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« Equilibria in discrete and continuous second price all-pay auctions, convergence or yoyo phenomena »

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Equilibria in discrete and continuous second price all-pay auctions, convergence or yoyo phenomena.

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Abstract

The paper is about mixed strategy Nash equilibria in discrete second price all-pay auctions with a limit budget. Two players fight over a prize of value V. Each player submits a bid lower or equal to M, the limit budget. The prize goes to the highest bidder but both bidders pay the lowest bid. V, M and the bids are integers. The paper studies the convergence of the mixed Nash equilibrium probability distribution in the discrete auction to the mixed Nash equilibrium probability distribution in the more well-known continuous second price all-pay auction –or static war of attrition.

We establish that the- expected- convergence between discrete and continuous equilibrium distributions is in no way automatic. Both distributions converge for V odd and large, but, for even values of V, the discrete distribution is quite strange and obeys a singular yoyo phenomenon: the probabilities assigned to two adjacent bids are quite different, one probability being much lower than the continuous one, the adjacent probability being much larger. So the discrete probabilities, for V even, don't converge to the continuous ones. Yet there is a convergence, when turning to sums: the sums of the discrete probabilities of two adjacent bids converge to the sums of the continuous probabilities of the same two bids for large values of V.

It is shown in the paper that the yoyo phenomenon doesn't disappear - it is even strengthened- when switching to lower natural bid increments, like 0.5 or 0.1. More generally, it is shown that convergence is an exception rather than the rule and that it requires a special link between V, M and the bid increment. It follows a lack of continuity between the discrete Nash equilibria and the continuous Nash equilibria.

Keywords: discrete game, continuous game, second price all-pay auction, Nash equilibrium, increment.

JEL Classification: C72 - D44

1. Introduction

The paper is about second price all-pay auctions with a limit budget. Two players fight over a prize of value V. Each player submits a bid lower or equal to M, the limit budget. The prize goes to the highest bidder but both bidders pay the lowest bid.

The continuous version of this game is isomorphic to the static war of attrition in continuous time, where each player has to choose a time t in the interval [0,1] to leave the game (1 plays the role of M); staying in the game is costly (the cost increases in time) but, as soon as one player leaves the game, the game stops and the other gets the prize (this amounts to saying that

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if player i bids less than player j, player j gets the prize and both pay player i's bid) (see for example Hendricks, Weiss and Wilson (1988)). So the mixed symmetric Nash equilibrium of this game has been studied by several authors, among them Hendricks et al. (1988).
In this paper, we are more particularly interested in the discrete version of the game, where V, M and the bids are integers. More specifically, we study the convergence of the mixed symmetric Nash equilibrium probability distribution in the discrete game to the mixed symmetric Nash equilibrium probability distribution in the continuous game.
We are motivated by the fact that experiments and class-room experiments of –both first and second price- all-pay auctions often focus on a discrete game, but often compare the players' behaviour to the mixed equilibrium of the continuous game. For example Hörisch & Kirchkamp (2010), Bilodeau et al.(2004) work with seconds, Gneezy and Smorodinsky (2006) work with an integer credit of points, Noussair and Silver (2006), Lugovskyy et al. (2010) work with experimental currency (an integer amount).
So it becomes crucial to know if the mixed Nash equilibria in the discrete games converge to the equilibria in the continuous game. We establish that this- yet expected- convergence, for second price all-pay auctions, is in no way automatic. Both distributions converge for V odd and large, but, for even values of V, the discrete distribution is quite strange and obeys a singular yoyo phenomenon: the probabilities assigned to two adjacent bids are quite different, one probability being much lower than the continuous one, the adjacent probability being much larger. So the discrete probabilities, for V even, don't converge to the continuous ones. Yet there is some convergence, when turning to sums: the sums of the discrete probabilities of two adjacent bids converge to the sums of the continuous probabilities of the same two bids for V large.
In fact the nice convergence obtained for V odd and a bid increment equal to 1 is an exception rather than the rule. This result namely explains that the yoyo phenomenon doesn't disappear - it is even strengthened- when switching to very natural lower bid increments, like 0.5 or 0.1. It also highlights a lack of continuity between the discrete Nash equilibria and the continuous Nash equilibria.
Section 2 recalls the mixed Nash equilibrium in the continuous game. Section 3 studies the mixed Nash equilibria in the discrete game. It establishes the convergence of the continuous and discrete equilibria when V is odd and large, M, V and the bids being integers, before turning to the strange discrete equilibrium distribution for even values of V. It details this distribution, the oscillations of the probabilities, their shape and evolution, the absence of convergence of the discrete probabilities to the continuous ones, but also the convergence of the sums of the discrete probabilities of two adjacent bids to the sums of the continuous probabilities of the same bids. Section 3 also highlights that the yoyo phenomenon is strengthened when switching to lower natural bid increments, like 0.5 and 0.1. Finally it generalizes the approach by establishing that convergence requires a special link between V, M and the bid increment. It derives that convergence is rather a rare event, even if the discrete game goes to the continuous game, i.e. if the bid increment goes to 0. Section 4 illustrates the obtained results. Section 5 concludes.

2. Mixed Nash equilibria in the continuous second price all-pay auction.

Two players have a limit budget M. They fight over a prize of value V. Each player i submits a bid b_i, i =1, 2 lower or equal to M. The prize goes to the highest bidder but both bidders pay the lowest bid. In case of a tie, the prize goes to each bidder with probability ½. Usually, we fix M≥V. Yet, for the mixed Nash equilibrium, we only need M>V/2.

In this section, we work with the continuous version of the game, where M, V and the bids are real numbers. This game, for M≥V, like the discrete version of the game, has a lot of asymmetric pure strategy Nash equilibria where one player bids 0 and the other plays a bid in [V, M]. Yet in this paper we only focus on the symmetric mixed Nash equilibrium.

We briefly recall the structure of this equilibrium.

Folk result

All the bids in [M-V/2, M[are weakly dominated by M.

The symmetric mixed Nash equilibrium in the continuous game is given by: b is played with probability f(b)db, with $f(b)= e^{-b/V}/V$ *for b in[0, M-V/2], M is an atom played with probability* $f(M) = 1-F(M-V/2) = e^{1/2-M/V}$*, and b in]M-V/2,M[is played with probability 0, where f(x) and F(x) are the density probability distribution and the cumulative probability distribution.*

Proof see Appendix 1

Let us comment on these equilibria. Figures 1a and 1b give the general form of the probability distribution, for V=15 and M=25 (Figure 1a), for V=8 and M=12 (Figure 1b). Both distributions will be compared to discrete distributions in section 4.

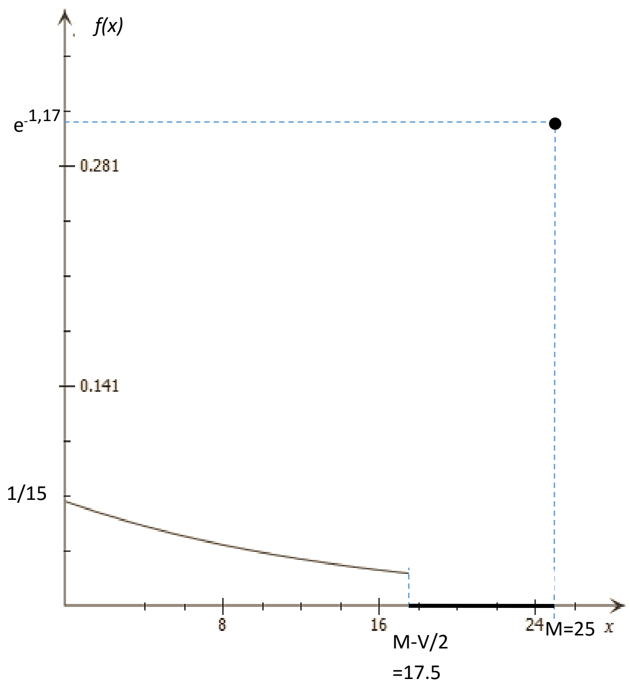

Figure 1a

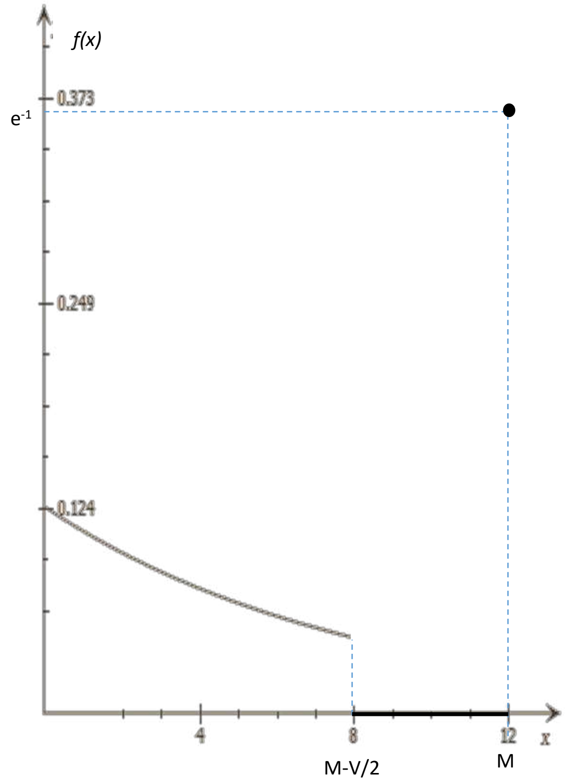

Figure 1b

Both figures are similar. There is an atom on M, which is played with probability $e^{1/2-M/V}$. This probability is quite large as long as the ration M/V is small (for example lower or equal to 2), but falls fast when M/V grows. The probabilities on bids b from 0 to M-V/2 are given by a decreasing exponential function f(b)= $e^{-b/V}/V$ that assigns probability db/V to the bid 0 and becomes flatter when V becomes large. There is no direct link between f(0) and f(M). We namely observe that f(0) doesn't depend on M, f(0)→0 when V→∞ whereas f(M) goes to 0 only if M/V→∞; for example, if M and V go to ∞ but M/V=2, then f(0)→0 but f(M)→$e^{-1.5} \neq 0$.

3. Nash equilibria in the discrete second price all-pay auction, partial convergence and yoyo phenomena

In experiments the game is seldom continuous. Usually players work with seconds, with experimental (integer) currency, credit of points. So the possible bids are seldom real numbers, but rather integers or at most decimal numbers (for example when the players are allowed to bids dollars and cents). That is why we now focus on discrete second price all-pay auctions, where M and V are integers.

We first focus on the case where the bids are integers too, so we fix the bid increment equal to 1. We call q_i the probability a player assigns to bid i (i is equal to M or goes from 0 to M-V/2-1 if V is even, to M-V/2-1/2 if V is odd). We get the following results.

Proposition 1

V, M and the bids are integers. For both even and odd values of V, the main recurrence equations that define the probabilities are:

$q_i = 2q_{i+1}/V+q_{i+2}$ *i from 0 to M-V/2-5/2 (V odd).* (1a)

$q_i = 2q_{i+1}/V+q_{i+2}$ *i from 0 to M-V/2-3 (V even).* (1b)

The additional equations are:

$q_{M-V/2-1/2} = q_M/V$ *and* $q_{M-V/2-3/2} = q_M(1/V+2/V^2)$ *for V odd.* (2a)

$q_{M-V/2-1} = 2q_M/V$ *and* $q_{M-V/2-2} = 4q_M/V^2$ *for V even.* (2b)

$\sum_{i=0}^{M-\frac{V}{2}-1/2} q_i + q_M = 1$ *for V odd.* (3a)

$\sum_{i=0}^{M-\frac{V}{2}-1} q_i + q_M = 1$ *for V even.* (3b)

Proof see Appendix 2

Proposition 2

V, M and the bids are integers and V is an odd number:

- q_i decreases in i, i being an integer in [0, M-V/2-1/2].

- When V becomes large, q_i=f(i) solves the equations (1a), (2a) and (3a) (f(i) being the density function in the continuous equilibrium). The discrete mixed Nash equilibrium goes to the continuous mixed Nash equilibrium for large values of V.

Proof see Appendix 3

Proposition 3

V, M and the bids are integers and V is an even number:

- $q_i=f(i)$ solves the recurrence equations (1b), but the discrete mixed Nash equilibrium doesn't get close to the continuous mixed Nash equilibrium for large values of V.

- For large values of V, q_i+q_{i+1} and $f(i)+f(i+1)$ converge to a same value, i from 0 to M-V/2 -2, at least if M-V/2 is even. So we get a convergence in probabilities when summing the probabilities two by two.

- The probabilities obey a yoyo phenomenon. For large values of V and $M<V^{1.9}/2+V/2$, $q_i \rightarrow 2q_M/V$ for i = M-V/2-1-2k (k from 0 to E((M-V/2-1)/2) and $q_{M-V/2-2i} \rightarrow 4iq_M/V^2$ (→0 for high bids (small values of i)), i from 1 to E((M-V/2)/2) (where E(x) is the integer part of x) .

Proof see Appendix 4

Let us comment on these three propositions. Working with a bid increment equal to 1, which seems a good way of proceeding given that M and V are integers, gives contrasted results, depending on whether V is odd or even. When V is odd we get a nice convergence of the discrete distribution to the continuous one, at least for large values of V (but we show in section 4 that this convergence is good even for small values of V). By contrast, when V is even, we get no convergence. More precisely, when V is large, we get a yoyo phenomenon: among the probabilities of two adjacent bids, especially for high bids, one probability goes to 0, whereas the other is close to $2q_M/V$, which keeps away from 0. Moreover the probabilities q_i, i odd, and the probabilities q_i, i even, evolve in a quite different way, the ones being all close to $2q_M/V$, the others decreasing linearly in i. This yoyo phenomenon, established for V large, is in fact observed even for small values of V (see section 4).

The strong difference in the results obtained for V odd and even seems due to the fact that the two largest played bids (after M) are M-V/2-1/2 and M-V/2-3/2 when V is odd, and M-V/2-1 and M-V/2-2 when V is even. So we could be tempted to switch to another bid increment such that we work with the same largest bids (after M) regardless of the fact that V is odd or even. An a priori good way to do would be to switch to a bid increment 0.5: in that case, the two largest played bids (after M) are M-V/2-0.5 and M-V/2-1 for V odd and even. Another spontaneous way to do would be to switch to a bid increment 0.1: in that case, the two largest played bids (after M) are M-V/2-0.1 and M-V/2-0.2 for V odd and even. Yet this way to do is not a good idea.

Proposition 4

V and M are integers.

For a bid increment equal to 0.5 the main recurrence equations are:

$q_i= q_{i+0.5}/V+q_{i+1}$ *i from 0 to M-V/2-1.5 (i =0.5l, l being an integer).* *(4a)*

The additional equations are: $q_{M-V/2-0.5} = q_M/V$ *and* $q_{M-V/2-1} = q_M/V^2$. *(4b)*

For a bid increment equal to 0.1 the main recurrence equations are:

$q_i= 0.2q_{i+0.1}/V+q_{i+0.2}$ *i from 0 to M-V/2-0.3 (i=0.1l, l being an integer).* *(5a)*

The additional equations are: $q_{M-V/2-0.1} = 0.2q_M/V$ *and* $q_{M-V/2-0.2} =0.04q_M/V^2$. *(5b)*

The yoyo phenomenon, described in proposition 3 is observed for both increments, for V odd and even. The strength of the yoyo phenomenon even increases when the bid increment decreases.

Proof see Appendix 5

So clearly, working with smaller bid increments such that it doesn't matter if V is odd or even is not a good idea, given that we get a stronger yoyo phenomenon for both odd and even values of V. More generally, the nice convergence observed for V odd and a bid increment equal to 1 is rather a rare event.

Proposition 5

M and V are integers, I is the bid increment and r is the remainder of the division of M-V/2 by I. The discrete Nash equilibrium converges to the continuous Nash equilibrium for large values of V if and only if r is equal to I/2 for large values of V. It follows that the discrete Nash equilibrium can't converge to the continuous one for I= 10^{-N}, regardless of N, where N is an integer higher or equal to 1. It derives a lack of continuity between the discrete equilibria and the continuous equilibria, in that we do not get the convergence of the discrete Nash equilibrium to the continuous equilibrium when the discrete game converges to the continuous one, i.e. when the bid increment goes to 0.

Proof see Appendix 6

Given that M and V are integers, it derives from proposition 5 that a possible convergence of the discrete Nash equilibrium to the continuous Nash equilibrium is not often observed. It is possible for I = 1 and V odd, given that r = 0.5 = I/2. But, for example, it isn't possible for the other studied cases, I = 0.5 or I = 0.1 because r = 0 regardless of V, and I = 1 and V even because r=0. In the same way, when looking for the increments I=0.1l, l being an integer, I= 0.3 will not allow convergence, because (M-V/2-0.15)/0.3 can't be an integer; a similar observation holds for I=0.7 and I=0.9. For I=0.4 it is possible to get values of V and M that ensure that (M-V/2-0.2)/0.4 is an integer but we get a special class of values V and M; the same is true for I=0.8. Things are different for I=0.2. In that case, r is equal to 0.1 provided that V is odd. When switching to larger bid increments, for example I=2, it is sufficient that V is even and that M-V/2 is odd to get convergence. But, as follows from these observations, convergence is not easy to get. Especially, when I =10^{-N}, N being an integer higher or equal to 1, then r is equal to 0 (because M and V are integers, so M-V/2 is an integer or a multiple of 0.5). Given that N can be very large, this means that, even for very small bid increments, i.e. for a discrete game that goes to the continuous one, there are discrete equilibria that do not converge to the continuous equilibria. So there is a lack of continuity between the discrete equilibria and the continuous equilibria, a result that was not expected.

Proposition 6

M and V are integers, I is the bid increment and r is the remainder of the division of M-V/2 by I. When r is equal to 0, then the discrete mixed Nash equilibrium doesn't get close to the continuous mixed Nash equilibrium for large values of V but, at least when (M-V/2)/I is even, *the sums of the discrete equilibrium probabilities of two adjacent bids converge to the sums of the continuous equilibrium probabilities of the same two bids. Yet the yoyo phenomenon is strengthened when I decreases.*

Proof see Appendix 6

Proposition 6 shows that the partial convergence result obtained for a bid increment 1 and even integers V extends to all bid increments I such that the remainder of the division of M-V/2 by I is zero. The bid increment 0.5 obeys this rule. More generally, if M and V are integers, all the increments 10^{-N}, with N an integer higher or equal to 1, ensure the partial convergence result.

4. Illustrations

Let us illustrate some obtained results and some others features of the discrete Nash equilibria. We first work with odd values of V and integer bids, to illustrate the nice convergence between the continuous mixed Nash equilibrium and the discrete mixed Nash equilibrium. We study the two cases V=9 and M=12, V=15 and M=25, in order to discuss the obtained convergence.

For V=9 and M=12 we get the results in table 1.

Bids	0	1	2	3	4	5	6	7	8 9 10 11	12
Discrete NE q_i	$\frac{11051129}{104252249}$ = 0,106004	$\frac{9660231}{104252249}$ = 0,092662	$\frac{8904411}{104252249}$ = 0,085412	$\frac{7681473}{104252249}$ = 0,073682	$\frac{7197417}{104252249}$ = 0,069038	$\frac{6082047}{104252249}$ = 0,05834	$\frac{5845851}{104252249}$ = 0,056074	$\frac{4782969}{104252249}$ = 0,045879	0	$\frac{43046721}{104252249}$ = 0,412909
Adj.cont.NE Adj f(i)	0,100967	0,090349	0,080848	0,072346	0,064738	0,057930	0,051838	0,046386	0	0,434598
Cont. NE f(i)	0,111111	0,099425	0,088971	0,079615	0,071242	0,063750	0,057046	0.051047	0	0,434598
q_i/Adj f(i)	1,049888	1,025601	1,056452	1,018467	1,066422	1,007078	1,081716	0,98907		0,950094

Table 1: NE means Nash equilibrium

Let us comment on table 1. The discrete Nash equilibrium probabilities, respectively the continuous Nash equilibrium probabilities, are given in the second row, respectively in the fourth row. To compare q_i and f(i) we have to take into account that, in the continuous game, the bids (real numbers) are played with probability f(b)db, with the exception of bid 12. As a consequence, we weight the probabilities f(i), i from 0 to 7, by multiplying them by (1-f(12))/($\sum_{i=0}^{7} f(i)$)(=0.9087), to get the adjusted continuous Nash equilibrium probabilities Adj f(i) in row 3[1]. The comparison of row 2 and row 3 shows that the discrete probabilities and the adjusted continuous probabilities are similar, despite V is not large. To evaluate the convergence we calculate $(\sum_{i=0}^{6} \frac{q_i}{Adj\ f(i)} + \frac{Adj\ f(7)}{q_7} + \frac{f(12)}{q_{12}})/9$ (we inverse the ratios when they are below 1, in order to not lower the differences). We get a rather good mean ratio 1.041.

Observe that the convergence is not uniform. We here focus on the bids different from 12. Despite all the discrete probabilities get close to the adjusted continuous ones, there is a kind of little oscillation as regards the ratios: 1.049888>1.025601<1.056452>1.018467

[1] The weight disappears (goes to 1) for V large: $(1-e^{1/2-M/V})/\sum_{i=0}^{M-\frac{V}{2}-1/2} f(i) = \frac{(1-e^{\frac{1}{2}-\frac{M}{V}})V(1-e^{-\frac{1}{V}})}{(1-e^{-\frac{M-\frac{V}{2}+\frac{1}{2}}{V}})}$ →1 for V large.

<1.066422>1.007078<1.081716>1/0.98907. We observe also that the ratios $\frac{q_i}{Adj\ f(i)}$ get worse (grow) when i increases for even values of i, whereas the ratios $\frac{q_i}{Adj\ f(i)}$ get better (decrease) when i increases up to 5 for odd values of i, and then get worse. And the ratios $\frac{q_i}{Adj\ f(i)}$ for i odd are all closer to 1 than the best ratio $\frac{q_i}{Adj\ f(i)}$ for i even.

The convergence of the probabilities in rows 2 and 3 and the little oscillation phenomenon are respectively illustrated in figures 2a and 2b and figure 4.

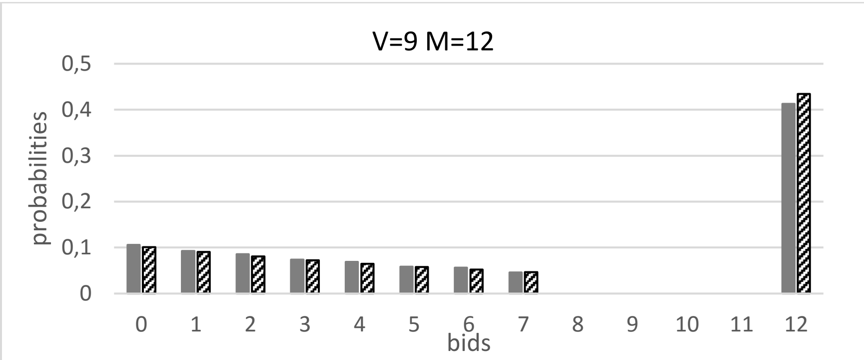

Figure 2a : The full columns are the discrete Nash equilibrium probabilities, the shaded columns are the adjusted continuous Nash equilibrium probabilities

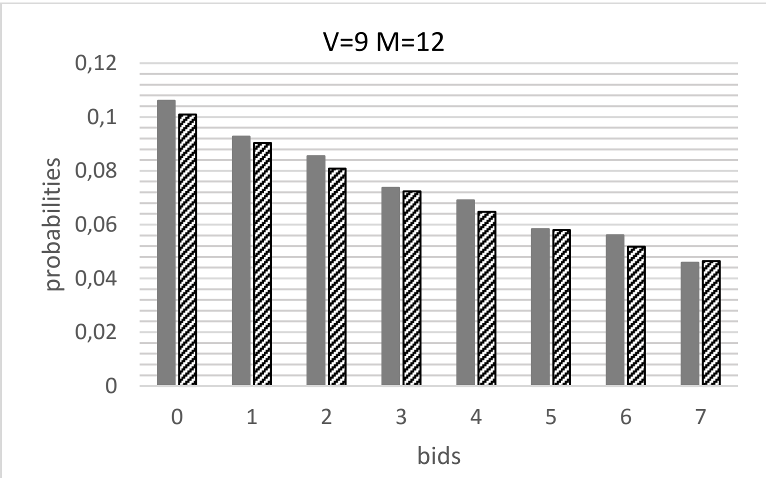

Figure 2b : It focuses on the probabilities assigned to the bids 0 to 7 to better highlight the good convergence between the discrete Nash equilibrium probabilities (full columns) and the adjusted continuous Nash equilibrium probabilities (shaded columns). The little asymmetric oscillation phenomenon is reflected in the fact that both columns for bids 1, 3, 5 and 7 are very similar whereas there is a (small) larger difference between both columns for bids 0, 2, 4 and 6.

Now we switch to the case V=15 and M=25 in order to illustrate a stronger convergence between the discrete and continuous Nash equilibria. The results are in table 2.

Bids	0	1	2	3	4	5	6	7	8	9
q_i	0,064649	0,060065	0,05664	0,052513	0,049638	0,045894	0,043519	0,040091	0,038174	0,035002
Adj f(i)	0,063551	0,059452	0,055618	0,052031	0,048675	0,045536	0,042599	0,039852	0,037282	0,034878
f(i)	0,066667	0,062367	0,058345	0,054582	0,051062	0,047769	0,044688	0,041806	0,03911	0,036587
q_i/Adj f(i)	1,017277	1,010311	1,018375	1,009264	1,019784	1,007862	1,021597	1,005997	1,023926	1,003555

Bids	10	11	12	13	14	15	16	17	18 19 20 21 22 23 24	25
q_i	0,033507	0,030534	0,029436	0,026609	0,025888	0,023158	0,0228	0,020118	0	0,301765
Adj f(i)	0,032628	0,030524	0,028555	0,026714	0,02499 1	0,023379	0,021871	0,020461	0	0,311403
f(i)	0,034228	0,03202	0,029955	0,028023	0,026216	0,024525	0,022944	0,021464	0	0,311403
q_i/Adj f(i)	1,02694	1,000328	1,030853	0,996069	1,035893	0,990547	1,042476	0,983236		0,96905

Table 2

The continuous Nash equilibrium probabilities in row 4 are multiplied by (1-f(25))/($\sum_{i=0}^{17} f(i)$)(=0.953262) to get the adjusted continuous Nash equilibrium probabilities in row 3. We observe that the discrete Nash equilibrium probabilities nicely converge to these probabilities: $(\sum_{i=0,\neq 13,15}^{16} \frac{q_i}{Adj\ f(i)} + \frac{Adj\ f(13)}{q_{13}} + \frac{Adj\ f(15)}{q_{15}} + \frac{Adj\ f(17)}{q_{17}} + \frac{f(25)}{q_{25}})/19 = 1.0177$. Observe that the convergence is again not uniform, with the same little oscillation phenomenon as in the previous study, but with a lower magnitude. When focusing on the bids from 0 to 17, we again observe that the ratios $\frac{q_i}{Adj\ f(i)}$ get worse (increase) when i increases for even values of i, and that the ratios $\frac{q_i}{Adj\ f(i)}$ get better (decrease) when i increases up to 11 for odd values of i, and then get worse. Finally we observe again that the ratios $\frac{q_i}{Adj\ f(i)}$ for i odd are all closer to 1 than the best ratio $\frac{q_i}{Adj\ f(i)}$ for i even.

The strong convergence of the probabilities q_i and Adj f(i) is illustrated in figures 3a and 3b, and the little oscillation phenomenon can be observed in figure 4.

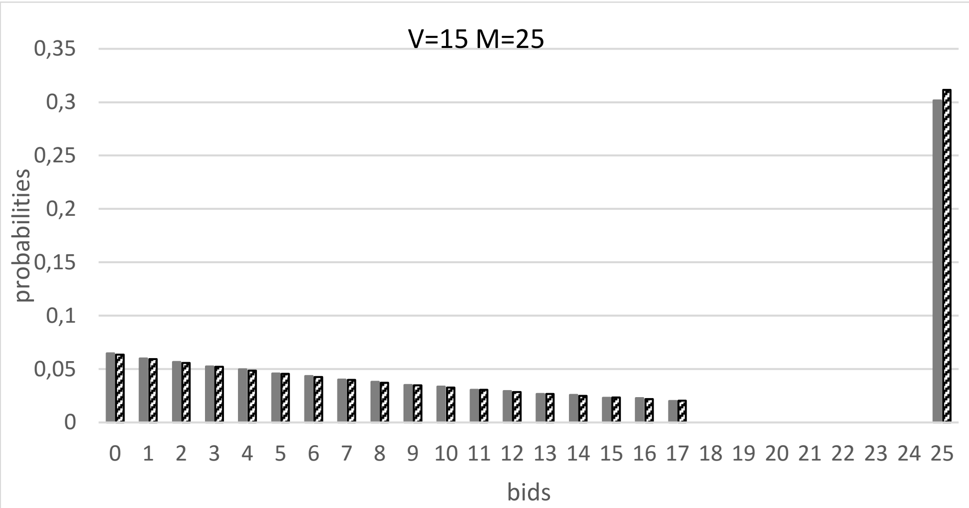

Figure 3a : The full columns are the discrete Nash equilibrium probabilities, the shaded columns are the adjusted continuous Nash equilibrium probabilities.

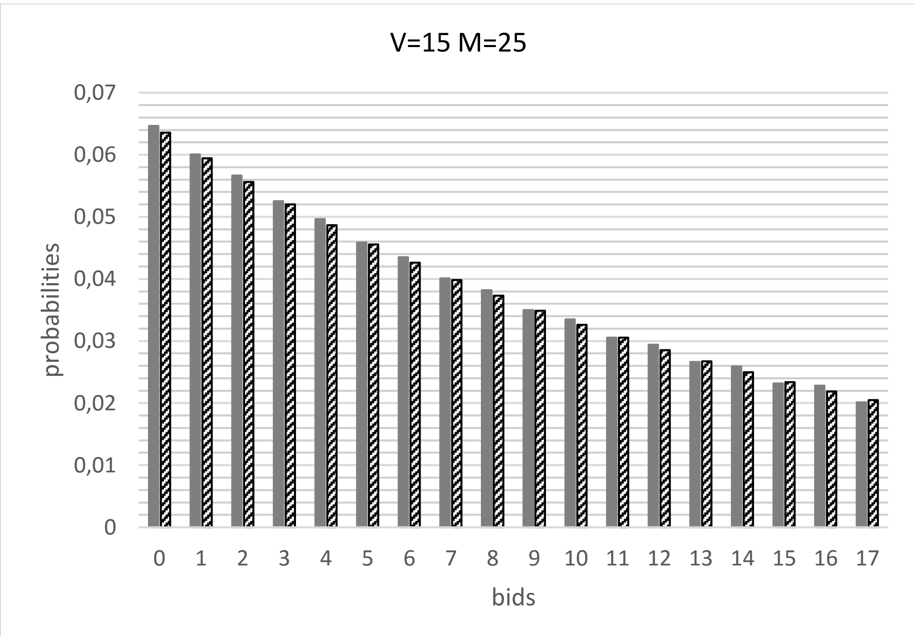

Figure 3b : It focuses on the probabilities assigned to the bids 0 to 17 to better highlight the strong convergence between the discrete Nash equilibrium probabilities (full columns) and the adjusted continuous Nash equilibrium probabilities (shaded columns). The little asymmetric oscillation phenomenon is reflected in the fact that both columns for bids 1, 3, 5, 7, 9, 11, 13 and 15 are very similar whereas there is a (small) larger difference between both columns for bids 0, 2, 4, 6, 8, 10, 12, 14 and 16.

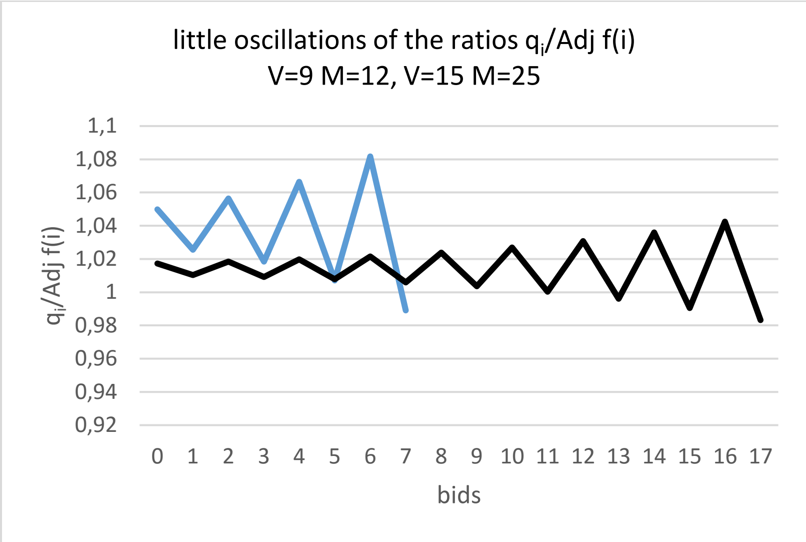

Figure 4 : The short curve represents the ratios $q_i/Adj\ f(i)$ for V=9 and M=12, the long curve represents the same ratios for V=15 and M=25. We observe the better convergence for V=15 and M=25 (in that the ratios are closer to 1), the little oscillation phenomenon, the fact that $q_i/Adj\ f(i)$ gets worse when i increases and is even, and the fact that $q_i/Adj\ f(i)$ is systematically closer to 1 for i odd.

To summarize, for V odd, the shape of the discrete Nash equilibrium probability distribution is the same than the shape of the continuous Nash equilibrium probability distribution even for V rather small (we would get this result even for very small values of V, like 3). And the discrete and continuous probabilities converge when V grows: yet the convergence is not uniform and exhibits a kind of little oscillation phenomenon that may foreshadow the strong yoyo phenomenon observed for even values of V.

As a matter of fact things are quite different for V even (the bids being still integers). We consider the two cases V=8 and M=12, V=24 and M=30, in order to show that (only) the sums of discrete probabilities 2 by 2 converges to the sums of continuous probabilities 2 by 2 when V is large, but that the difference between a discrete probability and the continuous one does not decrease with V.

We first study the case V=8 and M=12. The results are given in table 3a.

bids	0	1	2	3	4	5	6	7	8 9 10 11	12
q_i	19041/ 186613 =0,102035	22852/ 186613 =0,122457	1904/ 26659 =0,071420	19520/ 186613 0,104601	8448/ 186613 =0,04527	17408/ 186613 =0,093284	4096/ 186613 =0,021949	16384/ 186613 =0,087797	0	65536/ 186613 =0,351187
Adj f(i)	0,117503	0,103696	0,091511	0,080759	0,071269	0,062895	0,055505	0,048983	0	0,367879
f(i)	0,125	0,110312	0,097350	0,085911	0,075816	0,066908	0,059046	0,052108	0	0,367879
q_i/ Adj f(i)	0,868361	1,180923	0,780453	1,295224	0,635199	1,483170	0,395442	1,792397		0,954626
		q_1/q_0= 1,200147	q_2/q_1= 0,5832	q_3/q_2= 1,46459	q_4/q_3= 0,432787	q_5/q_4= 2,060614	q_6/q_5= 0,235292	q_7/q_6= 4,000046		

Table 3a

2 bids	0 + 1	2 + 3	4 + 5	6 + 7
q_i+q_{i+1}	0,224492	0,176021	0,138554	0,109746
Adj f(i) + Adj f(i+1)	0,221199	0,17227	0,134164	0,104488
(q_i+q_{i+1}) / (Adj f(i) + Adj f(i+1))	1, 014887	1,021774	1,032721	1,050322

Table 3b: "+" means "and"

We immediately observe that the discrete Nash equilibrium probabilities do not converge to the adjusted Nash equilibrium continuous probabilities (= 0.940025f(b), with 0.940025 = (1-f(12))/($\sum_{i=0}^{7} f(i)$)). We observe the yoyo phenomenon as regards the probabilities q_i, i.e. $q_0 < q_1 > q_2 < q_3 > q_4 < q_5 > q_6 < q_7$. The fact that the yoyo starts with the bid 0 is due to the fact that 0≥M-3V/2. We can evaluate the strength of this yoyo phenomenon by calculating the ratios q_{i+1}/q_i, i from 0 to 6 (see table 3a, figure 5a and figure 7). We can also observe that the ratios q_i/Adj f(i) go more and more away from 1 when i increases (see table 3a and figure 8). Yet they more or less regularly oscillate around the axe y=1. This explains a good convergence between the sums of discrete probabilities 2 by 2 and the sums of adjusted continuous probabilities 2 by 2 (see table 3b). We namely get $\sum_{i=0,2,4,6} \frac{\frac{q_i+q_{i+1}}{Adj\,f(i)+Adj\,f(i+1)}}{4}$ =1.0299 which expresses a good convergence. We can however observe that the ratio $\frac{q_i+q_{i+1}}{Adj\,f(i)+Adj\,f(i+1)}$ increases with i, so is better for i low (see table 3b, figure 5b).

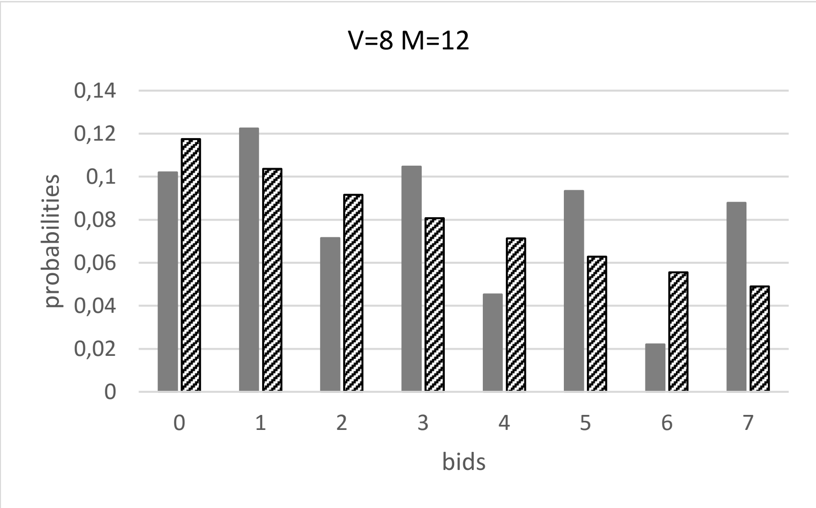

Figure 5a : The full columns are the discrete Nash equilibrium probabilities, the shaded columns are the adjusted continuous Nash equilibrium probabilities. We only focus on the bids from 0 to 7 to highlight the divergence and the yoyo phenomenon.

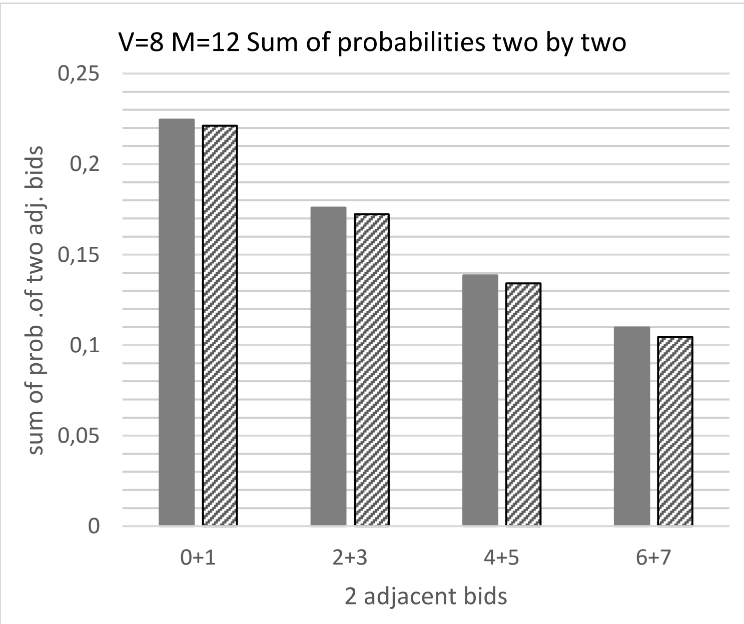

Figure 5b : The full columns are the sums of the discrete Nash equilibrium probabilities of two adjacent bids q_0+q_1, q_2+q_3, q_4+q_5, q_6+q_7, the shaded columns are the sums of the adjusted continuous Nash equilibrium probabilities of the same bids. We focus on the bids from 0 to 7 to show the good convergence of the sums.

We now switch to the case V=24 and M=30. We get the results in table 4a and table 4b.

bids	0	1	2	3	4	5	6
q_i	0,031834	0,048847	0,027763	0,046533	0,023885	0,044543	0,020173
Adj f(i)	0,040811	0,039145	0,037548	0,036015	0,034545	0,033136	0,031783
f(i)	0,041667	0,039966	0,038335	0,036771	0,03527	0,033831	0,03245
q_i/Adj f(i)	0,7800	1,2478	0,7394	1,2920	0,6914	1,3442	0,6347
		q_1/q_0= 1,534429	q_2/q_1= 0,568367	q_3/q_2= 1,676080	q_4/q_3= 0,513292	q_5/q_4= 1,864894	q_6/q_5= 0,452888

bids	7	8	9	10	11	12	13
q_i	0,042862	0,016601	0,041479	0,013145	0,040383	0,00978	0,039568
Adj f(i)	0,030486	0,029242	0,028049	0,026904	0,025806	0,024753	0,023743
f(i)	0,031126	0,029855	0,028637	0,027468	0,026347	0,025272	0,024241
q_i/Adj f(i)	1,4059	0,5677	1,4788	0,4886	1,5649	0,3951	1,6665
	q_7/q_6= 2,124721	q_8/q_7= 0,387313	q_9/q_8= 2,498584	q_{10}/q_9= 0,316907	q_{11}/q_{10}= 3,072119	q_{12}/q_{11}= 0,242181	q_{13}/q_{12}= 4,045808

bids	14	15	16	17	18 19 20 21 22 23 24 25 26 27 28 29	30
q_i	0,006482	0,039028	0,00323	0,038759	0	0,465105
Adj f(i)	0,022773	0,021845	0,020952	0,020097	0	0,472367
f(i)	0,023251	0,022303	0,021392	0,020519	0	0 ,472367
q_i/Adj f(i)	0,2846	1,7866	0,1542	1,9285		0,9846
	q_{14}/q_{13}= 0,163819	q_{15}/q_{14}= 6,020981	q_{16}/q_{15}= 0,082761	q_{17}/q_{16}= 11,999690		

Table 4a

Sum of bids	0+1	2+3	4+5	6+7	8+9	10+11	12+13	14+15	16+17
q_i+q_{i+1}	0,080681	0,074296	0,068428	0,063035	0,05808	0,053528	0,049348	0,04551	0,041989
Adj f(i) + Adj f(i+1)	0,079956	0,073563	0,067681	0,062269	0,057290	0,05271	0,048496	0,044618	0,041049
(q_i+q_{i+1}) / (Adj f(i) + Adj f(i+1))	1,009067	1,009964	1,011037	1,0123	1,013789	1,0155	1,017568	1,019992	1,0228

Table 4b

This case strengthens the facts observed in the previous study. A higher V doesn't lower the divergence between the discrete probabilities and the adjusted continuous probabilities (=0.979453f(i), where 0.979453=(1-f(30))/($\sum_{i=0}^{17} f(i)$)) (see table 4a and figure 6a). It doesn't diminish the yoyo phenomenon. On the contrary, whereas the ratios q_{i+1}/q_i, i from 0 to 5 are close to the ones obtained for V=8 and M=12, they become much more chaotic for i from 7 to 16 (only q_7/q_6 is significantly better for V=24 than for V=8, see table 4a and figure 7). What is more, the ratios q_i/Adj f(i), quite similar for i from 0 to 5 for V=8 and V=24, go more away from 1 for high bids for V=24 than all the ratios obtained for V=8 (see figure 8): only q_5/Adj f(5), q_6/Adj f(6) and q_7/Adj f(7) are significantly better for V=24 than for V=8.

Yet we can also observe that the oscillations of the ratios q_i/Adj f(i) become quite symmetric around the axe y=1 (more than for V=8 M=12), which explains a very strong convergence of the discrete and adjusted continuous sums of probabilities of two adjacent bids (see table 4b and figure 6b).We get the good ratio $\sum_{i=0}^{8} \frac{\frac{q_{2i}+q_{2i+1}}{Adj\,f(2i)+Adj\,f(2i+1)}}{9}$ =1.0147. And we again observe that the ratios $\frac{q_i+q_{i+1}}{Adj\,f(i)+Adj\,f(i+1)}$ increase with i, so are better for i low.

The stronger convergence of the discrete and adjusted continuous sums of probabilities for V=24, M=30 than for V=8, M=12, namely stems from the fact that $\frac{q_0+q_1}{Adj\,f(0)+Adj\,f(1)}$ = 1.009067 (V=24, M=30)< $\frac{q_0+q_1}{Adj\,f(0)+Adj\,f(1)}$ = 1.014887 (V=8, M=12) and $\frac{q_{16}+q_{17}}{Adj\,f(16)+Adj\,f(17)}$ = 1.0228 (V=24, M=30) < $\frac{q_6+q_7}{Adj\,f(6)+Adj\,f(7)}$ = 1.050322 (V=8, M=12).

This case also partly illustrates the results obtained for V large, i.e. $q_i \to 2q_M/V \simeq 0.0388$ for i odd from 1 to M-V/2-1= 17, and $q_i \to 2(M-V/2-i)q_M/V^2$ = (16.74378- i0.93021)/24^2 for i even from 0 to M-V/2-2 (given that M=30<<$24^{1.9}$/2+12 (=$V^{1.9}$/2+V/2)). This is illustrated in figure 6c. We namely observe that q_i, i odd, is quasi constant, close to 0.0388, at least for high values of i, whereas q_i, i even, linearly decreases and goes to 0 for high values of i (q_{16}= 0.00323, q_{14}= 0.006482, q_{12}= 0.00978).

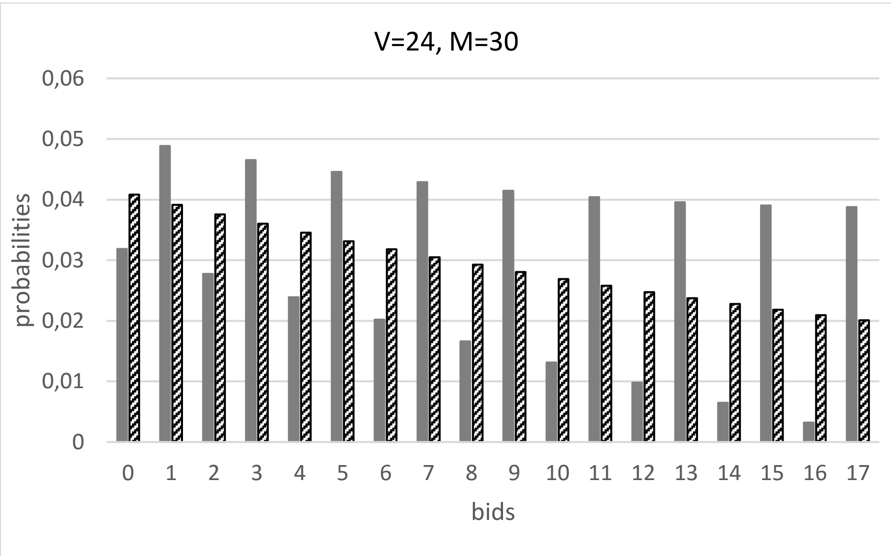

Figure 6a : The full columns are the discrete Nash equilibrium probabilities, the shaded columns are the adjusted continuous Nash equilibrium probabilities. We only focus on the bids from 0 to 17 to highlight the divergence and the strong yoyo phenomenon.

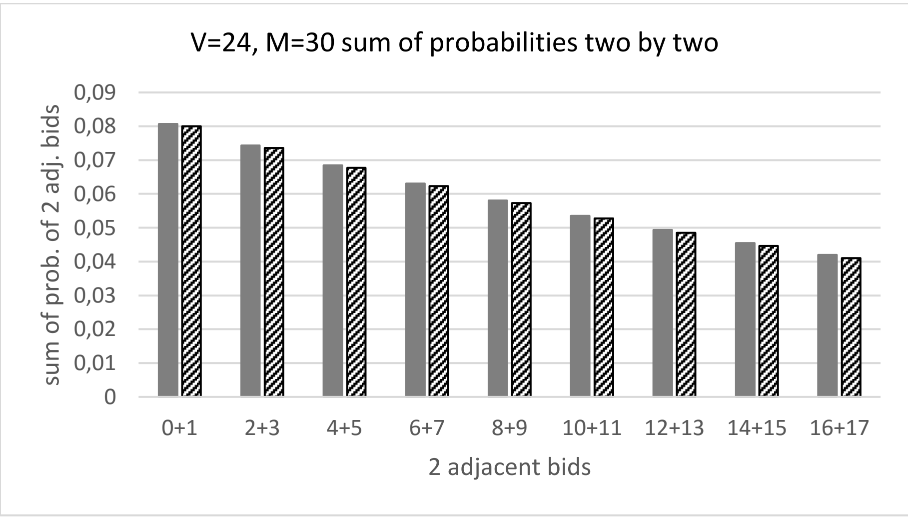

Figure 6b : The full columns are the sums of the discrete Nash equilibrium probabilities of two adjacent bids, the shaded columns are the sums of the adjusted continuous Nash equilibrium probabilities of the same bids. We only focus on the bids from 0 to 17 to show the good convergence.

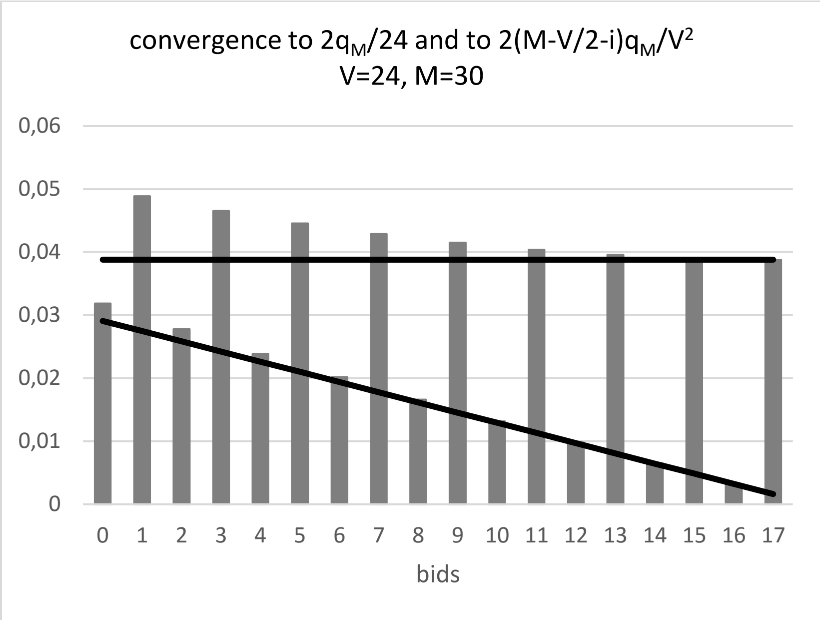

Figure 6c : The histogram gives the discrete Nash equilibrium probabilities, the horizontal line is $2q_M/24$ = 0.0388, and the decreasing line is $(16.74378-i0.93021)/24^2$. We observe that q_i, i even, is quite close to $(16.74378-i0.93021)/24^2$. The convergence of q_i, i odd, to 0.0388 is better for i large (V is not large enough to get a better convergence for smaller values of i).

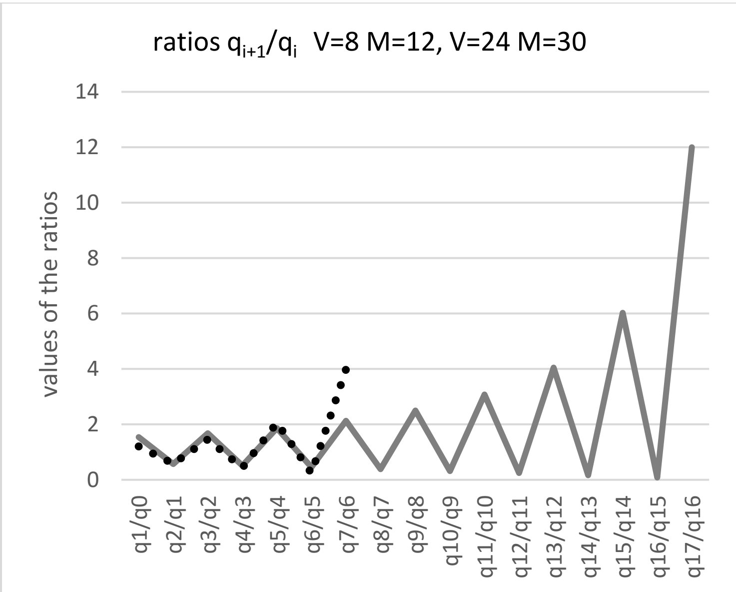

Figure 7 : The short dotted curve represents the ratios q_{i+1}/q_i for V=8 and M=12, the long curve represents the same ratios for V=24 and M=30. The two curves start in a similar way but the long curve ends in a much more chaotic way. So we easily observe that higher values of V do not bring adjacent probabilities closer. Only q_7/q_6 is significantly better for the larger value V=24 (perhaps due to a kind of end effect for V=8)

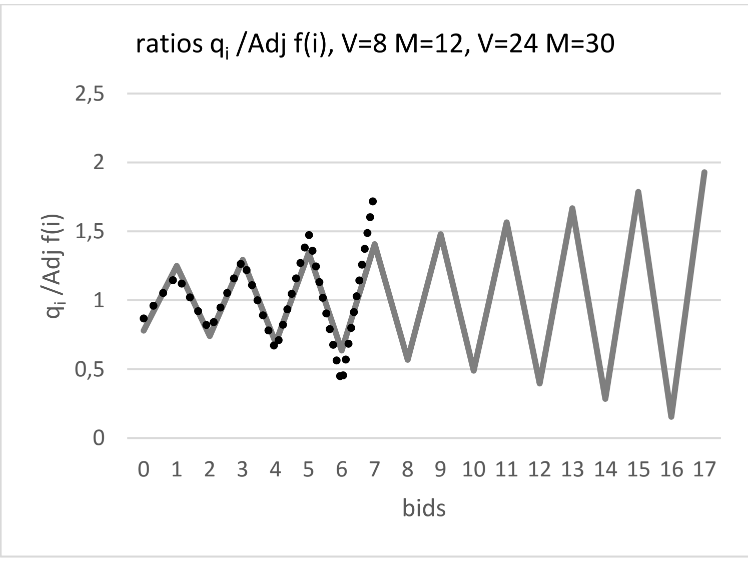

Figure 8 : The short dotted curve represents the ratios q_i/Adj f(i) for V=8 and M=12, the long curve represents the same ratios for V=24 and M=30. The two curves start in a similar way, but the long curve leads to oscillations of much larger magnitude. So we observe that higher values of V do not bring the ratios closer to 1. Only q_5/Adj f(5), q_6/Adj f(6) and q_7/ Adj f(7) are significantly better for the larger value V=24.

We now illustrate that choosing a smaller increment, such that it doesn't matter to work with an odd or an even V, namely 0.5 or 0.1, does not help the discrete Nash equilibrium to get closer to the continuous Nash equilibrium. We study two cases, V=9, M=12 and a bid increment 0.5, and V=8, M=12 and the same bid increment 0.5. These two cases also partly illustrate proposition 5 and proposition 6.

For V=9, M=12 and a bid increment 0.5, we get the results in table 5. So, by contrast to the obtained convergence of the discrete equilibrium to the continuous equilibrium when the bid increment is equal to 1, we now get a yoyo phenomenon as regards the discrete probabilities, which precludes their convergence to the continuous probabilities (see figure 9 compared to figure 2b). This result is in accordance with propositions 4, 5 and 6. The difference in the obtained convergence is due to the fact that we switch from the bid increment I=1 and a remainder of the division of M-V/2 by I equal to I/2=0.5, to the bid increment I=0.5 and a remainder of the division of M-V/2 by I equal to 0.

bids	0	0,5	1	1,5	2	2,5	3	3,5	4
q_i	0,063907	0,040103	0,059451	0,033498	0,055729	0,027306	0,052695	0,021451	0,050312

bids	4,5	5	5,5	6	6,5	7	7,5 8 8,5 9 9,5 10 10,5 11 11,5	12
q_i	0,01586	0,048549	0,010466	0,047386	0,005201	0,046809	0	0,421277

Table 5

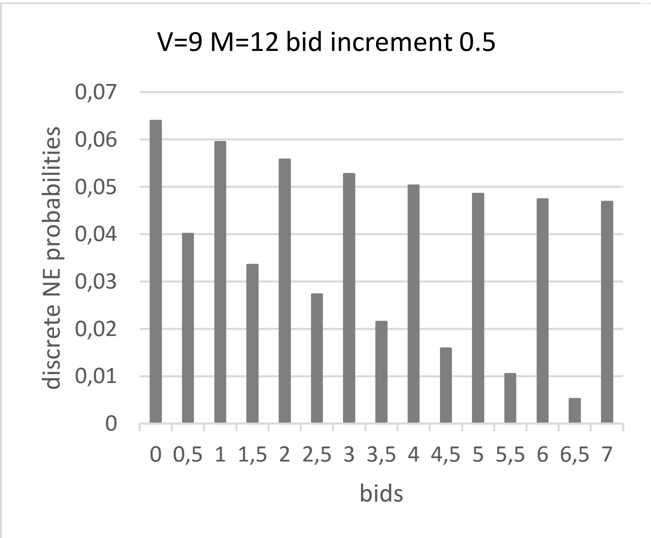

Figure 9 : For V=9 and M=12, the switch to the lower bid increment 0.5 yields a strong yoyo phenomenon as regards the discrete Nash equilibrium probabilities, which precludes the convergence of the discrete Nash equilibrium to the continuous Nash equilibrium.

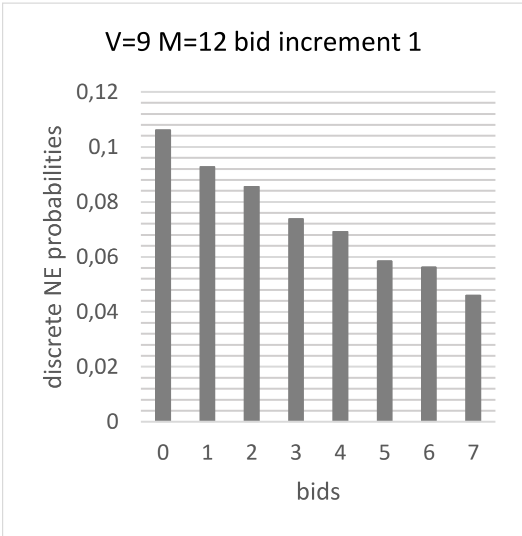

Figure 2b (with only the discrete Nash equilibrium probabilities)

For V=8, M=12 and a bid increment 0.5, we get the results in table 6a. It immediately follows that the yoyo phenomenon of the q_i is strengthened in comparison to the one observed for a bid increment equal to 1 (see table 3a). This can be observed by comparing figure 10a to figure 5a, but also by calculating the ratios $q_{i+0.5}/q_i$ (see table 6a) and by comparing them to the ratios in table 3a: there are more oscillations and the magnitude of the oscillations is larger for the smaller bid increment (see also figure 10b).

Yet, by contrast, we also observe in table 6b that the sums of the discrete Nash equilibrium probabilities of two adjacent bids converge to the sums of the adjusted continuous Nash equilibrium probabilities (=0.484695f(b) with 0.484695=(1-f(12))/($\sum_{i=0}^{7.5} f(i)$), i being a multiple of 0.5) of the same two bids, in accordance with proposition 6 (given that the remainder of the division of M-V/2 by the increment 0.5 is 0). And we can even observe that this convergence is better than for the bid increment 1. So we get, for the bid increment 0.5, $\sum_{i=0}^{7} \frac{\frac{q_i+q_{i+0.5}}{Adj\,f(i)+Adj\,f(i+0.5)}}{8}$= 1.0164, whereas we get $\sum_{i=0,2,4,6} \frac{\frac{q_i+q_{i+1}}{Adj\,f(i)+Adj\,f(i+1)}}{4}$= 1.0299 for the bid increment 1. So, switching from the bid increment 1 to the bid increment 0.5, for V=8 and M=12, leads to oscillations of the discrete probabilities of larger magnitude but to a better convergence of the sums 2 by 2.

bids	0	0,5	1	1,5	2	2,5	3	3,5	4
q_i	0,052553	0,065878	0,044318	0,060339	0,036775	0,055742	0,029808	0,052016	0,023306
f(i)	0,125	0,117427	0,110312	0,103629	0,097350	0,091452	0,085911	0,080706	0,075816
Adj f(i)	0,060587	0,056916	0,053468	0,050228	0,047185	0,044326	0,041641	0,039118	0,036748
		$q_{0.5}/q_0$= 1,253554	$q_1/q_{0.5}$= 0,672728	$q_{1.5}/q_1$= 1,361501	$q_2/q_{1.5}$= 0,609473	$q_{2.5}/q_2$= 1,515758	$q_3/q_{2.5}$= 0,534749	$q_{3.5}/q_3$= 1,745035	$q_4/q_{3.5}$= 0,448054

bids	4,5	5	5,5	6	6,5	7	7,5	8 8,5 9 9,5 10 10,5 11 11,5	12
q_i	0,049102	0,017168	0,046956	0,011298	0,045544	0,005605	0,044844	0	0.358748
f(i)	0,071223	0,066908	0,062854	0,059046	0,0554468	0,052108	0,048951	0	0,36788
Adj f(i)	0, 034522	0,03243	0,030465	0,028619	0,026885	0,025256	0,023726	0	0,36788
	$q_{4.5}/q_4$= 2,106839	$q_5/q_{4.5}$= 0,34964	$q_{5.5}/q_5$= 2,735089	$q_6/q_{5.5}$= 0,240608	$q_{6.5}/q_6$= 4,031156	$q_7/q_{6.5}$= 0,123068	$q_{7.5}/q_7$= 8,000714		

Table 6a

	0+0,5	1+1,5	2+2,5	3+3,5	4+4,5	5+5,5	6+6,5	7+7,5
$q_i+q_{i+0.5}$	0,118431	0,104657	0,092517	0,081824	0,072408	0,064124	0,056842	0,050449
Adj f(i)+Adj f(i+0.5)	0,117503	0,103696	0,091511	0,080759	0,071269	0,062895	0,055504	0,048982
$(q_i+q_{i+0.5})$/ (Adjf(i)+Adjf(i+0.5))	1,007898	1,009267	1,010993	1,013187	1,015982	1,019541	1,024106	1,029950

Table 6b

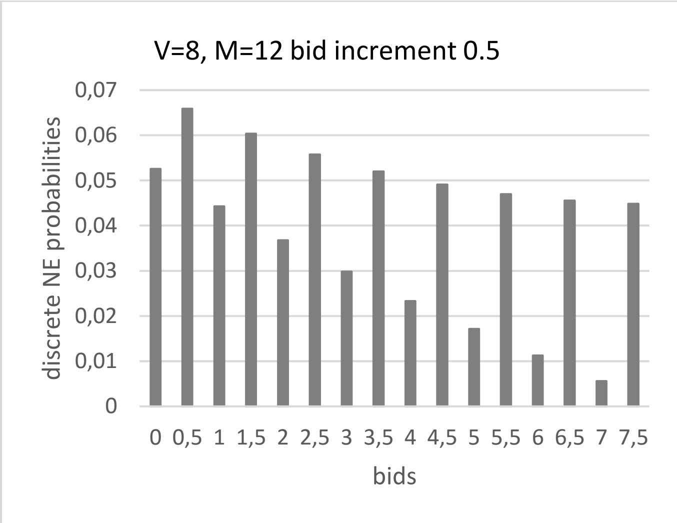

Figure 10a : The comparison with figure 5a highlights that for high bids the difference between the probabilities of two adjacent bids is stronger for a bid increment 0.5 than for the bid increment 1.

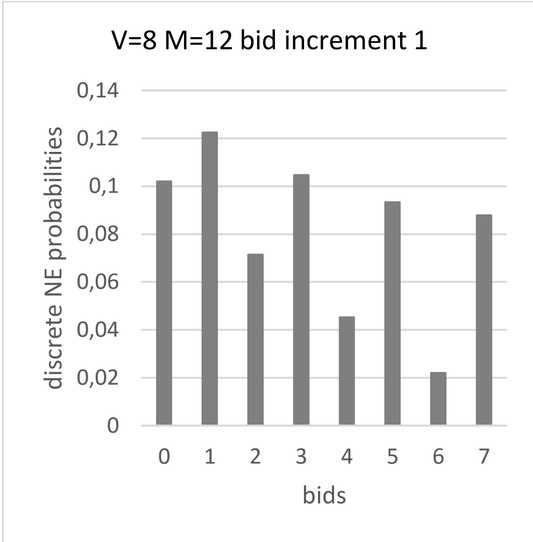

Figure 5a (with only the discrete Nash equilibrium probabilities)

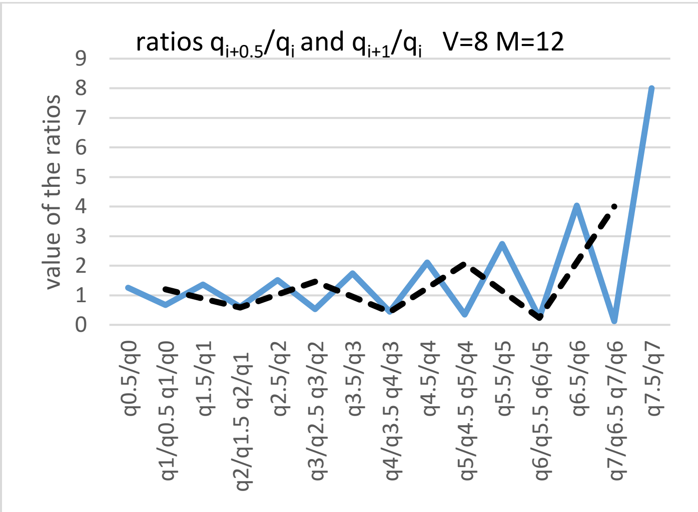

Figure 10b : The curve with more oscillations is obtained for the bid increment 0.5, the dashed curve with less oscillations is obtained for the bid increment 1. The magnitude of the oscillations is more important for the lower bid increment.

5. Concluding remarks

We could have expected that, given that the logic of the mixed Nash equilibrium is the same in the continuous game and in the discrete game – we just equalize the payoffs of the played bids, and the played bids are the bids that are not weakly dominated- that the convergence of the mixed Nash equilibrium of the discrete game to the mixed Nash equilibrium of the continuous game automatically follows.

Yet we established in proposition 5 that this convergence is rather an exception than the rule. To get convergence, we need that the remainder of the division of M-V/2 by the bid increment is half of the increment. So for example, when M and V are integers, convergence holds for odd values of V and a bid increment equal to 1, at least for large values of V. Proposition 5 namely implies that, when M and V are integers, convergence can't happen for the very small increments 10^{-N}, with N an integer ≥1, which proves that, even if the discrete game converges to the continuous game, i.e. if the bid increment goes to 0, the discrete Nash equilibrium may not converge to the continuous one. This lack of continuity between discrete second price all-pay auctions equilibria and continuous second price all-pay auctions equilibria is an unexpected result.

We also established in propositions 3, 4 and 6, that if the remainder of the division of M-V/2 by the bid increment is 0, then we get a kind of partial convergence, in that the sums of the discrete equilibrium probabilities of two adjacent bids go to the sums of the continuous equilibrium probabilities of the same two bids at least for large values of V. Yet we also showed that in that case a yoyo phenomenon arises, the probabilities of two adjacent bids being strikingly different. And this yoyo phenomenon is strengthened when one switches to lower bid increments. So, when the discrete game goes to the continuous one, i.e. when the bid increment goes to 0, the discrete equilibrium probabilities more and more diverge from the continuous ones, even if we get the convergence of the sums of the probabilities of two adjacent bids.

Well, these results lead us to two additional remarks.

First, in experiments, we often compare the observed players' behaviour to the mixed Nash equilibrium of the continuous game because this equilibrium is common knowledge and easy to calculate. But we often work with a discrete game, with integer values for V and M. It follows that we have to choose the bid increment very carefully, if we want to compare the players' behaviour to the right Nash equilibrium. Only few choices are possible. For example, working with odd values of V and a bid increment equal to 1 is a good way to do.

Second, the obtained equilibria in the discrete games show how strange the mixed Nash equilibrium may be. The yoyo phenomenon is really not intuitive. Why should a bid be played with a very low probability when its two adjacent bids are played with a much higher probability? Real players will surely not play in this way. This raises questions as regards the behavioral meaning of some mixed Nash equilibria (see Umbhauer 2017).

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Proofs

Appendix 1: out of Umbhauer (2016)

We set M>V/2. The weak dominance of a bid in [M-V/2, M[by M is obvious.
Let us turn to the equilibrium.
Given the weak dominance, it is conjectured that the Nash equilibrium strategy is a density function f(.) that decreases from 0 to M-V/2 and has an atom on M.
Call $f_2(.)$ player 2's equilibrium strategy. Suppose that player 1 plays b. She wins the auction each time player 2 bids less than b. So she gets:
$G(b) = M+\int_0^b (V-x)f_2(x)dx - b(\int_b^{M-V/2} f_2(x)dx + f_2(M))$
We check that a player gets the same payoff with M and M-V/2, regardless of the opponents' equilibrium distribution:
$G(M)= M+\int_0^{M-V/2}(V-x)f_2(x)dx + (\frac{V}{2}-M)f_2(M)= G(M-V/2)$
G(b) has to be constant for each b in [0, M-V/2] ∪ {M}. So G'(b) = 0 for b in [0, M-V/2] .
We get: $(V-b)f_2(b)-F_2(M-V/2)+F_2(b)-f_2(M)+bf_2(b) = 0$
where $F_2(.)$ is the cumulative distribution of the density function $f_2(.)$.
By construction $f_2(M) = 1-F_2(M-V/2)$, so we get the differential equation: $Vf_2(b)-1+F_2(b) = 0$ whose solution is: $F_2(b)= 1+Ke^{-b/V}$ where K is a constant determined as follows:
$F_2(0)=0$ because there is no atom on 0, so 1+K = 0 and K = -1.
It follows $F_2(b) = 1- e^{-b/V}$ for b in [0, M-V/2], $f_2(M) = 1-F_2(M-V/2) = e^{1/2-M/V}$ (<1) ,
$f_2(b) = e^{-b/V}/V$ for b in[0, M-V/2] (and $f_2(b)=0$ for b in]M-V/2, M[)
By symmetry, we get $f_1(b) = e^{-b/V}/V$ for b in[0, M-V/2], $f_1(M) = 1-F_1(M-V/2) = e^{1/2-M/V}$ (and $f_1(b)=0$ for b in]M-V/2, M[ ).

Appendix 2: Proof of proposition 1

Case 1: V is an odd integer
Consider player 1. The bids from M-V/2+1/2 to M-1 are dominated by M, so they are played with probability 0.
M-V/2-1/2 and M lead to the same payoff regardless of the bid b played by player 2, except b= M-V/2-1/2 and b=M. So player 1 gets the same payoff with both bids if and only if:
$q_{M-V/2-1/2}(V+1/2)+q_M(V/2+1/2) = q_{M-V/2-1/2}(3V/2+1/2)+q_M(V/2)$
hence $\boxed{q_{M-V/2-1/2} = q_M/V}$ *(2a)*
Now compare the bids M-V/2-3/2 and M-V/2-1/2. Both lead to the same payoff, except if player 2 bids M-V/2-3/2, M-V/2-1/2 or M.
We need:

$q_{M-V/2-3/2}(V+3/2)+q_{M-V/2-1/2}(V/2+3/2)+q_M(V/2+3/2) = q_{M-V/2-3/2}(3V/2+3/2)+q_{M-V/2-1/2}(V+1/2)+q_M(V/2+1/2)$

$q_{M-V/2-3/2}(-V/2)+q_{M-V/2-1/2}(-V/2+1)+q_M = 0$

Hence $\boxed{q_{M-V/2-3/2} = q_M(1/V +2/V^2).}$ *(2a)*

More generally M-V/2-1/2-j and M-V/2-1/2-j-1, for j from 1 to M-V/2-3/2, lead to the same payoff, except if player 2 bids M-V/2-1/2-k or M, with k going from 0 to j+1.

We need:

$q_{M-V/2-1/2-j-1}(V+3/2+j)+q_{M-V/2-1/2-j}(V/2+3/2+j)+\sum_{k=0}^{j-1} q_{M-\frac{V}{2}-\frac{1}{2}-k}(\frac{V}{2}+3/2+j)+q_M(V/2+3/2+j) =$

$q_{M-V/2-1/2-j-1}(3V/2+j+3/2)+q_{M-V/2-1/2-j}(V+1/2+j)+\sum_{k=0}^{j-1} q_{M-\frac{V}{2}-\frac{1}{2}-k}(\frac{V}{2}+\frac{1}{2}+j)+q_M(V/2+1/2+j)$

Hence $q_{M-V/2-3/2-j}(-V/2)+q_{M-V/2-1/2-j}(-V/2+1)+\sum_{k=0}^{j-1} q_{M-\frac{V}{2}-\frac{1}{2}-k}+q_M = 0$

We also get:

$q_{M-V/2-(j+1)-3/2}(-V/2)+q_{M-V/2-(j+1)-1/2}(-V/2+1)+q_{M-V/2-j-1/2}+\sum_{k=0}^{j-1} q_{M-\frac{V}{2}-\frac{1}{2}-k}+q_M = 0$

It follows $q_{M-V/2-j-5/2}= 2q_{M-V/2-j-3/2}/V+q_{M-V/2-j-1/2}$ for any j from 0 to M-V/2-5/2.

i.e. $\boxed{q_i = 2q_{i+1}/V+q_{i+2} \text{ for i from 0 to M-V/2-5/2.}}$ *(1a)*

Case 2 : V is an even integer

The study is similar.

Consider player 1. The bids from M-V/2 to M-1 are dominated by M, so they are played with probability 0.

M-V/2-1 and M lead to the same payoff except if player 2 bids M-V/2-1 or M. So player 1 gets the same payoff with both bids if and only if:

$q_{M-V/2-1}(V+1)+q_M(V/2+1) = q_{M-V/2-1}(3V/2+1)+q_M(V/2)$

so $\boxed{q_{M-V/2-1} = 2q_M/V}$ *(2b)*

Now compare the bids M-V/2-1 and M-V/2-2. Both lead to the same payoff, except if player 2 bids M-V/2-2, M-V/2-1 or M.

So we need:

$q_{M-V/2-2}(V+2)+q_{M-V/2-1}(V/2+2)+q_M(V/2+2) = q_{M-V/2-2}(3V/2+2)+q_{M-V/2-1}(V+1)+q_M(V/2+1)$

so $q_{M-V/2-2}(-V/2)+q_{M-V/2-1}(-V/2+1)+q_M= 0$

We get $\boxed{q_{M-V/2-2} =2q_{M-V/2-1}/V =4q_M/V^2}$ *(2b)*

More generally M-V/2-j and M-V/2-j-1, for j from 1 to M-V/2-1, lead to the same payoff, except if player 2 bids M-V/2-k or M, with k going from 1 to j+1.

So we need: $q_{M-V/2-j-1}(V+j+1)+q_{M-V/2-j}(V/2+j+1)+\sum_{k=1}^{j-1} q_{M-\frac{V}{2}-k}(\frac{V}{2}+j+1)+q_M(V/2+j+1) =$

$q_{M-V/2-j-1}(3V/2+j+1)+q_{M-V/2-j}(V+j)+\sum_{k=1}^{j-1} q_{M-\frac{V}{2}-k}(\frac{V}{2}+j)+q_M(V/2+j)$

Hence $q_{M-V/2-j-1}(-V/2)+q_{M-V/2-j}(-V/2+1)+\sum_{k=1}^{j-1} q_{M-\frac{V}{2}-k}+q_M = 0$

In the same way we get: $q_{M-V/2-j-2}(-V/2)+q_{M-V/2-j-1}(-V/2+1)+q_{M-V/2-j}+\sum_{k=1}^{j-1} q_{M-\frac{V}{2}-k}+q_M = 0$

It follows $q_{M-V/2-j-2}= 2q_{M-V/2-j-1}/V+q_{M-V/2-j}$ for j from 1 to M-V/2-2.

i.e. $\boxed{q_i= 2q_{i+1}/V+q_{i+2}}$ for i from 0 to M-V/2 -3 *(1b)*

Appendix 3: Proof of proposition 2

We first show that q_i is decreasing in i, for i from 0 to M-V/2-1/2.
Suppose that $q_{i+3}<q_{i+2}<q_{i+1}$. We already have $q_{M-V/2-1/2} = q_M/V < q_{M-V/2-3/2} = q_M(1/V+2/V^2) < q_{M-V/2-5/2} = q_M(1/V+2/V^2+4/V^3)$. So we just have to show that $q_{i+3}<q_{i+2}<q_{i+1}$ implies $q_i>q_{i+1}$.
We have: $q_i = 2q_{i+1}/V+q_{i+2} > 2q_{i+2}/V+q_{i+3}$
Given that $2q_{i+2}/V+q_{i+3} =q_{i+1}$, we get $q_i >q_{i+1}$.

We now show that the discrete equilibrium converges to the continuous one when V is large.

When V is large, we get with Taylor's theorem (quadratic approximation): $2e^{-1/V}/V+e^{-2/V}\simeq 2/V-2/V^2+1/V^3+1-2/V+2/V^2 = 1+1/V^3 \rightarrow 1$.
It follows $e^{-i/V}/V\simeq 2\, e^{-(i+1)/V}/V^2+e^{-(i+2)/V}/V$.
So, by replacing q_k with $f(k) = e^{-k/V}/V$ in q_i and $2q_{i+1}/V+q_{i+2}$ we get, for V large,
$q_i\simeq 2q_{i+1}/V+q_{i+2}$.
It follows that f(k) checks the main recurrence equations (1a) for large values of V.

Moreover, we have $q_{M-V/2-1/2} = q_M/V$ and $q_{M-V/2-3/2} = q_M(1/V+2/V^2)$. We show that these both equations hold when replacing q_M with $f(M)=e^{1/2-M/V}$, $q_{M-V/2-1/2}$ with $f(M-V/2-1/2)= e^{-M/V+1/2+1/(2V)}/V$ and $q_{M-V/2-3/2}$ with $f(M-V/2-3/2)=e^{-M/V+1/2+3/(2V)}/V$.
We just have to show that $e^{-M/V+1/2+1/(2V)}/V \simeq e^{1/2-M/V}/V$ and $e^{-M/V+1/2+3/(2V)}/V \simeq e^{1/2-M/V}(1/V+1/V^2)$, i.e. $e^{1/(2V)} \simeq 1$ and $e^{3/(2V)} \simeq 1+1/V$
$e^{1/(2V)}\simeq 1+1/(2V)+1/(8V^2)$ (Taylor's theorem, quadratic approximation)
$e^{3/(2V)}\simeq 1+3/(2V)+9/(8V^2)$
$0.5/V+1/(8V^2)$ and $0.5/V+9/(8V^2)$ go to 0 for V large (compared to 1), so we get the convergence for large values of V.

Now observe that, if q_{i+1} and q_{i+2} respectively converge to f(i+1) and f(i+2), then $q_i= 2q_{i+1}/V+q_{i+2}\rightarrow 2f(i+1)/V+f(i+2)$. Yet we also know that f(i) checks the main recurrence equations, so that $2f(i+1)/V+f(i+2)\rightarrow f(i)$. It follows that $q_i\rightarrow f(i)$ for V large.

In addition, by replacing q_i with $e^{-i/V}/V$ for i from 0 to M-V/2-1/2 and q_M with $e^{1/2-M/V}$ we check that $q_0+q_1+q_2+q_3+\ldots+q_{M-V/2-3/2}+q_{M-V/2-1/2} = (1+e^{-1/V}+e^{-2/V}+e^{-3/V}+\ldots\ldots+e^{-(M-V/2-1/2)/V})/V =$
$(1-e^{-(M-V/2+1/2)/V})/((1-e^{-1/V})V)$. We have $V(1-e^{-1/V}) \simeq V(1-(1-1/V)) = 1$ for V large and $(1- e^{-(M-V/2+1/2)/V}) = (1-e^{-(M-V/2)/V})$ for V large, given that $e^{-1/(2V)} \simeq 1-1/2V\rightarrow 1$ for V large. So
$q_0+q_1+q_2+q_3+\ldots+q_{M-V/2-3/2}+q_{M-V/2-1/2} \rightarrow 1-e^{1/2-M/V} = 1-q_M$ (3a)

Putting the results together ensures that for V large, $e^{-i/V}/V$ for q_i, i from 0 to M-V/2-1/2 and $e^{1/2-M/V}$ for q_M solve the discrete equations (1a), (2a) and (3a). By construction, the system of equations defining q_i, i from 0 to M, has a unique solution. It follows the convergence of the discrete equilibrium probabilities to the continuous equilibrium probabilities.

Appendix 4: Proof of proposition 3

Given the proof in Appendix 3, we know that the main equation $q_i=2q_{i+1}/V+q_{i+2}$ is satisfied for $q_i = e^{-i/V}/V$, for i from 0 to M-V/2-3. But the convergence of the discrete probabilities to the continuous probabilities isn't possible because the equation $q_{M-V/2-1} = 2q_M/V$ isn't true for $q_M = e^{1/2-M/V}$ and $q_{M-V/2-1} = e^{-M/V+1/2+1/V}/V$, even for V large. As a matter of fact $2\neq 1+ 1/V+1/(2V^2)$

even for V large. Moreover the discrete probability distribution is not uniformly decreasing. For example:
$q_{M-V/2-2} = 4q_M/V^2 < q_{M-V/2-1} = 2q_M/V$ for V>2.

Yet, by replacing $q_{M-V/2-1}$ and $q_{M-V/2-2}$ with $e^{1/V+1/2-M/V}/V$ and $e^{2/V+1/2-M/V}/V$ and q_M with $e^{1/2-M/V}$, we get $2q_M/V+4q_M/V^2 = e^{1/2-M/V}(2/V+4/V^2)$ and $e^{1/V+1/2-M/V}/V+e^{2/V+1/2-M/V}/V = e^{1/2-M/V}(e^{1/V}+e^{2/V})/V \simeq e^{1/2-M/V}(2/V+3/V^2+5/(2V^3))$ (Taylor's theorem, quadratic approximation). So, given that $-1/V^2+2.5/V^3 \rightarrow 0$ for V large (compared to 2/V), we observe that $q_{M-V/2-1}+q_{M-V/2-2}$ and $e^{1/V+1/2-M/V}/V+e^{2/V+1/2-M/V}/V$ converge to a same value for large values of V.
Let us fix $q_M = e^{1/2-M/V}$.
$q_{M-V/2-3}$ and $q_{M-V/2-2}$ are uniquely defined, with $q_{M-V/2-2}+q_{M-V/2-3} = q_{M-V/2-2}+2q_{M-V/2-2}/V+q_{M-V/2-1} = e^{1/2-M/V}(2/V+4/V^2+8/V^3)$. And we also observe that $e^{2/V+1/2-M/V}/V+e^{3/V+1/2-M/V}/V \rightarrow e^{1/2-M/V}(e^{2/V}+e^{3/V})/V$ which goes to $e^{1/2-M/V}(2/V+5/V^2+6.5/V^3)$. Given that $1/V^2-1.5/V^3 \rightarrow 0$ for V large (compared to 2/V), $q_{M-V/2-3}+q_{M-V/2-2}$ and $e^{2/V+1/2-M/V}/V+e^{3/V+1/2-M/V}/V$ converge to a same value for large values of V.
Now observe that if $e^{-(i+2)/V}/V+e^{-(i+3)/V}/V$ and $q_{i+2}+q_{i+3}$ converge to a same value, and if $e^{-(i+1)/V}/V+e^{-(i+2)/V}/V$ and $q_{i+1}+q_{i+2}$ converge to a same value, then q_i+q_{i+1} and $e^{-i/V}/V+e^{-(i+1)/V}/V$ also converge to a same value for i from 0 to M-V/2-4. Let us show it:
On the one side, $q_i+q_{i+1} = 2q_{i+1}/V+q_{i+2}+2q_{i+2}/V+q_{i+3} = 2(q_{i+1}+q_{i+2})/V+q_{i+2}+q_{i+3} \rightarrow$
$2(e^{-(i+1)/V}/V+e^{-(i+2)/V}/V)/V+e^{-(i+2)/V}/V+e^{-(i+3)/V}/V = (e^{-i/V}/V)[2(e^{-1/V}/V+e^{-2/V}/V)+e^{-2/V}+e^{-3/V}] \rightarrow$
$(e^{-i/V}/V)(2-1/V+0.5/V^2+5/V^3)$.
On the other side, $e^{-i/V}/V+e^{-(i+1)/V}/V = (e^{-i/V}/V)(1+e^{-1/V}) = (e^{-i/V}/V)(2-1/V+0.5/V^2)$.
So q_i+q_{i+1} and $e^{-i/V}/V+e^{-(i+1)/V}/V$ converge to a same value for large values of V.

Finally, at least for M-V/2 even, $q_M = e^{1/2-M/V}$ is the good assumption because the system of discrete probabilities has a unique solution and it checks:
$(q_0+q_1)+(q_2+q_3)+\ldots(q_{M-V/2-2}+q_{M-V/2-1}) = (1+e^{-1/V}+e^{-2/V}+e^{-3/V}+\ldots+e^{-(M-V/2-1)/V})/V =$
$(1-e^{-(M-V/2)/V})/((1-e^{-1/V})V) = 1-e^{1/2-M/V} = 1-q_M$ for V large (because $1-e^{-1/V} \simeq 1-(1-1/V) = 1/V$ for V large).

To summarize, by setting $q_M = e^{1/2-M/V}$, we observe that $e^{-i/V}/V+e^{-(i+1)/V}/V$ and q_i+q_{i+1} converge for V large, for i from 0 to M-V/2-2. And we observe that if $e^{-i/V}/V+e^{-(i+1)/V}/V$ and q_i+q_{i+1} converge, then $q_M = e^{1/2-M/V}$ is consistent with the equations (1b), (2b) and (3b). So, given that this system of equations has a unique solution, we have found that the solution of the set of equations checks $q_M = e^{1/2-M/V}$ and $q_i+q_{i+1} = f(i)+f(i+1)$ for i from 0 to M-V-2 for V large.
So we have a partial convergence result between the discrete Nash equilibrium and the continuous Nash equilibrium, but only by summing the adjacent probabilities two by two.

Let us now focus on the strange values taken by the probabilities for large values of V.
$q_{M-V/2-1} = 2q_M/V$
$q_{M-V/2-2} = 4q_M/V^2 \rightarrow 0$ (in comparison with $q_{M-V/2-1}$) for large values of V
$q_{M-V/2-3} = 2q_{M-V/2-2}/V+q_{M-V/2-1} \rightarrow 2q_M/V$
$q_{M-V/2-4} = 2q_{M-V/2-3}/V+q_{M-V/2-2} \rightarrow 8q_M/V^2 \rightarrow 0$
$q_{M-V/2-5} = 2q_{M-V/2-4}/V+q_{M-V/2-3} \rightarrow 2q_M/V$
And so on. By recurrence, if $q_{M-V/2-i} = 2q_M/V$ (i odd) and $q_{M-V/2-i-1} \rightarrow 2(i+1)q_M/V^2$, we get :
$q_{M-V/2-i-2} = 2q_{M-V/2-i-1}/V+q_{M-V/2-i} \rightarrow 2q_M/V$ (provided $4(i+1)q_M/V^3 \rightarrow 0$)

and $q_{M-V/2-i-3}= 2q_{M-V/2-i-2}/V+q_{M-V/2-i-1} \rightarrow 4q_M/V^2+2(i+1)q_M/V^2 =2(i+3)q_M/V^2$.
Observe that $4(i+1)q_M/V^3 \rightarrow 0$ (in comparison with $q_{M-V/2-1}$) means that $2(i+1)<V^x$ with $x<2$. So, given that $i+1 \leq M -V/2$, for $4(i+1)q_M/V^3$ to go to 0, it is enough that $M<V^x/2+V/2$ with $x < 2$. So we can fix $x = 1.9$ for example. Observe also that $q_{M-V/2-2j}$ increases in j, whereas $q_{M-V/2-i}$, for i odd, keeps close to $2q_M/V$, provided that M is lower than $V^{x<2}/2+V/2$. So we get a kind of yoyo phenomenon. For large values of k, the probabilities q_k alternate between 0 and $2q_M/V$. Things change for q_i with i small: $q_{M-V/2-2j}$ can become larger than $q_{M-V/2-i}$, i odd, if $M>3V/2$ and continues increasing linearly in j, whereas $q_{M-V/2-i}$, i odd, sticks to $2q_M/V$ if V large and $M < V^{x<2}/2+V/2$.

Appendix 5: Proof of proposition 4

We first focus on a bid increment I equal to 0.5. In that case, r, the remainder of the division of M-V/2 by I is 0 and we apply the results obtained in Appendix 6. The two highest played bids (after M) are M-V/2-0.5 and M-V/2-1.

We get $\boxed{q_{M-V/2-0.5} = 2Iq_M/V=q_M/V \text{ and } q_{M-V/2-1} = 4I^2q_M/V^2=q_M/V^2}$ *(4b)*

And the main recurrence equations are:

$q_i = 2Iq_{i+0.5}/V+q_{i+1} = q_{i+0.5}/V+q_{i+1}$
for i from 0 to M-V/2 -1.5, i being a multiple of 0.5. *(4a)*

For large values of V we get again:

$q_{M-V/2-0.5} = q_M/V$
$q_{M-V/2-1} = q_M/V^2 \rightarrow 0$ (in comparison with $q_{M-V/2 -0.5}$) for large values of V
$q_{M-V/2-1.5} = q_{M-V/2-1}/V+q_{M-V/2-0.5} \rightarrow q_M/V$
$q_{M-V/2-2} \rightarrow q_M/V^2+q_M/V^2 \rightarrow 2q_M/V^2 \rightarrow 0$
$q_{M-V/2 -2.5} \rightarrow q_M/V$

And so on. By recurrence it is easy to establish that $q_{M-V/2-i-0.5} = q_M/V$ (i integer) and $q_{M-V/2-i} \rightarrow iq_M/V^2$ (i integer) provided that $M<V^{x<2}+V/2$. So again we observe the yoyo phenomenon. Let us add that $q_{M-V/2-1}/q_{M-V/2-0.5}=1/V$ whereas $q_{M-V/2-2}/q_{M-V/2-1}=2/V$ when the bid increment is 1, so the contrast between two adjacent probabilities q_i for i large is larger.

We now focus on a bid increment I equal to 0.1. In that case, r, the remainder of the division of M-V/2 by I is again 0 and we apply the results obtained in Appendix 6. The two highest played bids (after M) are M-V/2-0.1 and M-V/2-0.2.

We get $\boxed{q_{M-V/2-0.1} = 2Iq_M/V=0.2q_M/V \text{ and } q_{M-V/2-0.2} = 4I^2q_M/V^2=0.04q_M/V^2}$ *(5b)*

And the main recurrence equations are:

$q_i = 2Iq_{i+0.1}/V+q_{i+0.2} = 0.2q_{i+0.1}/V+q_{i+0.2}$
for i from 0 to M-V/2-0.3, i being a multiple of 0.1. *(5a)*

For large values of V we get again:

$q_{M-V/2-0.1}=0.2q_M/V$
$q_{M-V/2-0.2} =0.04q_M/V^2 \rightarrow 0$ (in comparison with $q_{M-V/2-0.1}$) for large values of V
$q_{M-V/2-0.3} = 0.2q_{M-V/2-0.2}/V + q_{M-V/2-0.1} \rightarrow 0.2q_M/V$
$q_{M-V/2-0.4} \rightarrow 0.04q_M/V^2 +0.04q_M/V^2 \rightarrow 0.08q_M/V^2 \rightarrow 0$
$q_{M-V/2-0.5} \rightarrow 0.2q_M/V$

And so on. By recurrence it is easy to establish that, provided $M<V^{x<2}/0.2+V/2$, $q_{M-V/2-i/10} = 0.2q_M/V$ (i being an odd integer) and $q_{M-V/2-i/10} \rightarrow 0.02iq_M/V^2$ (i being an even integer). So we observe again the yoyo phenomenon. And this yoyo phenomenon is stronger than for a bid increment equal to 0.5 or 1 because on the one hand two adjacent bids are closer (bid increment 0.1), on the other hand the difference in probabilities between two adjacent high bids is larger: $(0.04q_M/V^2)/(0.2q_M/V) = 0.2/V$ (whereas we get 1/V for the bid increment 0.5 and 2/V for the bid increment 1).

Appendix 6: Proof of proposition 5 and proposition 6

I is the increment and r is the remainder or the division of M-V/2 by I.
So the two highest played bids after M are M-V/2-r and M-V/2-r-I, if $r \neq 0$. If $r = 0$, the two highest played bids after M are M-V/2-I and M-V/2-2I.

Case 1: r≠0 (Proof of proposition 5)

M-V/2-r and M lead to the same payoff, except if the opponent bids M-V/2-r or M. So we need:
$q_{M-V/2-r}(V+r)+q_M(V/2+r) = q_{M-V/2-r}(3V/2+r)+q_M(V/2)$
so $\boxed{q_{M-V/2-r} = 2rq_M/V}$ *(6a)*
The bids M-V/2-r and M-V/2-r-I lead to the same payoff, except if the opponent bids M-V/2-r, M-V/2-r-I or M. So we need:
$q_{M-V/2-r-I}(V+r+I)+q_{M-V/2-r}(V/2+r+I)+q_M(V/2+r+I)=$
$q_{M-V/2-r-I}(3V/2+r+I)+q_{M-V/2-r}(V+r)+q_M(V/2+r)$
so $q_{M-V/2-r-I}(-V/2)+q_{M-V/2-r}(-V/2+I)+Iq_M = 0$
We get $\boxed{q_{M-V/2-r-I} = [2(I-r)/V +4rI/V^2]q_M}$ *(6b)*

More generally M-V/2-r-j and M-V/2-r-j-I (with j a multiple of I, j≥I)) lead to the same payoff, except if player 2 plays M-V/2-r-k or M, with k going from 0 to j+I (j and k being multiples of I). So we need:
$q_{M-V/2-r-j-I}(V+r+j+I)+ q_{M-V/2-r-j}(V/2+r+j+I)+ \sum_{k=0}^{j-I} q_{M-\frac{V}{2}-r-k}(\frac{V}{2} + r + j + I)+ q_M(V/2+r+j+I) =$
$q_{M-V/2-r-j-I}(3V/2+r+j+I)+q_{M-V/2-r-j}(V+r+j)+ \sum_{k=0}^{j-I} q_{M-\frac{V}{2}-r-k}(\frac{V}{2} + r + j)+ q_M(V/2+r+j)$
Hence $q_{M-V/2-r-j-I}(-V/2)+q_{M-V/2-r-j}(-V/2+I)+ I\sum_{k=0}^{j-I} q_{M-\frac{V}{2}-r-k}+Iq_M = 0$ (k being a multiple of I).
In a similar way we get:
$q_{M-V/2-r-j-2I}(-V/2)+ q_{M-V/2-r-j-I}(-V/2+I)+Iq_{M-V/2-r-j} +I\sum_{k=0}^{j-I} q_{M-\frac{V}{2}-r-k}+Iq_M = 0$
(k being a multiple of I).
It follows $q_{M-V/2-r-j-2I}= 2Iq_{M-V/2-r-j-I}/V+q_{M-V/2-r-j}$ for j from 0 to M-V/2-r-2I
(j being a multiple of I)
So the main recurrence equations are:
$\boxed{q_i = 2Iq_{i+I}/V+q_{i+2I} \text{ for } i \text{ from 0 to } M\text{-}V/2\text{-}r\text{-}2I \text{ (i being a multiple of I)}}$. *(6c)*

To study the convergence of the discrete Nash equilibrium to the continuous Nash equilibrium we have to take into account that the bid kI, in the continuous equilibrium, is only played with probability f(kI)db. So we have to weight (adjust) the continuous probabilities by multiplying them by (1- f(M))/(f(0)+f(I)+…..+f(M-V/2-r)) . We get:

$f(0)+f(I)+\ldots..+f(M-V/2-r)) = (1+e^{-I/V}+\ldots.+e^{-(I/V)((M-V/2-r)/I)})/V= \frac{(1-e^{-\frac{\left(\frac{I}{V}\right)\left(M-\frac{V}{2}-r+I\right)}{I}})}{V(1-e^{-\frac{I}{V}})} \rightarrow$ $(1-f(M))/I$ for large values of V. So, convergence requires that $q_i \rightarrow If(i)$ for i from 0 to M-V/2- r, i being a multiple of I, and that $q_M \rightarrow f(M)$.

To get convergence, we need a decreasing discrete probability distribution.
So we need: $q_{M-V/2-r-2I} > q_{M-V/2-r-I} > q_{M-V/2-r}$
i.e.: $2rq_M/V<[2(I-r)/V+4rI/V^2]q_M$
and $[2(I-r)/V+4rI/V^2]q_M< [2r/V+4I(I-r)/V^2+8rI^2/V^3]q_M$
(given that $q_{M-V/2-r-2I}=2I\, q_{M-V/2-r-I}/V+ q_{M-V/2-r}= [2r/V+4I(I-r)/V^2+8rI^2/V^3]q_M$)
Convergence will only be obtained for V large, so the above inequations have to be true for large values of V, so we get:
$2rq_M/V<[2(I-r)/V+4rI/V^2]q_M \Rightarrow r \leq I/2$
And $[2(I-r)/V+4rI/V^2]q_M <[2r/V+(4I(I-r))/V^2 + (8rI^2)/V^3]q_M \Rightarrow r \geq I/2$
It derives r = I/2, so convergence can only be obtained for $\boxed{r = I/2}$. *(6d)*
Let us check that the probability distribution is decreasing for r=I/2.
It remains to show that if $q_i>q_{i+I}>q_{i+2I}$, then $q_{i-I} > q_i$ (i being a multiple of I).
We have $q_{i-I}= 2Iq_i/V +q_{i+I} > 2Iq_{i+I}/V +q_{i+2I}$. So we get $q_{i-I} > q_i$.

Convergence to the continuous Nash equilibrium requires that the equations (6c) are solved when replacing q_i with If(i). So we have to show that If(i) goes to $2I^2f(i+I)/V+If(i+2I)$.
This convergence holds for V large. As a matter of fact $2Ie^{-I/V}/V+e^{-2I/V}$ goes to 1 if V is large ($2Ie^{-I/V}/V+e^{-2I/V}\simeq 1+I^3/V^3 \rightarrow 1$ (Taylor's theorem, quadratic approximation)) and it follows $Ie^{-i/V}/V\simeq 2I^2e^{-(i+I)/V}/V^2+Ie^{-(i+2I)/V}/V$.

Let us now show that $q_{M-V/2-r}$ (= $2rq_M/V$) converges to If(M-V/2-r) provided that q_M is close to $e^{1/2-M/V}$.
$2rq_M/V \rightarrow 2re^{1/2-M/V}/V = Ie^{1/2-M/V}/V$ and $If(M-V/2-r) = Ie^{(1/2-M/V+r/V)}/V$. Convergence follows for large values of V.
Let us also show that $q_{M-V/2-r-I}$ (= $[2(I-r)/V +4rI/V^2]q_M$) goes to If(M-V/2-r-I) provided that q_M is close to $e^{1/2-M/V}$.
$[2(I-r)/V+4rI/V^2]q_M = (I/V+2I^2/V^2)q_M \rightarrow (I/V+2I^2/V^2)e^{1/2-M/V}$ and $If(M-V/2-r-I) = Ie^{(1/2-M/V+r/V+I/V)}/V$. Convergence follows for large values of V.
Now observe that if q_{i+I} and q_{i+2I} respectively converge to If(i+I) and If(i+2I), then $q_i= 2Iq_{i+I}/V+q_{i+2I} \rightarrow 2I^2f(i+I)/V+If(i+2I)$. Yet we also know that that If(i) checks the main recurrence equations (6c), so that $2I^2f(i+I)/V+If(i+2I) \rightarrow If(i)$. It follows that $q_i \rightarrow If(I)$ for V large, i being a multiple of I.

Finally let us check that $e^{1/2-M/V}$ is close to $1-q_0-q_I-\ldots-q_{M-V/2-r}$.
We have $1-q_0-q_I-\ldots-q_{M-V/2-r} \rightarrow 1-I(f(0)+f(I)+\ldots..+f(M-V/2-r))= 1-I(1-f(M))/I= f(M)$. So q_M is close to f(M) as assumed.

Putting all the results together ensures that, if $r\neq 0$, then the discrete Nash equilibrium converges to the continuous Nash equilibrium only if r=I/2 for large values of V.

Case 2: r= 0 (Proof of proposition 6)

M-V/2-I and M lead to the same payoff, except if the opponent bids M-V/2-I or M. So we need:
$q_{M\text{-}V/2\text{-}I}(V+I)+q_M(V/2+I) = q_{M\text{-}V/2\text{-}I}(3V/2+I)+q_M(V/2)$

It follows: $\boxed{q_{M\text{-}V/2\text{-}I} = 2Iq_M/V}$ *(7a)*

The bids M-V/2-I and M-V/2-2I lead to the same payoff, except if the opponent bids M-V/2-I, M-V/2-2I or M. So we need:

$q_{M\text{-}V/2\text{-}2I}(V+2I)+q_{M\text{-}V/2\text{-}I}(V/2+2I)+q_M(V/2+2I)=$

$q_{M\text{-}V/2\text{-}2I}(3V/2+2I)+q_{M\text{-}V/2\text{-}I}(V+I)+q_M(V/2+I).$

It follows $q_{M\text{-}V/2\text{-}2I}(-V/2)+q_{M\text{-}V/2\text{-}I}(-V/2+I)+Iq_M = 0$

So we get: $\boxed{q_{M\text{-}V/2\text{-}2I} = 4I^2q_M/V^2}$ *(7b)*

We also get (same proof than for r≠0) the main recurrence equations:

$\boxed{q_i = 2Iq_{i+I}/V+q_{i+2I} \text{ for } i \text{ from 0 to } M\text{-}V/2\text{-}3I \text{ (with } i \text{ a multiple of } I)}$. *(7c)*

It immediately follows that $q_{M\text{-}V/2\text{-}2I} < q_{M\text{-}V/2\text{-}I}$ for large values of V (V>2I). So we can't get a decreasing probability distribution, which precludes the convergence of the discrete equilibrium to the continuous equilibrium.

Yet we can establish that, for V large, $q_{M\text{-}V/2\text{-}I}+q_{M\text{-}V/2\text{-}2I}$ (= $2Iq_M/V+ 4I^2q_M/V^2$) goes to I(f(M-V/2-I)+f(M-V/2-2I)), provided that q_M is close to f(M) = $e^{1/2-M/V}$.
As a matter of fact I(f(M-V/2-I)+f(M-V/2-2I))= $Ie^{(1/2-M/V)}(e^{I/V}+e^{2I/V})/V \rightarrow 2I\, e^{(1/2-M/V)}/V$ for large values of V. And $2Iq_M/V+ 4I^2q_M/V^2 = (2Iq_M/V)(1+2I/V) \rightarrow 2I\, e^{(1/2-M/V)}/V$ for large values of V. In a similar way we can establish that $q_{M\text{-}V/2\text{-}2I}+q_{M\text{-}V/2\text{-}3I}$ converges to I(f(M-V/2-2I)+ f(M-V/2-3I)) for large values of V. And we establish that, if $q_{i+I}+q_{i+2I}$ converges to I(f(i+I)+f(i+2I)) and $q_{i+2I}+q_{i+3I}$ converges to I(f(i+2I)+f(i+3I)), then q_i+q_{i+I} converges to I(f(i)+f(i+I)), i being a multiple of I. As a matter of fact $q_i+q_{i+I} = 2I(q_{i+I}+q_{i+2I})/V+q_{i+2I}+q_{i+3I}$ goes to $2I^2(f(i+I)+f(i+2I))/V+I(f(i+2I)+f(i+3I))$ by assumption, so goes to $(I/V)e^{-i/V}(2-I/V+0.5(I/V)^2+2.5(I/V)^3)$. And I(f(i)+f(i+I)) goes to $(I/V)e^{-i/V}(2-I/V+0.5(I/V)^2)$, which ensures the convergence for large values of V.

And q_M close to $e^{1/2-M/V}$ is a good assumption when $q_i+q_{i+I} \rightarrow I(f(i)+f(i+I))$ because we get, at least if (M-V/2)/I is even:

$$(q_0+q_I)+(q_{2I}+q_{3I})+\ldots+(q_{M\text{-}V/2\text{-}2I}+q_{M\text{-}V/2\text{-}I}) = I(1+e^{-I/V}+\ldots+e^{-(I/V)((M-V/2)/I-1)})/V = \frac{I\left(1-e^{-\frac{\left(\frac{I}{V}\right)\left(M-\frac{V}{2}\right)}{I}}\right)}{V\left(1-e^{-\frac{I}{V}}\right)}$$

which goes to 1- $e^{1/2-M/V}$ for large values of V.

Let us observe that (1- f(M))/ (f(0)+f(I)+…..+f(M-V/2-I)) is the weight we have to assign to each f(kI) (with k an integer) to take into account that the bid kI, in the continuous equilibrium, is only played with probability f(kI)db. Given the above calculus, this weight is equal to $(1-e^{1/2-M/V})/\left[\frac{\left(1-e^{-\frac{\left(\frac{I}{V}\right)\left(M-\frac{V}{2}\right)}{I}}\right)}{V\left(1-e^{-\frac{I}{V}}\right)}\right]$= I (for large values of V).

Putting all the results together, we can conclude that the sums of the discrete probabilities of two adjacent bids go to the sums of the continuous probabilities of the same two bids.

Observe that, for large values of V, we get again:

$q_{M\text{-}V/2\text{-}I} = 2Iq_M/V$

$q_{M-V/2-2I} = 4I^2q_M/V^2 \rightarrow 0$ (in comparison with $q_{M-V/2-I}$) for large values of V
$q_{M-V/2-3I} = 2Iq_{M-V/2-2I}/V + q_{M-V/2-I} \rightarrow 2Iq_M/V$
$q_{M-V/2-4I} \rightarrow 4I^2q_M/V^2 + 4I^2q_M/V^2 \rightarrow 8I^2q_M/V^2 \rightarrow 0$
$q_{M-V/2-5I} \rightarrow 2Iq_M/V$
And so on. So we get again the yoyo phenomenon and, given that $(4I^2q_M/V^2)/(2Iq_M/V) = 2I/V$, the phenomenon is strengthened when I decreases.