical value for β, γ or η. Since η is the main economic information, capturing the pollution impact on consumption demand, we keep it as principal bifurcation parameter. γ and β capture the epidemiological side of the model but depend on the pollution level and can not fixed exogenously. Conversely, a is an exogenous variable driving the pollution level: it is eligible as additional bifurcation parameter to study the bifurcations of codimension-two.

[7] MATCONT is an equilibrium continuation package for MATLAB.

Focus on a standard parametrization:

Parameter	A	B_β	B_γ	ε_β	ε_γ	α	δ	ε	θ	τ	b	m
Value	1	1	1	1	-1	0.33	0.025	4	0.01	0.015	0.001	0.001

(30)

α, δ and θ take usual quarterly values while the low value of elasticity of intertemporal substitution ($1/\varepsilon < 1$) captures the so-called compensation effect ($\varepsilon_{cP} > 0$). τ, b and m satisfies Assumption 4.

According to (29), to each value of η_H may correspond multiple values of a. If we fix $\eta_H = 11.617$, we obtain $a = 0.002881863$ and $a = 0.00295$. In other terms, we expect to detect two Hopf bifurcations at these two values for a such that both the pairs $(a(\eta_H), \eta_H)$ belong to the same Hopf bifurcation curve in the (a, η)-plane. This curve is analytically given by

$$\eta_H(a) \equiv \frac{\varepsilon}{\varepsilon - 1} \frac{1}{2a\alpha\kappa} \left[1 + \frac{\theta - a}{\tilde{\gamma}(a) - \tilde{\beta}(a)}\right] \left(E_-(a) + \sqrt{E_+^2(a) + 4D(a)\frac{\tilde{\gamma}(a) - \tilde{\beta}(a)}{\theta - a}}\right) \tag{31}$$

where $\tilde{\beta}(a) \equiv \beta(P(a))$ and $\tilde{\gamma}(a) \equiv \gamma(P(a))$, and it is geometrically represented in Figure 1 by MATCONT, where H, GH, BP and ZH denote respectively a Hopf, a Generalized-Hopf, a transcritical (Branch Point) and a Gavrilov-Guckenheimer (Zero-Hopf) bifurcation.

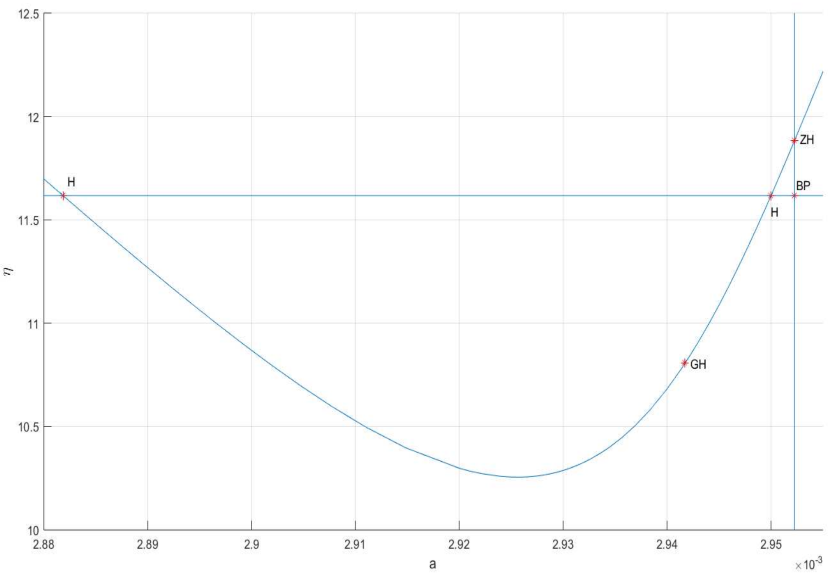

Fig. 1. Equilibrium continuation

We perform an equilibrium continuation. We fix $\eta = 11.617$ and we move along the horizontal line HBP in Figure 1 from the left to the right, by increasing a from the lower Hopf critical point (H with $a = 0.00295$) to the higher Hopf critical point (H with $a = 0.0028818558$) and, eventually, to the transcritical critical value (BP with $a = 0.002952$). The following table summarizes the codimension-one bifurcations and provides also their steady state values.

Bifurcation	Hopf (H)	Hopf (H)	transcritical (BP)
a	0.0028818558	0.00295	0.002952
l_1	$-2.307495 * 10^{-5}$	$8.388102 * 10^{-5}$	
Steady state	$\mu = 0.054460537$ $k = 27.835574$ $l = 0.98403822$ $P = 1.0080777$	$\mu = 0.039004$ $k = 27.835574$ $l = 0.999490$ $P = 1.000255$	$\mu = 0.03857979$ $k = 27.835574$ $l = 1$ $P = 1$
Eigenvalues	$\lambda_1 = -0.0195198$ $\lambda_2 = 0.0105472$ $\lambda_3 = -0.0179195i$ $\lambda_4 = 0.0179195i$	$\lambda_1 = -0.00140323$ $\lambda_2 = 0.00794332$ $\lambda_3 = -0.0138706i$ $\lambda_4 = 0.0138706i$	$\lambda_1 = 0$ $\lambda_2 = 0.00751033$ $\lambda_3 = -0.000231292 - 0.0136657i$ $\lambda_4 = -0.000231292 + 0.0136657i$

The lower Hopf bifurcation is supercritical and the corresponding limit cycle around the endemic steady state in Figure 2 is stable (negative first Lyapunov coefficient l_1).

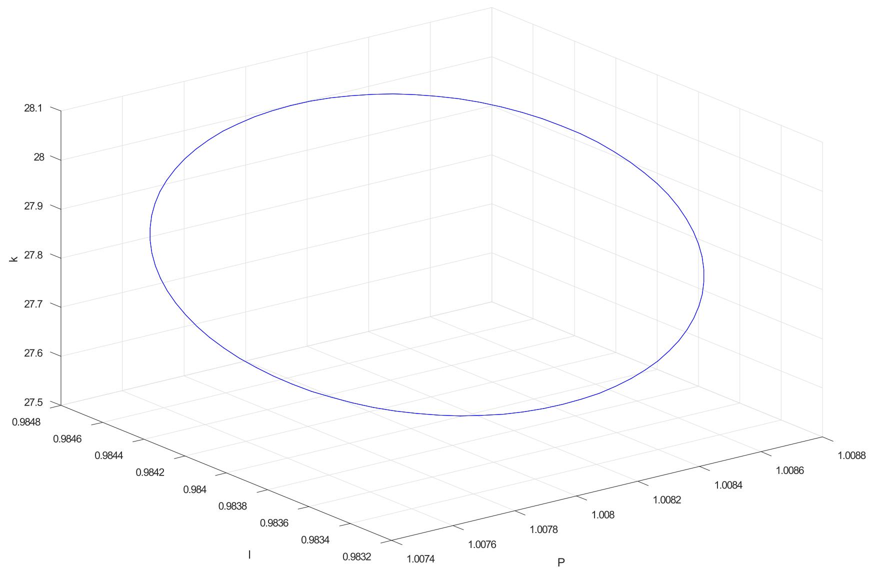

Fig. 2. Stable limit cycle

Conversely, the higher Hopf bifurcation is subcritical and the corresponding limit cycle around the endemic steady state in Figure 3 is unstable (positive l_1).

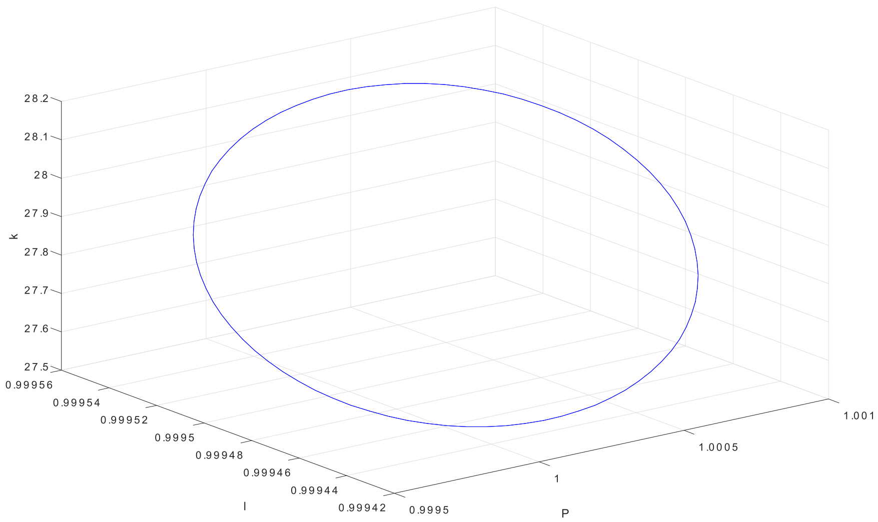

Fig. 3. Unstable limit cycle

As seen above, the lower and the higher Hopf bifurcation belong to the same U-shaped curve. Since the corresponding limit cycles have opposite stability properties (the one is stable while the other is unstable), we expect a generalized-Hopf bifurcation taking place somewhere along the Hopf bifurcation curve when the first Lyapunov coefficient crosses zero moving from $l_1 = -2.307495 * 10^{-5} < 0$ to $l_1 = 8.388102 * 10^{-5} > 0$. Indeed, at a generalized-Hopf critical point, the Hopf bifurcation from supercritical becomes subcritical with $l_1 = 0$.

In order to draw the Hopf bifurcation curve, we relax η and we ask MATCONT to represent all the points (a, η) at which a Hopf bifurcation occur (function (31) and Figure 1). As expected, MATCONT detects a generalized-Hopf bifurcation (GH) in an intermediary point. The following table summarizes the relevant informations about the codimension-two generalized-Hopf bifurcation:

Bifurcation	Generalized-Hopf (GH)
Codimension two	$a = 0.0029417316$ $\eta = 10.806627$
l_2	$-1.430066 * 10^{-6}$
Steady state	$\mu = 0.040481846$ $k = 27.835574$ $l = 0.99762158$ $P = 1.001913$
Eigenvalues	$\lambda_1 = -0.00545026$ $\lambda_2 = 0.0101273$ $\lambda_3 = -0.0134511i$ $\lambda_4 = 0.0134511i$

where l_2 denotes the second Lyapunov coefficient.[8] $l_2 \neq 0$ ensures that the generalized-Hopf bifurcation is non-degenerated (see Section 8.3 in Kuznetsov (1998) for more details).

To complete the bifurcation analysis, we draw also the transcritical bifurcation curve, that is, the set of points in the (a, η)-space at which a transcritical bifurcation arises. This locus is given by the vertical line in Figure 1. Interestingly, when conditions for both the Hopf and the transcritical bifurcation are jointly satisfied, the system undergoes a Gavrilov-Guckenheimer bifurcation (ZH). The following table summarizes all the relevant informations:

Bifurcation Type	Gavrilov-Guckenheimer (ZH)
Codimension two	$a = 0.002952$ $\eta = 11.882664$
Normal form coefficients	$s = -1$ $\theta = -7.575717 * 10^{-2}$ $E(0) = 1$
Steady state	$\mu = 0.038579789$ $k = 27.835574$ $l = 1$ $P = 1$
Eigenvalues	$\lambda_1 = 0$ $\lambda_2 = 0.00704774$ $\lambda_3 = -0.0141091i$ $\lambda_4 = 0.0141091i$

Consider the coefficients of the normal form presented in the Section 8.5 of Kuznetsov (1998): $s = -1$ jointly with $\theta = -7.575717 * 10^{-2} < 0$ means that the limit cycle around the endemic steady state in Figure 4 is unstable,

[8] MATCONT computes l_2 when a generalized-Hopf bifurcation is detected.

while $E(0) \neq 0$ ensures that the Gavrilov-Guckenheimer bifurcation is non-degenerated.[9]

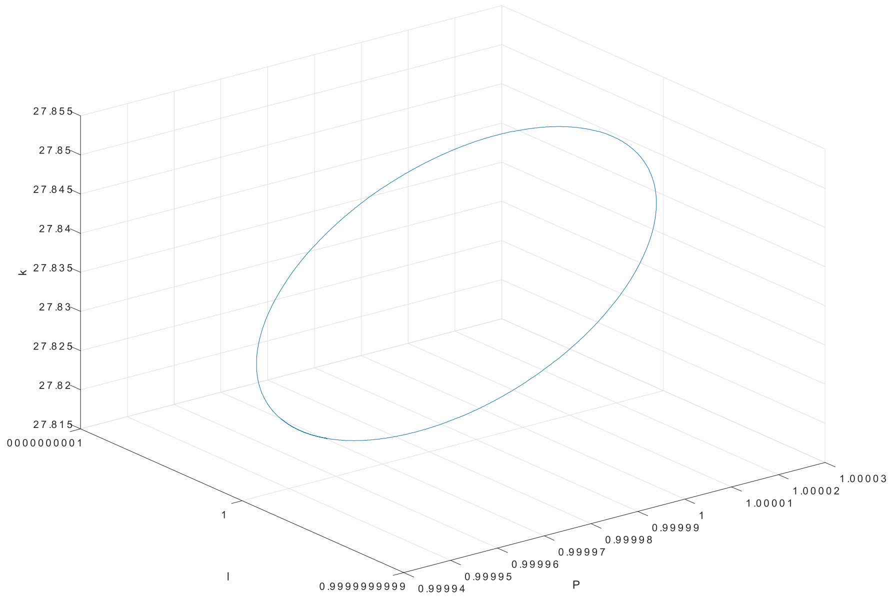

Fig. 4. Limit cycle at the GG bifurcation point

In sections 4 and 5, we have considered the destabilizing effect of pollution. A limit cycle means also a fluctuation in pollution associated to intertemporal inequality. Indeed, in this case, some generation enjoys a low pollution level while the following experiences a high level. A government interested in inequality reduction can tune the green tax to avoid any bifurcation giving rise to cycles. Figure 5 represents the bifurcation value η_H as a function of τ in the

[9] MATCONT computes s, θ and $E(0)$ when a Gavrilov-Guckenheimer bifurcation is detected.

interval $(0, 0.2)$ according to calibration (30) with $a = 0.00295$.

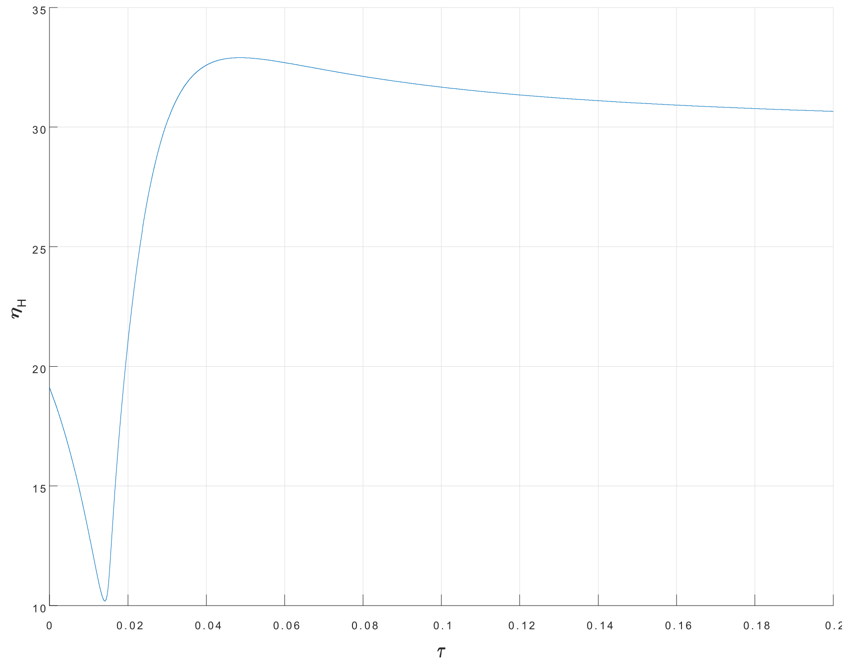

Fig. 5. η_H depending on τ

The curve in Figure 5 is generated by (29) and illustrates the ambiguous impact of taxation on the Hopf critical value. If we assume that low values for the aversion to pollution η are more plausible, then the introduction of a green tax (with rates around 1%) can promote the occurrence of undesirable cycles for future generations. In addition, the ambiguous effect of taxation on macroeconomic stability can be compatible with the its unambiguous impact on capital intensity, labor supply and pollution level (Proposition 5). In other terms, a well-intentioned policy for a cleaner world may lead to a higher macroeconomic volatility and a larger intertemporal inequality. This unpleasant consequence could be reinterpreted as a new form of green paradox.

7 Conclusion

In this paper, we have developed a unified framework, at the crossroad of economy, ecology and epidemiology, where a pollution externality, coming from production, affects both the spread of infectious diseases and the consumption demand. We have embedded a standard SIS model into a Ramsey model. This unified framework allows us to recover either in the short or the long run some important results of different strands of literature: (1) on the epidemiological side, the coexistence of a disease-free and an endemic steady state; (2) on the ecological side, the possible occurrence of limit cycles when pollution increases the consumption demand and pollution has a strong inertia. Moreover, the complex interplay between the pollution effect on disease transmission and consumption demand implies richer dynamics, namely the occurrence of a Gavrilov-Guckenheimer bifurcation. To convince the reader about the plausibility of our theoretical results, we have provided a numerical illustration of the preservation of the limit cycle after the Gavrilov-Guckenheimer bifurcation. Finally, we have pointed out the ambiguous role of a green tax: a higher green-tax rate can lower the Hopf bifurcation degree of pollution aversion in the utility function, making the occurrence of limit cycles more likely. In this sense, a heavier green tax intended to clean the environment, can promote the macroeconomic volatility. Such unpleasant effect of the green tax can be viewed as a new facet of the Green Paradox.

8 Appendix

Proof of Proposition 1

The consumer's Hamiltonian function writes

$$H \equiv e^{-\theta t} u(c, P) + \lambda \left[(r - \delta) h + \bar{\omega} - c \right]$$

The first-order conditions are given by $\partial H / \partial \lambda = (r - \delta) h + \bar{\omega} - c = \dot{h}$, $\partial H / \partial h = \lambda (r - \delta) = -\dot{\lambda}$, $\partial H / \partial c = e^{-\theta t} u_c - \lambda = 0$. Setting $\mu \equiv e^{\theta t} \lambda$, we find $\dot{\mu} - \theta \mu = e^{\theta t} \dot{\lambda}$ and, therefore, $\mu (r - \delta - \theta) = -\dot{\mu}$. Finally, the budget constraint becomes binding because of the multiplier positivity. ■

Proof of Lemma 3

(20) jointly with Assumption 1 yields $\lim_{l \to 0} g(l) = +\infty$ and $\lim_{l \to +\infty} g(l) = -\infty$. We observe that g is a continuous function. According to the Intermediate Value Theorem, there exists at least one value $l > 0$ such that $g(l) = 0$. Monotonicity of $g(l)$ for any $l > 0$ is a sufficient condition to get a unique positive value. Monotonicity is also satisfyied under Assumption 1:

$$g'(l) = \left[\gamma'(P) - l \beta'(P) \right] \frac{b - m\tau}{a} f(k^*) - \beta(P) < 0$$

■

Proof of Proposition 4

Simply consider (19), (20) and (21) jointly with Lemma (3). ■

Proof of Proposition 5

Totally differentiating system (15)-(18) at the endemic steady state, we obtain

$$\begin{bmatrix} \frac{\tau}{\mu} \frac{d\mu}{d\tau} \\ \frac{\tau}{k} \frac{dk}{d\tau} \\ \frac{\tau}{l} \frac{dl}{d\tau} \\ \frac{\tau}{P} \frac{dP}{d\tau} \end{bmatrix} = \begin{bmatrix} 0 & (1 - \tau) \frac{1 - \alpha}{\sigma} & 0 & 0 \\ -\kappa \frac{1}{\varepsilon_{cc}} & \theta & \kappa + \beta - \gamma & d(\beta - \gamma) + \kappa \frac{\varepsilon_{cP}}{\varepsilon_{cc}} \\ 0 & 0 & 1 & d \\ 0 & \alpha & 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} -\tau \\ \frac{\tau}{1 - \tau} \frac{\theta + \delta}{\alpha} \\ 0 \\ \frac{m\tau}{b - m\tau} \end{bmatrix}$$

Moreover,

$$W^* = \int_0^{+\infty} e^{-\theta t} u\left(c^*, P^*\right) dt = u\left(c^*, P^*\right) \int_0^{+\infty} e^{-\theta t} dt = \frac{1}{\theta} u\left(c^*, P^*\right)$$

■

Proof of Proposition 6

$b \geq m$ implies $\varphi < 1$ and, then, according to (23),

$$\frac{\tau}{c^*}\frac{\partial c^*}{\partial \tau} = \frac{1+(1-\varphi)d}{1+d}\frac{\tau}{k^*}\frac{\partial k^*}{\partial \tau} < 0$$

■

Proof of Proposition 7

According to (22) and (23),

$$\frac{\tau}{W^*}\frac{\partial W^*}{\partial \tau} = \left[\varepsilon_c \frac{1+(1-\varphi)d}{1+d} + \varepsilon_P \frac{\varphi}{1+d}\right]\frac{\tau}{k^*}\frac{\partial k^*}{\partial \tau} \quad (32)$$

Under Assumption 5, $\varphi \in (0,1)$. The RHS of (32) is positive iff (24) holds. ■

Proof of Proposition 8

A bifurcation of the saddle-node family occurs if and only if $D = 0$ (see Bosi and Desmarchelier, 2017). In our model, the type of bifurcation the system experiences, is a transcritical because two steady states coalesce while exchanging their stability properties (see Proposition 4). We observe that $D = 0$ if and only if $\beta = \gamma$. ■

Proof of Proposition 9

According to Bosi and Desmarchelier (2017), a limit cycle generically arises through a Hopf bifurcation if and only if

$$S_2 = \frac{S_3}{T} + \frac{T}{S_3} D \quad (33)$$

and T and S_3 have the same sign. We observe that $S_2 = a_2 x + b_2$ and $S_3/T = x + b_3$, where

$$\begin{aligned} a_2 &\equiv 1 + \frac{\theta - a}{\gamma - \beta} \\ x &\equiv a\alpha\eta\kappa \frac{\varepsilon - 1}{\varepsilon} \frac{\gamma - \beta}{\theta - a + \gamma - \beta} \end{aligned}$$

and b_2 and b_3 are given by (27) and (28).

Notice that a_2, b_2, b_3 and D do not depend of η because $\gamma^* = \gamma\left(P^*\right)$, $\beta^* = \beta\left(P^*\right)$ and P^* does not depend of η:

$$P^* = \left[\frac{b - m\tau}{a}\frac{B_\gamma}{B_\beta} f\left(k^*\right)\right]^{\frac{1}{1+d}}$$

Therefore, (33) becomes

$$x^2 + \left(z + b_3\right) x + y + z b_3 = 0$$

where

$$y \equiv \frac{D}{1 - a_2} \text{ and } z \equiv \frac{b_3 - b_2}{1 - a_2}$$

The solutions are

$$x_{\pm} = \frac{-z - b_3 \pm \sqrt{(z - b_3)^2 - 4y}}{2}$$

Under the assumption $a < \theta$, we have $(z - b_3)^2 - 4y > 0$. We observe that

$$\left(\frac{S_3}{T}\right)_{\pm} = x_{\pm} + b_3 = \frac{b_3 - z \pm \sqrt{(b_3 - z)^2 - 4y}}{2}$$

and, since $y < 0$, $(S_3/T)_- < 0 < (S_3/T)_+$. The occurrence of a Hopf bifurcation requires the same sign for T and S_3, that is $S_3/T > 0$. Therefore, we consider only $(S_3/T)_+$.

The bifurcation point is given by

$$\eta_H \equiv \frac{\varepsilon}{\varepsilon - 1} \frac{a_2}{a\alpha\kappa} x_+ = \frac{\varepsilon}{1 - \varepsilon} \frac{a_2}{2a\alpha\kappa} \left[z + b_3 - \sqrt{(z - b_3)^2 - 4y}\right]$$

where the RHS does not depend of η. This espression is identical to (29). ■

Proof of Proposition 10

According to Bosi and Desmarchelier (2017), a Bogdanov-Takens bifurcation occurs if and only if $D = S_3 = 0$. $D = 0$ implies $\beta = \gamma$ (see (26)) and, therefore, $S_3 = (1 - \alpha)(\theta + \delta) a\kappa/\varepsilon > 0$, which violates the condition for a Bogdanov-Takens bifurcation. ■

Proof of Proposition 11

As shown in Bosi and Desmarchelier (2017), a Gavrilov-Guckenheimer bifurcation of a four-dimensional system generically arises if and only if $D = 0$ and $S_3 = TS_2$ with $S_2 > 0$. $D = 0$ if and only if $\beta = \gamma$. Solving $S_3 = TS_2$ for η, we get $\eta = \eta_{GG}$. In addition,

$$S_2(\eta_{GG}) = \frac{a\kappa}{\varepsilon} \frac{(1 - \alpha)(\theta + \delta)}{\theta - a} > 0$$

because $a < \theta$ by assumption. ■

Proof of Proposition 12

At the endemic steady state, $\beta > \gamma$ and, therefore, $D < 0$. As shown in Bosi and Desmarchelier (2017), a necessary condition for the occurrence of a Double-Hopf bifurcation is generically $D > 0$. The proposition follows. ■

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