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Central bank accountability under adaptive learning

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Abstract

Using a New Keynesian model, we examine the accountability issue in a delegation framework where private agents form expectations through adaptive learning while the central bank is rational and optimally sets monetary policy under discretion. Learning gives rise to an incentive for the central bank to accommodate less the effect of inflation expectations and cost-push shocks on inflation and induces thus a deviation of endogenous variables from rational expectations equilibrium. To help the central bank to better manage the intratemporal tradeoff, the government should nominate a liberal central banker, i.e., set a negative optimal inflation penalty according to the value of learning coefficient. By reducing the deviation of the feedback effects of inflation expectations and cost-push shocks on inflation and the output gap from the corresponding ones under rational expectations, the optimal inflation penalty allows the economy to be more efficient and improves the social welfare. The main conclusions are valid under both constant- and decreasing-gain learning.

Keywords: Adaptive learning, optimal monetary policy, accountability, inflation penalty, rational expectations.

JEL Classification: E42, E52, E58

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1 Introduction

Over the past several decades, most of the research on monetary policy and central banking relies upon the hypothesis of model-consistent or rational expectations (RE). The RE benchmark has permitted the theoretical literature in monetary economics to make major advances and improvements in its analysis of dynamics.

However, the assumption that private agents always form RE and exclusively base their economic decisions on such expectations seems to be too strong and is even heroic under some circumstances for effective monetary policy decision-making. According to Bernanke (2004), modelling adaptive learning is highly relevant to the understanding of modern U.S. monetary history. In a complex economic environment, private agents may not have a full view about the functioning of the economy due to the costs of information collection and treatment and hence cannot form fully rational expectations, particularly when this environment is unstable and could be disturbed by important financial and structural changes as those observed since the recent global financial crisis. This explains that the literature on adaptive learning attracts a growing interest.[1] In a world where private agents could make mistakes that are inherent to their learning process, it is important for the central bank to properly consider private agents' expectations in its monetary policy decisions and in its relationship with the government. In fact, introducing expectations with learning has three main consequences (Gaspar, Smets and Vestin 2010). First, agents do their own regressions and forecasts, and inflation is no longer only caused by the game between the central bank and the private sector. Second, promises of a future policy do not affect the behavior of private agents because the latter focus on the past value of inflation. Finally, expectations with learning produce a non-linear model. These consequences should be seriously taken into account in monetary policy decisions and the institutional design of the central bank, given that models with constant- and decreasing-gain learning seem to provide a good fit to the expectations of professional forecasters about a range of variables (Markiewicz and Pick 2014). On the other hand, the modern theory of monetary policy and central bank governance is essentially based on the RE hypothesis. In

[1] See Evans and Honkapohja (2009), Zumpe (2011), and Woodford (2013) for a survey of the literature.

the light of empirical evidence of learning by private agents, some conclusions about central banking should be reexamined.

Many central banks, including the Fed and the ECB, have adopted an inflation target near 2%. However, we have observed that the recent trend of inflation in many industrial countries tends to zero despite the extreme accommodative monetary policy put into place. Blanchard, Dell'Ariccia and Mauro (2010) argue that the solution to this problem consists in setting either a policy interest rate largely under zero or a 4% inflation target. However, if a 2% inflation target does not allow the inflation to be high enough, why a 4% target would make the difference? One possible explication of policy failure to bring inflation to its target level is that private agents follow a learning algorithm and may ignore in some circumstances the central bank's inflation target. Thus, an alternative solution to low inflation is to change the institutional design of the central bank conceived for the case in which private agents form RE (that perfectly integrate the inflation target) by taking into account the implications of learning. One likely consequence is that the central bank should be more accountable for the output-gap stabilization. To make this possible, the government should nominate a more liberal central banker.

This paper contributes to the literature that focuses on the consequences of non-rational expectations for monetary policy by considering the issue of central bank governance in a framework where private agents form expectations with adaptive learning. We examine how adaptive learning affects the way the government should delegate monetary policy decision to an independent central bank. The latter's incentives are affected by the weight on the target objective, which can be interpreted as the degree of central bank conservatism (Rogoff 1985) and is optimally set by the government. Under RE, central bank accountability can neither be ensured by insufficiently powered incentive schemes nor by excessively powered one, meaning that there is an optimal weight to place on the achievement of inflation target (Walsh 2003).

Our main findings are that, when private agents form expectations with adaptive learning, a higher inflation penalty weakens the feedback effects of inflation expectations and cost-push shocks on inflation while strengthening those on the output gap and the policy interest rate.

Moreover, as long as the learning algorithm is characterized by a positive learning coefficient, the government sets a negative optimal inflation penalty that is negatively correlated with the learning coefficient, implying that the government should nominate a more liberal central banker than under RE. This contrasts with the result found by Walsh (2003) who shows that inflation penalty should be positive if the central bank is subject to unobservable political pressures for greater economic expansion, while it is comparable to that obtained by Dai and Spyromitros (2010) who find that when private agents form RE, the optimal inflation penalty should be negative when the central bank has a preference for policy robustness.

We have obtained closed-form solutions that provide a better understanding of policy tradeoffs in the presence of accountability issue. In our framework, the government could affect both the intratemporal tradeoff between inflation and the output gap but also an intertemporal tradeoff introduced by a departure from RE. In the current period, the central bank stabilizes the economy in a way to better anchor future inflation expectations, thus reducing the future intratemporal trade-off. The institutional design that consists to nominate a liberal central banker according to our analysis helps the central bank to achieve the intratemporal tradeoff and could substantially improve the social welfare.

Our paper is related to a growing literature that applies learning to macroeconomic models, in particular several strands of literature that examine the consequences of adaptive learning applied to monetary policy. These studies demonstrate the relevance of introducing adaptive learning for monetary policy analysis and design. Marcet and Nicolini (2003) have shown that the process of learning matches remarkably well some major stylized facts observed during the hyperinflations of the 1980's, while Slobodyan and Wouters (2012) have reported that expectations based on small forecasting models are closely related to the survey evidence on inflation expectations, and the adaptive learning model with an inertial Taylor rule fits the data better than the one with RE. A number of studies (Bullard and Mitra, 2002, Evans and Honkapohja 2003, 2006) find that Taylor rules, which are optimal or ensure determinacy under RE, can lead to instability if private expectations slightly deviate from rationality by following adaptive learning. Machado (2013) suggests that, under adaptive learning, a direct monetary policy response to asset prices is not desirable under common interest rate rules. In general, departures from RE increase the potential for instability in the economy. This strengthens the importance of anchoring inflation expectations. Ferrero (2007) find that by strongly reacting to private agents' inflation expectations formed with adaptive learning, a central bank increases the speed of convergence and thus shortens the transition length to the RE equilibrium. Gaspar, Smets and Vestin (2010) find that the commitment rule under RE is robust when expectations are formed with adaptive learning. Marzioni (2014) shows that the economic dynamics are less volatile if the central bank takes into account the impact of signals, i.e., the communication of its own forecasts, on private agents' prior expectations estimated in conformity with the adaptive learning scheme. Our paper is closely related to Molnár and Santoro (2014) who explore issues of intertemporal and intratemporal tradeoffs that arise when a rational central bank should optimally conduct monetary policy while private agents form expectations with adaptive learning. To the difference of existing studies, we take into account the institutional design of the central bank.

The remainder of the paper is organized as follows. The next section presents the model. Section 3 solves the model under discretionary monetary policy. Section 4 analyzes how inflation penalty influences the effects of constant-gain learning on the feedback coefficients of endogenous variables in response to inflation expectations and cost-push shocks and determines the optimal level of inflation penalty. Section 5 discusses the implications of decreasing-gain learning for central bank accountability. The last section concludes.

2 The model

The theoretical framework is based on a standard New Keynesian model that is widely used in the recent literature in monetary policy (Clarida, Galí and Gertler 1999). It is characterized by optimizing private-sector behavior and nominal rigidities. This model consists of an aggregate demand specification (or IS equation) derived from the representative household's optimal consumption decision and a forward-looking inflation adjustment (or Phillips curve) equation.

2.1 Aggregate demand and supply

The New Keynesian IS equation is given by

$$x_t = E_t^* x_{t+1} - \sigma^{-1} \left(r_t - E_t^* \pi_{t+1} \right) \tag{1}$$

where x_t stands for the output gap, r_t the nominal interest rate and π_t inflation. Here, σ represents the risk aversion for households. The expectation operator E_t^* represents private agents' expectations conditional on the information set available at time t. The asterisk on the expectations operator in (1) reflects the fact that the private sector may form expectations which could be rational or not. To simplify the analysis, we assume that there is no demand shock in the IS equation. Indeed, since the central bank can neutralize a demand shock by setting the interest rate, introducing such shocks does not modify the results.

The forward-looking New Keynesian Phillips curve is:

$$\pi_t = \beta E_t^* \pi_{t+1} + \kappa x_t + e_t, \tag{2}$$

where $0 < \beta < 1$ is the discount factor. The composite parameter κ measures the output-gap elasticity for inflation and captures the effects of the output gap on real marginal costs and thus on inflation, and it is a function of structural parameters, i.e., $\kappa \equiv \frac{(1-\vartheta)(1-\vartheta\beta)}{\vartheta}(1+\varphi)$, where φ represents the inverse of the steady-state elasticity of labor supply and ϑ the share of firms that do not optimally adjust but simply update in period t their previous price by the steady-state inflation rate. The noise $e_t \sim N(0, \sigma_e^2)$ is an *iid* cost-push or supply shock. Assuming that shocks are serially uncorrelated allows the model to be tractable, once learning algorithms are introduced. Furthermore, this assumption is justified in the context of learning since as shown by Milani (2006, 2007), learning represents the main cause of persistence in inflation.

2.2 Institutional settings and policy objectives

We assume that the central bank (the agent) is independent and is delegated by the government (the principal) to implement the monetary policy without any external political interference. Under RE, this institutional setting would be credible for private agents and could avoid the inflation bias if the nominated central banker was conservative, with conservativeness referring to the relative importance that he/she assigns to price stability objective. To describe the relationship in this delegation framework, we distinguish the objective function of the government from that of the central bank. The government designs the targeting regime, by setting the central bank's target and the penalty associated with a failure to achieve the target, under which the central bank conducts monetary policy.[2]

The expected social loss function is assumed to take a standard form:

$$L_t^s = \frac{1}{2} E_t \sum_{i=0}^{+\infty} \beta^i [\pi_{t+i}^2 + \alpha x_{t+i}^2], \tag{3}$$

where $\alpha > 0$ is the relative weight assigned to the objective of output-gap stabilization. Social loss is a function of the variance of both inflation and the output gap.[3] The overly ambitious output target, which is common in the Barro-Gordon framework, is here absent in the formulation given in (3). Thus, discretionary monetary policy set to minimize social loss (3) would avoid an average inflation bias.

The central bank implements discretionary monetary policy to minimize the conditional expectation of the loss function:[4]

$$L_t^{CB} = \frac{1}{2} E_t \sum_{i=0}^{+\infty} \beta^i [(1+\tau)\pi_{t+i}^2 + \alpha x_{t+i}^2], \quad 1+\tau > 0, \tag{4}$$

[2] See Eijffinger and Masciandaro (2014) for a survey of the literature on central bank governance.

[3] The loss function (3) could be micro-founded by deriving the utility function of representative agent as in Woodford (2003).

[4] Issues of learning when monetary policy is under commitment have been studied by Evans and Honkapohja (2006), and Mele, Molnár and Santoro (2012). The first study shows that both rational expectations commitment equilibrium (RECE) and rational expectations discretionary equilibrium (REDE) are attainable, while the second suggests that the optimal monetary policy drives the economy far from the RECE, and to the REDE

where τ is the penalty (weight) inflicted by the government on the central banker for deviations from inflation target. The loss function represents a weighted average of the variance of inflation and the output gap around their respective target. The inflation target is set to zero for simplicity even though it characterizes, together with τ, alternative inflation targeting regimes. Under RE, setting inflation target at a positive level influences the linear penalty associated with inflation (Svensson 1997) and can offset any average inflation bias as in the optimal contract formulated by Walsh (1995). The condition $1 + \tau > 0$ implies that any deviation from the inflation target is a loss.

2.3 Learning rules of private expectations

Private agents' expectations are assumed to be formed according to an adaptive learning algorithm. This assumption relies on the idea that agents have no knowledge of the exact process governing the evolution of endogenous variables. However, to improve their decisions, they recursively estimate a Perceived Law of Motion (PLM) in the terminology of Evans and Honkapohja (2001), consistent with the law of motion that the central bank follows under RE. More precisely, private agents believe that steady-state levels of inflation and the output gap only depend on *iid* cost-push shocks and hence perceive their expected levels as constant knowing that the conditional and unconditional expectations of these variables are identical. This provides a justification for private agents to estimate these variables using sample means.

In accordance with the literature on learning algorithms (Marcet and Sargent 1989, Evans and Honkapohja 2001, Marcet and Nicolini 2003), private agents' expectations are assumed to be formed with following learning algorithms:

$$E_t \pi_{t+1} \equiv a_t = a_{t-1} + \gamma_t (\pi_{t-1} - a_{t-1}), \tag{5}$$

$$E_t x_{t+1} \equiv b_t = b_{t-1} + \gamma_t (x_{t-1} - b_{t-1}), \tag{6}$$

where $0 \leq \gamma_t \leq 1$ represents a deterministic sequence of learning gains that defines the speed of integration of new data into expectations, which are initially set at a_0 and b_0. This learning mechanism implies that inflation (output-gap) expectations are increasing with last period inflation (output gap).[5] In the case of decreasing-gain learning, as time goes by, private agents assign a decreasing importance to past values of inflation and the output gap in the formation of expectations. When $t \to \infty$, i.e., $\gamma \to 0$, the policymakers cannot manipulate future expectations by changing the current policy, hence they cannot intervene on intertemporal tradeoff.

As Preston (2005) highlights it, without the RE hypothesis, the structural equations forming the New Keynesian model should also include the forecast of macroeconomic conditions in many time periods to come (infinite horizon learning) and not only one period ahead (corresponding to the Euler-equation learning). However, following the learning literature, we adopt the Euler-equation learning to maintain analytical tractability.

Using (5), we rewrite (2) as

$$\pi_t = \beta \left[a_{t-1} + \gamma_t(\pi_{t-1} - a_{t-1})\right] + \kappa x_t + e_t. \tag{7}$$

This equation shows the dependence of inflation on the current shock, the current output-gap value, and the past values of inflation and especially of expected inflation. The past expectations based on learning themselves contain a share of past inflation shocks. This will be explicitly shown below by defining the Actual Law of Motion (ALM).

3 The equilibrium under monetary discretion

Assuming that expectations are based on learning is an alternative way of conceiving how private agents interact with monetary policy compared to the RE hypothesis. Expectations with learning modify the central bank's trade-off between inflation and the output gap. Notably, they give rise to a greater incentive for the central bank to influence the current inflation but also a larger room to maneuver through manipulating the output gap. Thus, there could be good grounds for an inflation penalty to be inflicted to the central bank if the latter does not

[5] The learning process formulated in (5) and (6) is limited by the fact that they focus on past information and the forecast with one period ahead.

respect the inflation target fixed by the government. To put into evidence the role of learning in the presence of accountability issue, in the following, we solve the model first under RE and then under adaptive learning.

3.1 Benchmark equilibrium with rational expectations

We concisely show the rational expectations equilibrium (REE) solution when the central bank sets optimal monetary policy taking private agents' expectations as given. We use then this solution as a benchmark to illustrate how the equilibrium is modified by an optimal policy designed with private agents' beliefs being taken into account.

The central bank minimizes its loss function (4) subject to (2) taking inflation expectations as given. This leads to the optimal tradeoff rule between inflation and the output gap:

$$x_t = -\frac{(1+\tau)\kappa}{\alpha}\pi_t. \tag{8}$$

The targeting rule (8) implies the equilibrium solution of inflation and the output gap now depends on inflation penalty and the central bank's preferences for output-gap stabilization. The tradeoff between inflation and the output gap is affected by the presence of inflation penalty in the sense that higher the inflation penalty is, the costlier it is for the central bank to adjust the output gap.

Solving (1), (2) and (8) yields the ALM for inflation and the output gap, and the interest rate rule that implements the optimal monetary policy as follows:

$$\pi_t = \frac{\alpha\beta}{\alpha+\kappa^2(1+\tau)}E_t^*\pi_{t+1} + \frac{\alpha}{\alpha+\kappa^2(1+\tau)}e_t, \tag{9}$$

$$x_t = -\frac{\beta\kappa(1+\tau)}{\alpha+\kappa^2(1+\tau)}E_t^*\pi_{t+1} - \frac{\kappa(1+\tau)}{\alpha+\kappa^2(1+\tau)}e_t, \tag{10}$$

$$r_t = \sigma E_t^* x_{t+1} + \left[1+\frac{\sigma\beta\kappa(1+\tau)}{\alpha+\kappa^2(1+\tau)}\right]E_t^*\pi_{t+1} + \frac{\sigma\kappa(1+\tau)}{\alpha+\kappa^2(1+\tau)}e_t. \tag{11}$$

The ALMs defined by (9)-(11) correspond to the anticipated utility policy set by a policymaker who does not take into account the way private agents revise in the future their beliefs.[6]

[6] The anticipated utility (Kreps 1998) is commonly used in the learning literature. It is similar to expected

The system of equations (1), (2) and (8) has a unique non-explosive REE solution only in terms of exogenous state variable e_t, which is known as the "minimal state variable" solution (McCallum, 1983). Thus, in the case of RE, i.e., $E_t^* = E_t$, the solution of π_t takes the following form: $\pi_t = \zeta e_t$. The formation of RE conditional on the available information at t leads to :

$$E_t \pi_{t+1} = \zeta E_t e_{t+1} = 0. \tag{12}$$

Substituting $E_t \pi_{t+1} = 0$ into (9)-(11), we obtain the REE solution corresponding to the optimal discretionary monetary policy:

$$\pi_t = \frac{\alpha}{\alpha + \kappa^2(1+\tau)} e_t, \tag{13}$$

$$x_t = -\frac{(1+\tau)\kappa}{\alpha + \kappa^2(1+\tau)} e_t, \tag{14}$$

$$r_t = \frac{\sigma\kappa(1+\tau)}{\alpha + \kappa^2(1+\tau)} e_t. \tag{15}$$

The optimal level of inflation penalty is determined by minimizing (3) taking account of the solutions of π_t and x_t given by (13)-(14) as:

$$\tau = 0 \tag{16}$$

It is to underline that this result is obtained under the RE hypothesis and in the absence of inflation bias.

3.2 Equilibrium with learning

The deviation of private expectations from rationality implies that the expectations become state variables and hence their law of motion could affect monetary policy. Current monetary policy decision, given its effects on future inflation expectations, also has to consider future

utility except for two properties. First, private agents have no knowledge of the true model. Second, despite that private agents know that they are learning about the parameters or the state of the economy, they choose actions today in a way that is myopic with respect to the updating of their information set while ignoring that they will continue to learn in the future. Under RE, anticipated utility coincides with expected utility, implying that the current beliefs of private agents reflect the true model and the optimal anticipated utility policy would also maximize expected utility.

intratemporal tradeoffs between inflation and the output gap. Here, we assume that the central bank knows the exact learning algorithm followed by private agents and takes it into account when setting monetary policy. This hypothesis, even though it is quite strong, allows us to appreciate how the policy design could change if private agents depart from rationality.

The central bank's optimization problem consists of minimizing (4) subject to (1)-(2) and (5)-(6), with $E_t^* x_{t+i+1}$ being substituted by b_{t+i} and $E_t^* \pi_{t+i+1}$ by a_{t+i}. The Lagrangian of the central bank's optimization problem is:

$$\begin{aligned}\mathcal{L}_t^{CB} &= E_t \sum_{i=0}^{+\infty} \beta^i \Big\{ \tfrac{1}{2} \left[(1+\tau)\pi_{t+i}^2 + \alpha x_{t+i}^2 \right] \quad - \lambda_{1,t+i} \left[x_{t+i} - b_{t+i} + \sigma^{-1}(r_{t+i} - a_{t+i}) \right] \\ &\quad - \lambda_{2,t+i} \left[\pi_{t+i} - \beta a_{t+i} - \kappa x_{t+i} - u_{t+i} \right] - \lambda_{3,t+i} \left[a_{t+i+1} - a_{t+i} - \gamma_{t+i+1}(\pi_{t+i} - a_{t+i}) \right] \\ &\quad - \lambda_{4,t+i} \left[b_{t+i+1} - b_{t+i} - \gamma_{t+i+1}(x_{t+i} - b_{t+i}) \right] \Big\}, \end{aligned} \tag{17}$$

where $\lambda_{i,t}$, with i=1,...,4, are Lagrangian multipliers associated with (1), (2), (5) and (6), respectively. The first-order conditions of the central bank's optimization problem are obtained by deriving (17) with respect to r_t, π_t, x_t, a_{t+1} and b_{t+1}:

$$\lambda_{1,t} = 0, \tag{18}$$

$$(1+\tau)\pi_t - \lambda_{2,t} + \gamma_{t+1}\lambda_{3,t} = 0, \tag{19}$$

$$\alpha x_t - \lambda_{1,t} + \kappa\lambda_{2,t} + \gamma_{t+1}\lambda_{4,t} = 0, \tag{20}$$

$$\lambda_{3,t} = E_t \left[\frac{\beta}{\sigma}\lambda_{1,t+1} + \beta^2 \lambda_{2,t+1} + \beta\lambda_{3,t+1}(1-\gamma_{t+2}) \right], \tag{21}$$

$$\lambda_{4,t} = E_t \left[\beta\lambda_{1,t+1} + \beta\lambda_{4,t+1}(1-\gamma_{t+2}) \right]. \tag{22}$$

Substituting $\lambda_{1,t} = 0$ given by (18) into (22) leads to $\lambda_{4,t} = \beta(1-\gamma)E_t[\lambda_{4t+1}]$. The only bounded forward-looking solution is $\lambda_{4,t} = \lambda_{4,t+1} = 0$. Using these results into (20) yields $\lambda_{2,t} = -\frac{\alpha}{\kappa} x_t$ and $\lambda_{2,t+1} = -\dfrac{\alpha}{\kappa} x_{t+1}$. Substituting $\lambda_{2,t} = -\frac{\alpha}{\kappa} x_t$ into (19), we get:

$$(1+\tau)\pi_t + \frac{\alpha}{\kappa} x_t + \gamma_{t+1}\lambda_{3,t} = 0. \tag{23}$$

According to (23), only $\lambda_{3,t}$, the Lagrangian multiplier, associated with the evolution of inflation expectations, plays a role in the choice of optimal monetary policy. We note that when $\gamma = 0$, i.e., the case where expectations are constant, the rule defined by (23) becomes identical to (8), which is the optimality condition derived for setting the optimal monetary policy under discretion when the private sector forms RE.

Besides intratemporal tradeoff between the output gap and inflation observed in the benchmark case with RE, the learning effect induces an intertemporal tradeoff due to the feedback between monetary policy and inflation expectations. The term $\gamma_{t+1}\lambda_{3,t}$ in (23) distinguishes the optimal policy rule under learning from the one under RE by the fact that the optimal decision should now depend on inflation expectations. According to (5) and (23), γ_{t+1} represents the marginal effect of an increase in inflation on inflation expectations at $t + 1$, i.e., a_{t+1}; and $\lambda_{3,t}$ the marginal effect of an increase in inflation expectations on welfare loss. For $\gamma_{t+1} > 0$, the sign of $\lambda_{3,t}$ depends on the sign of current inflation expectations a_t. Given that inflation target is set to zero, a_t could be either positive or negative, depending on the nature of past shocks. When a_t is positive, an increase in a_t drives it further away from the target and hence reduces social welfare, implying that $\lambda_{3,t}$ is positive, and *vice versa*.

A change in inflation in the current period impacts future inflation expectations and thus will result in a variation of social welfare. Thus, the learning effect introduces an intertemporal tradeoff for the central bank between the stabilization in the current period and the stabilization in the following periods. This tradeoff is generated by the ability of the central bank to influence inflation expectations in future periods.

In the following, we successively consider constant- and decreasing-gain learning.

4 Inflation penalty under constant-gain learning

As extensively discussed in the learning literature (Evans and Honkapohja 2009), private agents would be more inclined to use a constant-gain learning algorithm if they believe in possible structural changes in the near future. In this section, we first analyze how constant-gain learning and inflation penalty interact with macroeconomic stabilization compared to the benchmark case where private agents form RE, and then examine how the government should set inflation penalty to improve the social welfare.

4.1 Equilibrium solution and the effect of learning

There exists a unique solution of the ALMs corresponding to the central bank's control problem under constant-gain learning (see Appendix A.1 for the proof). The ALM of inflation is given by:

$$\pi_t = c_\pi^{cg} a_t + d_\pi^{cg} e_t \tag{24}$$

where

$$c_\pi^{cg} = -\frac{p_0 + p_2 \left(c_\pi^{cg}\right)^2}{p_1}, \tag{25}$$

$$d_\pi^{cg} = \frac{\alpha}{\kappa^2(1+\tau) + \alpha + \alpha\beta^2\gamma^2(\beta - c_\pi^{cg}) + \beta\gamma(1-\gamma)\left\{\alpha\beta - \left[\alpha + \kappa^2(1+\tau)\right] c_\pi^{cg}\right\}}, \tag{26}$$

with

$$\begin{aligned}
p_2 &= \gamma\left\{\alpha\beta\left[1-\gamma(1-\beta)\right] + \beta\kappa^2(1-\gamma)(1+\tau)\right\} > 0, \\
p_1 &= (1-\gamma)\left\{\alpha\beta\left[1-\gamma(1-\beta)\right] + \beta\kappa^2(1-\gamma)(1+\tau)\right\} - \left\{\kappa^2(1+\tau) + \alpha + \alpha\beta^2\gamma\left[1-\gamma(1-\beta)\right]\right\} < 0, \\
p_0 &= \alpha\beta\left\{1 - \beta(1-\gamma)\left[1-\gamma(1-\beta)\right]\right\} > 0.
\end{aligned}$$

The solution for c_π^{cg} ensuring a non-explosive evolution of inflation is given by:

$$c_\pi^{cg} = \frac{-p_1 - \sqrt{p_1^2 - 4p_2p_0}}{2p_2}. \tag{27}$$

Given that the value of c_π^{cg} is in the interval $]0; \frac{\alpha\beta}{\alpha+\kappa^2(1+\tau)}[$, current inflation increases with inflation expectations (a_t) less than proportionally. Current inflation is indirectly influenced by the central bank's policy responses to past shocks. Thus, an increase in the learning coefficient γ has two opposite effects on c_π^{cg}. According to (5), a higher γ increases the positive correlation between current inflation π_t and future inflation expectations a_{t+1}, and hence the incentive for the central bank to lower c_π^{cg}, i.e., the feedback from a_t to π_t in (24). However, according to the same learning algorithm, an increase in γ attenuates the effect of a_t on a_{t+1}, thus allowing a greater c_π^{cg} without deteriorating social welfare. In general, the first effect dominates, i.e., $\frac{\partial c_\pi^{cg}}{\partial \gamma} < 0$ (Appendix A.2). When the learning coefficient is zero, (24) is reduced to the form given by (9) with exogenous inflation expectations.

Regarding the effect of cost-push shocks on current inflation, we notice that the higher and closer to 1 the learning coefficient is, the more inflation is influenced by current cost-push shocks. The denominator of d_π^{cg} is clearly decreasing in γ if $\frac{\partial c_\pi^{cg}}{\partial \gamma} < 0$.

Substituting π_t given by (24) into (2) yields the equilibrium output gap:

$$x_t = c_x^{cg} a_t + d_x^{cg} e_t, \tag{28}$$

with

$$c_x^{cg} = -\frac{\beta - c_\pi^{cg}}{\kappa}, \tag{29}$$

$$d_x^{cg} = -\frac{1 - d_\pi^{cg}}{\kappa}. \tag{30}$$

In response to an increase in private inflation expectations, the central bank sets a monetary policy that reduces the output gap, given that $c_\pi^{cg} < \frac{\alpha\beta}{\alpha+\kappa^2(1+\tau)}$ implies $c_x^{cg} < -\frac{\beta\kappa(1+\tau)}{\alpha+\kappa^2(1+\tau)}$. If the policy involves a contraction in the output gap, equation (2) implies an increase in current inflation smaller than that of inflation expectations and hence lower future inflation expectations.

Using (24) and (28) to eliminate π_t and x_t in equation (1), we get the ALM ruling the evolution of the interest rate:

$$r_t = \delta_r^{cg} b_t + c_r^{cg} a_t + d_r^{cg} e_t \tag{31}$$

where $\delta_r^{cg} = \sigma$, $c_r^{cg} = 1 + \dfrac{\sigma(\beta - c_\pi^{cg})}{\kappa}$ and $d_r^{cg} = \dfrac{\sigma(1 - d_\pi^{cg})}{\kappa}$. It is to notice that $c_r^{cg} > 1$ since $c_\pi^{cg} \leq \frac{\alpha\beta}{\alpha+\kappa^2(1+\tau)} < \beta$, $\forall\gamma$, meaning that the Taylor principle is verified.

The ALM (31) highlights that the interest rate is affected by both inflation and output-gap learning with their impact depending on the risk aversion of households. Thus, a higher σ mechanically increases the interest rate, all things being equal. Moreover, the feedback coefficient (c_π^{cg}) of inflation expectations associated with inflation in (24) is negatively correlated with the feedback coefficient (c_r^{cg}) of inflation expectations on the interest rate. A positive cost-push shock increases the interest rate. The higher the learning coefficient γ is, the greater the effect of the shock on the interest rate (smaller d_π^{cg} and hence greater d_r^{cg}). The central bank sets the coefficient associated with output-gap expectations b_t in (31) to $\delta_r^{cg} = \sigma$, thus fully neutralizing the effect of output-gap expectations on the output gap and hence inflation.

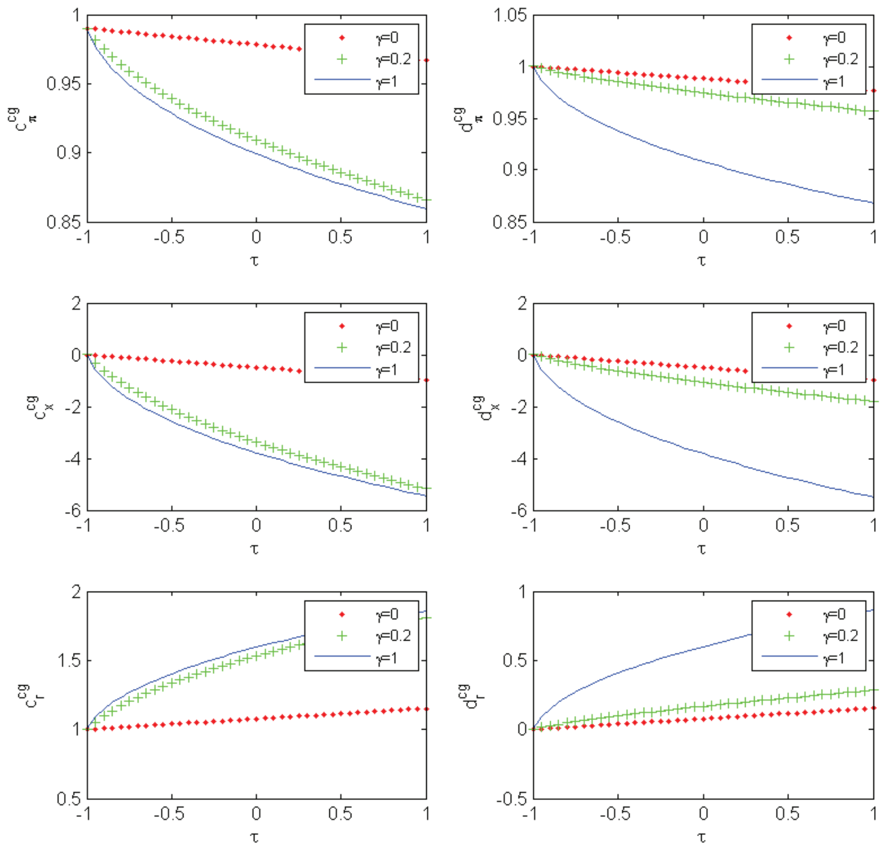

Figure 1: Feedback coefficient associated with inflation and cost-push shocks in the ALMs.

The values of c_π^{cg}, c_x^{cg} and c_r^{cg} are very sensitive to the value of γ for $\gamma = 0.2$, and are close to their values when $\gamma = 1$, while for $\gamma = 0.2$, the feedback coefficients d_π^{cg}, d_x^{cg} and d_r^{cg} stay close to their corresponding curves when $\gamma = 0$ (Figure 1). The latter are identical to those obtained under RE.

Result 1. *An increase in learning coefficient reduces (enlarges) the deviation of feedback coefficients of inflation expectations and cost-push shocks in the actual law of motion of inflation (the output gap and the policy interest rate) from the corresponding ones under rational expectations if* $\tau < \frac{\alpha\beta(1-\beta^2)}{\kappa^2} - 1$.

Proof. See Appendix A.2.

For the baseline parameter values, i.e., $\beta = 0.99$, $\kappa = 0.024$, $\alpha = 0.048$, we obtain $\tau < 0.6418$. It is to notice that the condition imposed on τ is obtained with $\gamma = 1$ and it can be considerably relaxed as γ decreases.

Comparing (13)-(15) with their corresponding equations (24), (28) and (31), we find that the feedback effect of inflation expectations on the ALM of inflation (the output gap) is lower (higher) in the case of learning than under RE, i.e., $c_\pi^{cg} < \frac{\alpha\beta}{\alpha+\kappa^2(1+\tau)}$ ($c_x^{cg} < -\frac{\beta\kappa}{\alpha+\kappa^2(1+\tau)}$ respectively). To make this possible, the interest rate under learning must react more strongly to inflation expectations, i.e., $c_r^{cg} > 1 + \frac{\sigma\beta\kappa}{\alpha+\kappa^2(1+\tau)}$. The ALMs of inflation and the output gap are independent of output-gap expectations under both learning and RE, while the interest rate under learning responds to output-gap expectations with the same coefficient as under RE and is independent of inflation penalty. Regarding the feedback coefficients associated with e_t in the ALMs, it is straightforward to show that $d_\pi^{cg} < \frac{\alpha}{\alpha+\kappa^2(1+\tau)}$, $d_x^{cg} < -\frac{\kappa}{\alpha+\kappa^2(1+\tau)}$ and $d_r^{cg} > \frac{\sigma\kappa}{\alpha+\kappa^2(1+\tau)}$, meaning that under learning, the current inflation and output gap respond more strongly while the interest rate reacts less to current cost-push shocks than under RE.

The learning gain coefficient determines the time horizon of private agents' expectations and hence the persistence of inflation and the flexibility in the central bank's discretionary policy. For $\gamma = 0$, i.e., inflation expectations are constant over time with $a_t = a_{t-1}$ and $b_t = b_{t-1}$, we obtain:

$$c_\pi^{cg} = \frac{\alpha\beta}{\alpha + \kappa^2(1+\tau)}, \tag{32}$$

$$d_\pi^{cg} = \frac{\alpha}{\alpha + \kappa^2(1+\tau)}. \tag{33}$$

Thus, the feedback coefficients of the ALMs for inflation, the output gap and the interest rate are identical to those associated with anticipated utility policy given by (9)-(11).

For $\gamma = 1$, i.e., inflation expectations are static or naive with $a_t = \pi_{t-1}$ and $b_t = x_{t-1}$, we have

$$\begin{aligned} c_{\pi}^{cg} &= \frac{\kappa^2(1+\tau) + \alpha + \alpha\beta^3 - \sqrt{\left[\kappa^2(1+\tau) + \alpha + \alpha\beta^3\right]^2 - 4\alpha^2\beta^3}}{2\alpha\beta^2}, \\ d_{\pi}^{cg} &= \frac{\alpha}{\alpha + \kappa^2(1+\tau) + \alpha\beta^2(\beta - c_{\pi}^{cg})}. \end{aligned}$$

As γ is equal to 1, inflation is self-sustained because private agents' inflation expectations are depending on past inflation. The effect is similar for γ near to 1, since in this case private agents form expectations on the basis of a very short horizon.

4.2 The effects of inflation penalty on the feedback coefficients of ALMs

Inflation penalty represents an extra incentive for the central bank to improve the tradeoff between inflation and the output gap.

We examine how an increase in inflation penalty could affect the ALMs by deriving the feedback coefficients in (24), (28) and (31) with respect to τ (Appendix A.3):

$$\frac{\partial c_{\pi}^{cg}}{\partial \tau} = \kappa \frac{\partial c_{x}^{cg}}{\partial \tau} = -\frac{\kappa}{\sigma}\frac{\partial c_{r}^{cg}}{\partial \tau} = \frac{c_{\pi}^{cg}\left(p_2 \frac{\partial p_1}{\partial \tau} - p_1 \frac{\partial p_2}{\partial \tau}\right) - p_0 \frac{\partial p_2}{\partial \tau}}{p_2\sqrt{p_1^2 - 4p_2p_0}} < 0,$$

$$\frac{\partial d_{\pi}^{cg}}{\partial \tau} = \kappa \frac{\partial d_{x}^{cg}}{\partial \tau} = -\frac{\kappa}{\sigma}\frac{\partial c_{r}^{dg}}{\partial \tau} = \frac{-\alpha\left\{\kappa^2 - \alpha\beta^2\gamma^2 \frac{\partial c_{\pi}^{cg}}{\partial \tau} - \beta\gamma(1-\gamma)\left[\alpha + \kappa^2(1+\tau)\right]\frac{\partial c_{\pi}^{cg}}{\partial \tau}\right\}}{\left\{\kappa^2(1+\tau) + \alpha + \alpha\beta^2\gamma^2(\beta - c_{\pi}^{cg}) + \beta\gamma(1-\gamma)\left\{\alpha\beta - \left[\alpha + \kappa^2(1+\tau)\right]c_{\pi}^{cg}\right\}\right\}^2} < 0,$$

$$\frac{\partial \delta_{r}^{cg}}{\partial \tau} = 0,$$

where $\frac{\partial p_1}{\partial \tau} = \kappa^2\left[\beta(1-\gamma)^2 - 1\right] < 0$, $\frac{\partial p_2}{\partial \tau} = \gamma\beta\kappa^2(1-\gamma) > 0$. The ALMs of inflation and the output gap are independent of output-gap expectations and the feedback effect of output-gap expectations on the interest rate is not affected by inflation penalty. An increase in inflation penalty induces a decrease in the feedback effects of inflation expectations and cost-push shocks on current inflation and the policy interest rate while it strengthens the feedback effects on the current output gap. Thus, a higher inflation penalty incites the central bank to focus more on reducing inflation due to current positive cost-push shocks but reduces the possibility for it to control future inflation expectations through the feedback between inflation and inflation expectations.

Figure 1 shows how these coefficients evolve with τ. Notably, a given past inflation or a positive cost-push shock induces lower current inflation and output gap but a higher interest rate if the government increases inflation penalty. For the parameter values in the calibration of Woodford (1999), i.e., $\beta = 0.99$, $\kappa = 0.024$, $\alpha = 0.048$ and $\sigma = 0.157$, the feedback coefficients c_π^{cg}, d_π^{cg}, c_x^{cg} and d_x^{cg} (c_r^{cg} and d_r^{cg}) are decreasing (increasing respectively) in inflation penalty. The feedback coefficients of the ALMs of inflation and the output gap (the interest rate) are decreasing (increasing) in the learning coefficient γ, meaning that a higher learning coefficient reinforces the effects of inflation penalty.

Result 2. *An increase in inflation penalty weakens (strengthens) the positive feedback effect of inflation expectations and cost-push shocks on inflation (the policy interest rate), and strengthens the negative feedback effects on the output gap.*

Proof. See Appendix A.3.

A positive inflation penalty set by the government incites the central bank to keep inflation at a lower level while accepting a greater output gap to minimize its loss function. By reducing the feedback effect of inflation expectations on current inflation, a positive inflation penalty could have a contractionary effect on future inflation expectations, with its importance depending on the learning coefficient according to (5). In the case of positive inflation expectations, the output moves away from its potential level especially since the central bank, undergoing a positive inflation penalty, must implement a restrictive policy that reduces not only current inflation but also the persistence of inflation. This leads to decreasing inflation expectations, which eventually lower the central bank's loss due to inflation penalty. We notice that since κ is very small, the impact of a positive inflation penalty on the feedback effects of inflation expectations and cost-push shocks in the ALM of the output gap is largely greater than that on the corresponding feedback effects in the ALM of inflation.

Indeed, if $\gamma = 0$, the central bank cannot influence private agents' expectations by varying the actual values of inflation since private agents behave as if their expectations were fixed. Since the action of the central bank on private expectations is limited, the effect of inflation penalty will be smaller in this case than when $\gamma > 0$. When $\gamma = 1$, an increase in inflation penalty makes the largest impact on the feedback effects of inflation expectations and cost-push shocks on the ALMs of endogenous variables. We notice that numerical simulations show that for $\gamma > 0.2$ ($\gamma > 0.5$), the impact of inflation penalty on the feedback coefficients of inflation expectations (cost-push shocks) in the ALMs is very close to the ones obtained when $\gamma = 1$.

Our simulation exercises show that when the learning coefficient is higher, the deviation of inflation from the REE is greater. Consequently, a decrease in inflation penalty allows the central bank to increase the private sector's expectations in a more significant manner, thus permitting a correction of the deviation.

4.3 The optimal level of inflation penalty

The feedback coefficients of inflation and the output gap are function of inflation penalty and learning coefficient. This implies that the contribution of their respective volatility to the social welfare loss also depends on these two parameters. Using (24) and (28), the volatility of inflation and the output gap are respectively given by $var(\pi_{t+i}) = (c_\pi^{cg})^2 E_t(a_{t+i}^2) + (d_\pi^{cg})^2 E_t(e_{t+i}^2)$ and $var(x_{t+i}) = \frac{(c_\pi^{cg}-\beta)^2}{\kappa^2} E_t(a_{t+i}^2) + \frac{(d_\pi^{cg}-1)^2}{\kappa^2} E_t(e_{t+i}^2)$. Given that $\beta < c_\pi^{cg}$ and $d_\pi^{cg} < 1$, the volatility of inflation is decreasing in inflation penalty while the volatility of the output gap is increasing with it. The social loss function (3) can be rewritten as

$$L_t^s = \frac{1}{2}\sum_{i=0}^{+\infty}\beta^i \left\{ \left[(c_\pi^{cg})^2 + \frac{\alpha(c_\pi^{cg}-\beta)^2}{\kappa^2} \right] E_t(a_{t+i}^2) + \left[(d_\pi^{cg})^2 + \frac{\alpha(d_\pi^{cg}-1)^2}{\kappa^2} \right] E_t(e_{t+i}^2) \right\} \quad (34)$$

In the case where $\gamma = 0$, using (29)-(30) and (32)-(33), the social loss function (34) becomes

$$L_t^s = \frac{1}{2} E_t \sum_{i=0}^{+\infty}\beta^i \left\{ \frac{\alpha^2\beta^2}{[\alpha+\kappa^2(1+\tau)]^2} + \frac{\alpha\kappa^2(1+\tau)^2}{[\alpha+\kappa^2(1+\tau)]^2} \right\} \left[\beta E_t(a_{t+i}^2) + E_t(e_{t+i}^2) \right]. \quad (35)$$

The government's optimal decision obtained by the minimization of (35) is to set $\tau = 0$.

Result 3. *When the learning algorithms* (5)-(6) *are characterized by a learning coefficient equal to zero, i.e.,* $\gamma = 0$, *the government sets the optimal inflation penalty to zero.*

The result 3 shows that the optimal inflation penalty set by the government when private agents ignore their expectations errors is the same as when private agents form RE. This is because under RE, inflation expectations are taken as given by the central bank and consequently not affected by the latter, the government cannot influence inflation expectations and hence current inflation by imposing an inflation penalty on the central bank. In the case of adaptive learning with a learning coefficient equal to zero, the fact that private agents form exogenous inflation expectations implies a similar decision problem for the government. Under RE, as in the absence of learning, the central bank cannot make an intertemporal tradeoff by influencing private future expectations. Meanwhile, the intratemporal tradeoff, induced by the imposition of an inflation penalty other than zero, will deteriorate the social welfare.

For $\gamma > 0$, the minimization of the social loss function (34) does not allow for an analytical solution of τ. As the choice of τ for a period is independent of those in other periods, to determine the optimal solution of τ, it is sufficient to consider the unconditional (or average) expected social loss function per period. We proceed to numerically simulate the social loss function by setting $\beta = 0.99$, $\kappa = 0.024$, $\alpha = 0.048$, $var(a) = 0.5$, and $var(e) = 0.5$. It follows that for $\gamma = 0.1$, $\gamma = 0.5$ and $\gamma = 1$, the optimal inflation penalty τ^* is -0.86, -0.94 and -0.96 respectively (Figure 2). We notice that the existence of an optimal negative inflation penalty does not depend on the initial past inflation.

Result 4. *When the learning algorithm is characterized by a positive constant-gain learning coefficient, the government sets an optimal inflation penalty such that* $\tau \in]-1; 0[$. *The higher is the learning coefficient, the lower is the optimal inflation penalty.*

Under learning with a positive learning coefficient, since private agents take into account the expectations errors when forming their expectations, the central bank can influence their future expectations by setting monetary policy. Compared to the REE, the equilibrium with adaptive learning is suboptimal given that inflation and output-gap expectations based on past information deviate from correct expectations formed with the knowledge of the law of distribution of cost-push shocks. Thus, when private agents are learning, choosing a central banker with the same preferences as the society is not socially optimal. In the delegation framework, the government can set a negative inflation penalty on the central bank to incite the latter to mimic the socially optimal equilibrium with RE, given that the difference between the feedback coefficients in ALMs of inflation and the output gap under adaptive learning and those corresponding to the anticipated utility policy is increasing in inflation penalty.

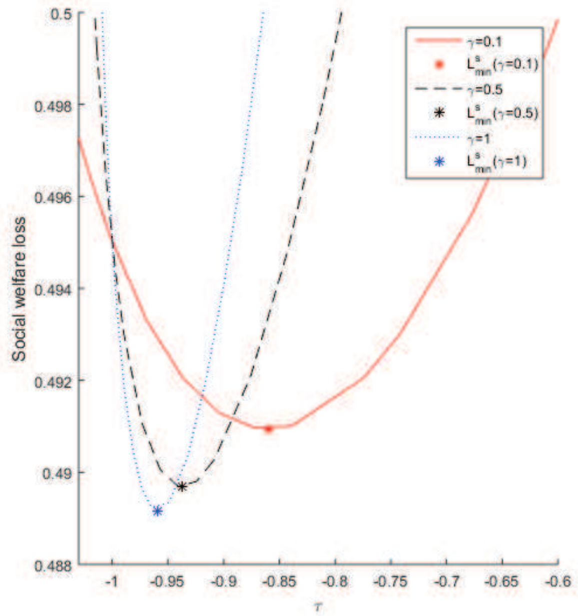

Figure 2: The social loss function with learning and inflation penalty

5 Implications of decreasing-gain learning

The main results obtained above are based on the assumption of constant-gain learning. However, agents could begin to learn with a decreasing-gain coefficient before stabilizing the latter's value. Indeed, constant-gain learning is more suitable for time-varying environments but decreasing-gain learning can be considered as the first step in the expectations process adopted by most economic agents (Berardi and Galimberti, 2013). Thus, we relax in this section the assumption of constant-gain learning to show if these results remain valid when learning gain is decreasing over time.[7]

Assume that learning gain characterizing algorithms (5) and (6) decreases with time such that $\gamma_t = \frac{1}{t}$. There exists a unique solution of the ALMs corresponding to the control problem of the central bank under the decreasing-gain learning, which is given by (Appendix A.4):

$$\pi_t = c^{dg}_{\pi,t} a_t + d^{dg}_{\pi,t} e_t \tag{36}$$

where

$$c^{dg}_{\pi,t} = \frac{\beta \left\{ \alpha - (1-\gamma_{t+1}) \left[\alpha(1+\gamma_{t+1}\beta)(\beta - c^{dg}_{\pi,t+1}) - \kappa^2(1+\tau) c^{dg}_{\pi,t+1} \right] \right\}}{\alpha + \alpha\beta\gamma_{t+1}(1+\gamma_{t+1}\beta)(\beta - c^{dg}_{\pi,t+1}) + \kappa^2(1+\tau)(1-\gamma_{t+1}\beta c^{dg}_{\pi,t+1})}, \tag{37}$$

$$d^{dg}_{\pi,t} = \frac{\alpha}{\alpha + \alpha\beta\gamma_{t+1}(1+\gamma_{t+1}\beta)(\beta - c^{dg}_{\pi,t+1}) + \kappa^2(1+\tau)(1-\gamma_{t+1}\beta c^{dg}_{\pi,t+1})}. \tag{38}$$

Current inflation is indirectly influenced by the central bank's policy responses to past shocks. The level of expectations and mostly the value of the learning coefficient $\gamma_{t+1} = \frac{1}{t+1}$ are crucial to the determination of current inflation. In the first period, we have $t = 1$ and $\gamma_t = 0.5$. The learning coefficient γ_{t+1} rapidly decreases over time. As $t \to +\infty$, $\gamma_{+\infty} \to 0$. This corresponds to the steady state where expectations are constant, i.e., private agents do not correct their expectations errors (absence of learning). This also leads to a constant value for inflation expectations. The feedback coefficients in the ALM of inflation, $c^{dg}_{\pi,t}$ and $d^{dg}_{\pi,t}$, will be identical to c^{cg}_{π} and d^{cg}_{π} given by (32)-(33).

The ALM of the output gap is obtained using into (2) and (36) as:

$$x_t = c^{dg}_{x} a_t + d^{dg}_{x} e_t \tag{39}$$

with $c^{dg}_{x,t} = -\frac{\beta - c^{dg}_{\pi,t}}{\kappa}$ and $d^{dg}_{x,t} = -\frac{1-d^{dg}_{\pi,t}}{\kappa}$.

[7] The relaxation of the assumption of constant-gain learning could be justified by the study of Milani (2014) who shows that private agents appear to have often switched to constant-gain learning, with a high constant-gain, during most of the 1970s and until the early 1980s, while reverting to a decreasing-gain later on.

The ALM ruling the interest rate is given by:

$$r_t = \delta_{r,t}^{dg} b_t + c_{r,t}^{dg} a_t + d_{r,t}^{dg} e_t \tag{40}$$

where $\delta_{r,t}^{dg} = \sigma$, $c_{r,t}^{dg} = 1 + \sigma \frac{\beta - c_{\pi,t}^{dg}}{\kappa}$ and $d_{r,t}^{dg} = \sigma \frac{1-d_{\pi,t}^{dg}}{\kappa}$.

We can show (Appendix A.5) that $c_{\pi,t}^{dg} < \frac{\alpha\beta}{\alpha+\kappa^2(1+\tau)}$, $d_{\pi,t}^{dg} < \frac{\alpha}{\alpha+\kappa^2(1+\tau)}$, $c_{x,t}^{dg} < -\frac{\beta\kappa}{\alpha+\kappa^2(1+\tau)}$, $d_{x,t}^{dg} < -\frac{\kappa}{\alpha+\kappa^2(1+\tau)}$, $c_{r,t}^{dg} > 1 + \frac{\sigma\beta\kappa}{\alpha+\kappa^2(1+\tau)}$ and $d_{r,t}^{dg} > \frac{\sigma\kappa}{\alpha+\kappa^2(1+\tau)}$. Furthermore, we have $\frac{\partial c_{\pi,t}^{dg}}{\partial t} > 0$, $\frac{\partial d_{\pi,t}^{dg}}{\partial t} > 0$, $\frac{\partial c_{x,t}^{dg}}{\partial t} > 0$, $\frac{\partial d_{x,t}^{dg}}{\partial t} > 0$, $\frac{\partial c_{r,t}^{dg}}{\partial t} < 0$, $\frac{\partial d_{r,t}^{dg}}{\partial t} < 0$.

As times goes on, the higher and closer to 0 the learning gain is, the more inflation is influenced by inflation expectations and current cost-push shocks while the inverse is true with the feedback coefficients in the ALMs of the output gap and the interest rate. Therefore, decreasing-gain learning leads the equilibrium solution to deviate from a more efficient REE, implying the possibility for the government to improve the social welfare by setting a negative inflation penalty that varies with time as the equilibria under decreasing-gain learning replicate the equilibria under learning with different constant gains.

Result 5. *As time goes on, the learning coefficient γ_t decreases from 1 to 0, implying that the feedback coefficients of inflation expectations and cost-push shocks in the ALMs of inflation and the output gap (the interest rate) increase (decrease) with time. For a given volatility of inflation and cost-push shocks, the optimal inflation penalty will increase from a value not far away from -1 to 0.*

Proof: See Appendix A.5.

Under decreasing-gain learning, as long as the economy is not in the steady state where $\lim_{t\to+\infty} \gamma_t \to 0$, private agents will adjust their expectations by correcting expectations errors, making possible for the central bank to influence their future expectations. Given the sub-optimality of transitory learning equilibria with regard to the REE, there is an incentive for the government to impose time-varying negative inflation penalty so that the learning equilibrium could be as close as possible to the REE.

The effects of cost-push shocks and expected inflation on inflation, the output gap and the interest rate under decreasing-gain learning are similar to those observed under constant-gain learning. Given that under decreasing-gain learning, the learning coefficient is decreasing with time, the effect of learning on the equilibrium is also decreasing with time. As a consequence, inflation penalty increases from a negative value to zero as the economy approaches its steady state where, with $\gamma_{t+1} \rightarrow 0$, private agents do not revise their previous expectations and the central bank is no more able to manipulate expectations, thus eliminating the possibility for the government to improve the social welfare by nominating a liberal central banker. During this process, as the learning coefficient decreases while inflation penalty rises, the impacts of inflation penalty on the equilibrium tends to approach those observed at the REE.

6 Conclusion

In this paper, we consider the issue of accountability of an independent central bank when private agents form inflation expectations with learning algorithms. Facing both the intratemporal tradeoff between inflation and the output gap and the intertemporal tradeoff between the stabilization in current and future periods, the central bank should stabilize the economy in a way to better anchor inflation expectations, thus easing future intratemporal tradeoffs. Introducing inflation penalty helps the central bank to better manage these tradeoffs and could substantially improve the social welfare.

We have shown that under adaptive learning, the optimal inflation penalty set by the government should be negative. This strengthens the feedback effects of inflation expectations and cost-push shocks on inflation, and weakens the feedback effects on the output gap and the policy interest rate, thus reducing the deviations of the feedback coefficients from their corresponding ones under rational expectations and making the economy more efficient. Moreover, the higher the learning gain coefficient is, the larger the deviation of feedback coefficients from the corresponding ones under rational expectations, reinforcing the need for a more liberal central banker. The main conclusions obtained with constant-gain learning remain valid under the assumption of decreasing-gain learning.

A APPENDIX

A.1 The equilibrium solution of inflation under learning

Using (23) to obtain $\lambda_{3,t}$ and $\lambda_{3,t+1}$ and substituting their expressions as well as $\lambda_{1,t+1} = 0$, $\lambda_{2,t+1} = -\frac{\alpha}{\kappa} x_{t+1}$ into (21), we get :

$$\frac{(1+\tau)}{\gamma}\pi_t + \frac{\alpha}{\gamma\kappa}x_t = \frac{\alpha\beta^2}{\kappa}E_t x_{t+1} + \frac{(1+\tau)\beta(1-\gamma)}{\gamma}E_t\pi_{t+1} + \frac{\alpha\beta(1-\gamma)}{\gamma\kappa}E_t x_{t+1}. \tag{A.1}$$

Using (2) and (5), we obtain:

$$x_t = \frac{1}{\kappa}\pi_t - \frac{\beta}{\kappa}a_t - \frac{1}{\kappa}e_t, \tag{A.2}$$

$$x_{t+1} = \frac{1}{\kappa}\pi_{t+1} - \frac{\beta}{\kappa}\left[a_t + \gamma(\pi_t - a_t)\right] - \frac{1}{\kappa}e_{t+1}. \tag{A.3}$$

Substituting x_t and x_{t+1} respectively given by (A.2) and (A.3) into (A.1) and arranging the terms lead to

$$E_t\pi_{t+1} = A_{11,t}\pi_t + A_{12,t}a_t + P_{1,t}e_t, \tag{A.4}$$

where

$$A_{11} \equiv \frac{\kappa^2(1+\tau) + \alpha + \alpha\beta^2\gamma\left[1-\gamma(1-\beta)\right]}{\alpha\beta\left[1-\gamma(1-\beta)\right] + \beta\kappa^2(1-\gamma)(1+\tau)}, \tag{A.5}$$

$$A_{12} \equiv -\frac{\alpha\beta\left\{1-\beta(1-\gamma)\left[1-\gamma(1-\beta)\right]\right\}}{\alpha\beta\left[1-\gamma(1-\beta)\right] + \beta\kappa^2(1-\gamma)(1+\tau)}, \tag{A.6}$$

$$P_1 \equiv -\frac{\alpha}{\alpha\beta\left[1-\gamma(1-\beta)\right] + \beta\kappa^2(1-\gamma)(1+\tau)}. \tag{A.7}$$

According to the proposition 1 from Blanchard and Kahn (1980), the solution of the ALM of inflation takes the following form:

$$\pi_t = c_\pi^{cg}a_t + d_\pi^{cg}e_t. \tag{A.8}$$

We obtain using (5) and (A.8):

$$E_t\pi_{t+1} = c_\pi^{cg}\left[(1-\gamma)a_t + \gamma\pi_t\right]. \tag{A.9}$$

Using equations (A.4) and (A.9) to eliminate $E_t\pi_{t+1}$ and arranging the terms yield:

$$\pi_t = \frac{A_{12} - c_\pi^{cg}(1-\gamma)}{c_\pi^{cg}\gamma - A_{11}}a_t + \frac{P_1}{c_\pi^{cg}\gamma - A_{11}}e_t. \tag{A.10}$$

This implies that:

$$c_\pi^{cg} = \frac{A_{12} - c_\pi^{cg}(1-\gamma)}{c_\pi^{cg}\gamma - A_{11}}, \tag{A.11}$$

and

$$d_{\pi,t}^{cg} = \frac{P_1}{c_\pi^{cg}\gamma - A_{11}}. \tag{A.12}$$

We gather equations (5), (6) and (A.4), while using (A.2) to substitute x_t to obtain the system of three equations:

$$E_t y_{t+1} = A_t y_t + P_t e_t,$$

where

$$y_t \equiv \left[\pi_t,\, a_t,\, b_t\right],\ A \equiv \begin{bmatrix} A_{11} & A_{12} & 0 \\ \gamma & 1-\gamma & 0 \\ \frac{\gamma}{\kappa} & \frac{-\beta\gamma}{\kappa} & 1-\gamma \end{bmatrix}, \text{ and } P \equiv \begin{bmatrix} P_1 \\ 0 \\ -\frac{\gamma}{\kappa} \end{bmatrix}.$$

The above system is subject to three boundary conditions: a_0, b_0, and $\lim_{s\to\infty}|E_t\pi_{t+s}| < \infty$. The eigenvalues of A are $1-\gamma$ and the two eigenvalues of A_1:

$$A_1 = \begin{bmatrix} A_{11} & A_{12} \\ \gamma & 1-\gamma \end{bmatrix}. \tag{A.13}$$

We can show that A_1 has an eigenvalue inside and one outside the unit circle.

Among infinite stochastic sequences satisfying equation (A.11), we focus on a non-explosive solution, i.e., the solution corresponding to the eigenvalue of A_1 given by (A.13) inside the unit circle.

It is straightforward to show that the trace and determinant of A_1 are both positive. Thus, for A_1 to have two real eigenvalues (μ_1, μ_2), one inside and one outside the unit circle, it is sufficient to show that $(1-\mu_1)(1-\mu_2) < 0$. This is equivalent to:

$$\mu_1 + \mu_2 > 1 + \mu_1\mu_2. \tag{A.14}$$

Knowing that $\mu_1 + \mu_2$ is equal to the trace of A_1 and $\mu_1\mu_2$ equal to its determinant, we can rewrite (A.14) as :

$$\frac{\kappa^2(1+\tau)+\alpha+\alpha\beta^2\gamma\left[1-\gamma(1-\beta)\right]}{\alpha\beta\left[1-\gamma(1-\beta)\right]+\beta\kappa^2(1-\gamma)(1+\tau)}+1-\gamma > 1+\frac{\kappa^2(1+\tau)+\alpha+\alpha\beta^2\gamma\left[1-\gamma(1-\beta)\right]}{\alpha\beta\left[1-\gamma(1-\beta)\right]+\beta\kappa^2(1-\gamma)(1+\tau)}(1-\gamma)$$

$$+\frac{\alpha\beta\left\{1-\beta(1-\gamma)\left[1-\gamma(1-\beta)\right]\right\}}{\alpha\beta\left[1-\gamma(1-\beta)\right]+\beta\kappa^2(1-\gamma)(1+\tau)}\gamma.$$

After simplification, we get:

$$\kappa^2(1+\tau)\left[1-\beta(1-\gamma)\right]+\alpha\left\{1+\beta(\gamma\beta^2)\right\} > 0,$$

which is always verified given that $\beta \in [0,1]$ and $\gamma \in [0,1]$.

Rewriting (A.11) as $c_\pi^{cg}c_\pi^{cg}\gamma - c_\pi^{cg}A_{11} - A_{12} + c_\pi^{cg}(1-\gamma) = 0$ and substituting A_{11} and A_{12} by their expressions, we obtain:

$$p_2(c_\pi^{cg})^2 + p_1 c_\pi^{cg} + p_0 = 0 \tag{A.15}$$

with

$$\begin{aligned}
p_0 &= \alpha\beta\left\{1-\beta(1-\gamma)\left[1-\gamma(1-\beta)\right]\right\} > 0,\\
p_1 &= (1-\gamma)\beta\left\{\alpha\left[1-\gamma(1-\beta)\right]+\kappa^2(1-\gamma)(1+\tau)\right\} - \left\{\kappa^2(1+\tau)+\alpha+\alpha\beta^2\gamma\left[1-\gamma(1-\beta)\right]\right\},\\
p_2 &= \gamma\left\{\alpha\beta\left[1-\gamma(1-\beta)\right]+\beta\kappa^2(1-\gamma)(1+\tau)\right\} > 0.
\end{aligned}$$

We rewrite p_1 as $p_1 = -\kappa^2(1+\tau)\left[1-\beta(1-\gamma)\right] - \alpha(1-\beta)\left\{1-\beta\left[1-\gamma(1-\beta)\right]\right\} - p_0 - p_2$, it follows immediately that $p_1 < 0$. Then, it is straightforward to show that the discriminant of the polynomial (A.15), , is positive.

To characterize the two solutions of c_π^{cg}, we rewrite (A.15) as:

$$c_\pi^{cg} = -\frac{p_0 + p_2\left(c_\pi^{cg}\right)^2}{p_1} \equiv f(c_\pi^{cg}) \qquad \text{(A.16)}$$

As $f(c_\pi^{cg})$ is strictly increasing for $c_\pi^{cg} \in [0,1]$ with $f'(c_\pi^{cg}) = -\frac{2p_2}{p_1}c_\pi^{cg} > 0$ for $c_\pi^{cg} \in [0,1]$. To prove $f(c_\pi^{cg}) : [0,1] \rightarrow]0,1[$, it is sufficient to show that $f(0) > 0$ and that $f(1) < 1$. It is straightforward to see that $f(0) = -\frac{p_0}{p_1} > 0$ and

$$f(1) = -\frac{p_0 + p_2}{p_1} = \frac{p_0 + p_2}{\kappa^2(1+\tau)\left[1-\beta(1-\gamma)\right] + \alpha(1-\beta)\left\{1-\beta\left[1-\gamma(1-\beta)\right]\right\} + p_0 + p_2} < 1.$$

Since $f(c_\pi^{cg}) : [0,1] \rightarrow]0,1[$ and $f(c_\pi^{cg})$ is strictly increasing, it follows from the theorem of Brouwer that there exists one unique solution of c_π^{cg} in the interval]0,1[. This solution corresponds to

$$c_\pi^{cg} = \frac{-p_1 - \sqrt{p_1^2 - 4p_2p_0}}{2p_2} \qquad \text{(A.17)}$$

The other possible solution $c_\pi^{cg} = \frac{-p_1 + \sqrt{p_1^2 - 4p_2p_0}}{2p_2}$ is larger than unit, which is excluded to avoid an explosive evolution of inflation.

Substituting A_{11} and P_1 into (A.12) leads to:

$$d_\pi^{cg} = \frac{\alpha}{\kappa^2(1+\tau) + \alpha + \alpha\beta^2\gamma^2(\beta - c_\pi^{cg}) + \beta\gamma(1-\gamma)\left\{\alpha\beta - \left[\alpha + \kappa^2(1+\tau)\right]c_\pi^{cg}\right\}}. \qquad \text{(A.18)}$$

We now show that $f(c_\pi^{cg}) : [0; \frac{\alpha\beta}{\alpha+\kappa^2(1+\tau)}] \rightarrow]0; \frac{\alpha\beta}{\alpha+\kappa^2(1+\tau)}[$. Knowing that $f(0) > 0$ and substituting c_π^{cg} by $\frac{\alpha\beta}{\alpha+\kappa^2(1+\tau)}$ into the function $f(c_\pi^{cg})$ defined by (A.19), we find

$$
\begin{aligned}
f(\frac{\alpha\beta}{\alpha+\kappa^2(1+\tau)}) &= -\frac{p_0+p_2\left[\frac{\alpha\beta}{\alpha+\kappa^2(1+\tau)}\right]^2}{p_1} \\
&= \frac{\frac{\alpha\beta}{\alpha+\kappa^2(1+\tau)}\left\{\frac{\alpha+\kappa^2(1+\tau)}{\alpha\beta}p_0+\frac{\alpha\beta}{\alpha+\kappa^2(1+\tau)}p_2\right\}}{-p_1}. \qquad \text{(A.19)}
\end{aligned}
$$

Using $p_2 = \frac{\alpha(1-\beta)+\kappa^2(1+\tau)}{\alpha+\kappa^2(1+\tau)}p_2 + \frac{\alpha\beta}{\alpha+\kappa^2(1+\tau)}p_2$, $p_0 = -\frac{\alpha(1-\beta)+\kappa^2(1+\tau)}{\alpha\beta}p_0 + \frac{\alpha+\kappa^2(1+\tau)}{\alpha\beta}p_0$ and the definition of p_0, p_1, and p_2 given above, we rewrite the denominator as

$$
\begin{aligned}
-p_1 &= \kappa^2(1+\tau)\left[1-\beta(1-\gamma)\right]+\alpha(1-\beta)\left\{1-\beta\left[1-\gamma(1-\beta)\right]\right\}+p_0+p_2 \\
&= \kappa^2(1+\tau)\left[1-\beta(1-\gamma)\right]+\alpha(1-\beta)\left\{1-\beta\left[1-\gamma(1-\beta)\right]\right\}+ \\
&\quad -\frac{\alpha(1-\beta)+\kappa^2(1+\tau)}{\alpha\beta}p_0+\frac{\alpha+\kappa^2(1+\tau)}{\alpha\beta}p_0+\frac{\alpha(1-\beta)+\kappa^2(1+\tau)}{\alpha+\kappa^2(1+\tau)}p_2+\frac{\alpha\beta}{\alpha+\kappa^2(1+\tau)}p_2 \\
&= -(1-\beta)p_2+\frac{\alpha(1-\beta)+\kappa^2(1+\tau)}{\alpha+\kappa^2(1+\tau)}p_2+\frac{\alpha+\kappa^2(1+\tau)}{\alpha\beta}p_0+\frac{\alpha\beta}{\alpha+\kappa^2(1+\tau)}p_2 \\
&= \frac{\beta\kappa^2(1+\tau)}{\alpha+\kappa^2(1+\tau)}p_2+\frac{\alpha+\kappa^2(1+\tau)}{\alpha\beta}p_0+\frac{\alpha\beta}{\alpha+\kappa^2(1+\tau)}p_2. \qquad \text{(A.20)}
\end{aligned}
$$

Substituting the above expression of $-p_1$ into (A.19), we obtain:

$$
f(\frac{\alpha\beta}{\alpha+\kappa^2(1+\tau)}) = \frac{\frac{\alpha\beta}{\alpha+\kappa^2(1+\tau)}\left\{\frac{\alpha+\kappa^2(1+\tau)}{\alpha\beta}p_0+\frac{\alpha\beta}{\alpha+\kappa^2(1+\tau)}p_2\right\}}{\frac{\beta\kappa^2(1+\tau)}{\alpha+\kappa^2(1+\tau)}p_2+\frac{\alpha+\kappa^2(1+\tau)}{\alpha\beta}p_0+\frac{\alpha\beta}{\alpha+\kappa^2(1+\tau)}p_2} < \frac{\alpha\beta}{\alpha+\kappa^2(1+\tau)}.
$$

Given that $f'(c_\pi^{cg}) = -\frac{2p_2}{p_1}c_\pi^{cg} > 0$ for $c_\pi^{cg} \in [0,1]$, $f(c_\pi^{cg})$ is strictly increasing in the interval $\left[0;\ \frac{\alpha\beta}{\alpha+\kappa^2(1+\tau)}\right]$. This property and the fact that $f(c_\pi^{cg}):\ [0;\ \frac{\alpha\beta}{\alpha+\kappa^2(1+\tau)}] \rightarrow]0;\ \frac{\alpha\beta}{\alpha+\kappa^2(1+\tau)}[$ imply that there is a unique solution for c_π^{cg} so that $0 < c_\pi^{cg} < \frac{\alpha\beta}{\alpha+\kappa^2(1+\tau)}$.

The case where $\gamma = 0$. We obtain by substituting $\gamma = 0$ into (A.5)-(A.7) :

$$
\begin{aligned}
A_{11} &\equiv \frac{1}{\beta}, \\
A_{12} &\equiv -\frac{\alpha(1-\beta)}{\alpha+\kappa^2(1+\tau)}, \\
P_1 &\equiv -\frac{\alpha}{\alpha\beta+\beta\kappa^2(1+\tau)}.
\end{aligned}
$$

It follows from (A.11)-(A.12) that

$$
\begin{aligned}
c_\pi^{cg} &= \frac{\alpha\beta}{\alpha+\kappa^2(1+\tau)}, \\
d_\pi^{cg} &= \frac{\alpha}{\alpha+\kappa^2(1+\tau)}.
\end{aligned}
$$

The case where $\gamma = 1$. Inserting $\gamma = 1$ into (A.5)-(A.7) yields

$$
\begin{aligned}
A_{11} &\equiv \frac{\kappa^2(1+\tau)+\alpha+\alpha\beta^3}{\alpha\beta^2}, \\
A_{12} &\equiv -\frac{1}{\beta}, \\
P_1 &\equiv -\frac{1}{\beta^2}.
\end{aligned}
$$

Substituting the latter into (A.5)-(A.7) leads to

$$
\begin{aligned}
c_\pi^{cg} &= \frac{\left\{\kappa^2(1+\tau)+\alpha+\alpha\beta^3\right\} - \sqrt{\left[\kappa^2(1+\tau)+\alpha+\alpha\beta^3\right]^2 - 4\alpha^2\beta^3}}{2\alpha\beta^2}, \\
d_\pi^{cg} &= \frac{\alpha}{\kappa^2(1+\tau)+\alpha+\alpha\beta^2(\beta-c_\pi^{cg})}.
\end{aligned}
$$

A.2 The effect of an increase in learning coefficient

Deriving p_0, p_1 and p_2 and using (A.20), we get

$$
\begin{aligned}
\frac{\partial p_2}{\partial \gamma} &= \alpha\beta\left[1-2\gamma(1-\beta)\right] + \beta\kappa^2(1-2\gamma)(1+\tau), \\
\frac{\partial p_0}{\partial \gamma} &= \alpha\beta^2\left[1-\gamma(1-\beta)\right] + \alpha\beta^2(1-\gamma)(1-\beta) > 0, \\
\frac{\partial p_1}{\partial \gamma} &= -\beta\frac{\partial p_2}{\partial \gamma} - \tfrac{\alpha+\kappa^2(1+\tau)}{\alpha\beta}\frac{\partial p_0}{\partial \gamma} \Longrightarrow \frac{\partial p_2}{\partial \gamma} = -\frac{1}{\beta}\frac{\partial p_1}{\partial \gamma} - \tfrac{\alpha+\kappa^2(1+\tau)}{\alpha\beta^2}\frac{\partial p_0}{\partial \gamma}.
\end{aligned}
$$

Alternatively, using the original expression of p_1, we obtain

$$
\frac{\partial p_1}{\partial \gamma} = -2\alpha\beta[1-\gamma(1-\beta^2)] - 2\beta\kappa^2(1-\gamma)(1+\tau) < 0.
$$

Deriving the solution of c_π^{cg} given by (A.17) that ensures a non-explosive evolution of inflation yields:

$$\frac{\partial c_\pi^{cg}}{\partial \gamma} = \frac{\left(-p_2 + \frac{-p_1 p_2}{\sqrt{p_1^2-4p_2p_0}}\right)\frac{\partial p_1}{\partial \gamma} + \frac{2p_2p_2}{\sqrt{p_1^2-4p_2p_0}}\frac{\partial p_0}{\partial \gamma} + \left(p_1 + \frac{p_1^2-2p_2p_0}{\sqrt{p_1^2-4p_2p_0}}\right)\frac{\partial p_2}{\partial \gamma}}{2p_2^2}.$$

Using $\frac{\partial p_2}{\partial \gamma} = -\frac{1}{\beta}\frac{\partial p_1}{\partial \gamma} - \frac{\alpha+\kappa^2(1+\tau)}{\alpha\beta^2}\frac{\partial p_0}{\partial \gamma}$, we get:

$$\frac{\partial c_\pi^{cg}}{\partial \gamma} = \frac{1}{2p_2^2}\left(F\frac{\partial p_1}{\partial \gamma} + G\frac{\partial p_2}{\partial \gamma}\right),$$

where

$$\begin{aligned} F &= -p_2 + \frac{-p_1p_2}{\sqrt{p_1^2-4p_2p_0}} - \frac{1}{\beta}p_1 - \frac{1}{\beta}\frac{p_1^2-2p_2p_0}{\sqrt{p_1^2-4p_2p_0}}, \\ G &= \frac{2p_2p_2}{\sqrt{p_1^2-4p_2p_0}} - \left(p_1 + \frac{p_1^2-2p_2p_0}{\sqrt{p_1^2-4p_2p_0}}\right)\frac{\alpha+\kappa^2(1+\tau)}{\alpha\beta^2}. \end{aligned}$$

Using $p_1 = -\beta p_2 - \frac{\alpha+\kappa^2(1+\tau)}{\alpha\beta}p_0$, after fastidious arrangements of terms, we finally obtain:

$$\frac{\partial c_\pi^{cg}}{\partial \gamma} = \frac{1 - \frac{\alpha+\kappa^2(1+\tau)}{\alpha\beta}c_\pi^{cg}}{\beta p_2\sqrt{p_1^2-4p_2p_0}}\left(p_0\frac{\partial p_1}{\partial \gamma} - p_1\frac{\partial p_0}{\partial \gamma}\right).$$

Using $c_\pi^{cg} < \frac{\alpha\beta}{\alpha+\kappa^2(1+\tau)}$, we obtain: $1-\frac{\alpha+\kappa^2(1+\tau)}{\alpha\beta}c_\pi^{cg} > 1-\frac{\alpha+\kappa^2(1+\tau)}{\alpha\beta}\frac{\alpha\beta}{\alpha+\kappa^2(1+\tau)} = 0$. To determine the sign of $H \equiv p_0\frac{\partial p_1}{\partial \gamma} - p_1\frac{\partial p_0}{\partial \gamma}$, we first check its value for and then its derivative with respect to γ. For $\gamma = 1$, we have : $\frac{\partial p_0}{\partial \gamma} = \alpha\beta^3 > 0$, $\frac{\partial p_1}{\partial \gamma} = -\beta^2\alpha(1+\beta) < 0$, $p_1 = -\left[\kappa^2(1+\tau) + \alpha + \alpha\beta^3\right]$ and $p_0 = \alpha\beta$. It is straightforward to show that if

$$\tau < \frac{\alpha\beta(1-\beta^2)}{\kappa^2} - 1,$$

we have:

$$H = -\alpha^2\beta^4(1-\beta^2) + \kappa^2(1+\tau)\alpha\beta^3 < 0.$$

Deriving H with respect to γ yields

$$\frac{\partial H}{\partial \gamma} = \frac{\partial p_0}{\partial \gamma}\frac{\partial p_1}{\partial \gamma} + p_0 \frac{\partial^2 p_1}{\partial^2 \gamma} - \frac{\partial p_1}{\partial \gamma}\frac{\partial p_0}{\partial \gamma} - p_1 \frac{\partial^2 p_0}{\partial^2 \gamma} = p_0 \frac{\partial^2 p_1}{\partial^2 \gamma} - p_1 \frac{\partial^2 p_0}{\partial^2 \gamma}$$

Deriving twice p_0 and p_1 with respect to γ for $\gamma = 1$ leads to

$$\begin{aligned} \frac{\partial^2 p_0}{\partial^2 \gamma} &= -2\alpha\beta^2(1-\beta) < 0, \\ \frac{\partial^2 p_1}{\partial^2 \gamma} &= 2\alpha\beta(1-\beta^2) + 2\beta\kappa^2(1+\tau). \end{aligned}$$

Using these second-order derivatives, we get

$$\frac{\partial H}{\partial \gamma} = 2\alpha^2\beta^3(1-\beta)\left\{1-\beta\left[1-\gamma(1-\beta)\right]\right\} + 2\alpha\beta^3\left[1-\beta(1-\gamma)\right]\kappa^2(1+\tau) > 0.$$

Consequently, for $\tau < \frac{\alpha\beta(1-\beta^2)}{\kappa^2} - 1$, given that $H < 0$ and $\frac{\partial H}{\partial \gamma} > 0$ for $\gamma = 1$, we conclude that

$$\frac{\partial c_\pi^{cg}}{\partial \gamma} < 0.$$

Using the definition of c_x^{cg}, d_x^{cg}, c_r^{cg} and d_r^{cg}, it is straightforward to show the sign of their partial derivative with respect to γ.

A.3 The effects of inflation penalty

Deriving p_0, p_1 and p_2 with respect to τ gives:

$$\begin{aligned} \frac{\partial p_0}{\partial \tau} &= 0, \\ \frac{\partial p_1}{\partial \tau} &= \kappa^2\left[\beta(1-\gamma)^2 - 1\right] < 0, \\ \frac{\partial p_2}{\partial \tau} &= \gamma\beta\kappa^2(1-\gamma) > 0. \end{aligned}$$

Deriving c_π^{cg} given by (A.17) with respect to τ and using the fact $\frac{\partial p_0}{\partial \tau} = 0$ and arranging the terms yield:

$$\frac{\partial c_\pi^{cg}}{\partial \tau} = \frac{c_\pi^{cg}\left(p_2\frac{\partial p_1}{\partial \tau} - p_1\frac{\partial p_2}{\partial \tau}\right) - p_0\frac{\partial p_2}{\partial \tau}}{p_2\sqrt{p_1^2 - 4p_2p_0}}.$$

Substituting p_1, p_2, $\frac{\partial p_1}{\partial \tau}$ and $\frac{\partial p_2}{\partial \tau}$ by their respective expression, we obtain

$$\left(p_2\frac{\partial p_1}{\partial \tau} - p_1\frac{\partial p_2}{\partial \tau}\right) = -\alpha\beta\gamma\beta\gamma\kappa^2\left\{1 - \beta(1-\gamma)\left[1 - \gamma(1-\beta)\right]\right\} < 0.$$

This result and the fact that $p_2 > 0$, $p_0 > 0$, $-p_1 - \sqrt{p_1^2 - 4p_2p_0} > 0$, $\frac{\partial p_1}{\partial \tau} < 0$ and $\frac{\partial p_2}{\partial \tau} > 0$ yield:

$$\frac{\partial c_\pi^{cg}}{\partial \tau} < 0.$$

Knowing the above result and deriving d_π^{cg} given by (A.18) with respect to τ, we get:

$$\frac{\partial d_\pi^{cg}}{\partial \tau} = \frac{-\alpha\left\{\kappa^2 - \alpha\beta^2\gamma^2\frac{\partial c_\pi^{cg}}{\partial \tau} - \beta\gamma(1-\gamma)\left[\alpha + \kappa^2(1+\tau)\right]\frac{\partial c_\pi^{cg}}{\partial \tau}\right\}}{\left\{\kappa^2(1+\tau) + \alpha + \alpha\beta^2\gamma^2(\beta - c_\pi^{cg}) + \beta\gamma(1-\gamma)\left\{\alpha\beta - \left[\alpha + \kappa^2(1+\tau)\right]c_\pi^{cg}\right\}\right\}^2} < 0.$$

Using the definition of c_x^{cg}, d_x^{cg}, c_r^{cg} and d_r^{cg}, it is straightforward to show the sign of their partial derivative with respect to τ.

A.4 The equilibrium solution of inflation under decreasing-gain learning

Using (23) to obtain $\lambda_{3,t}$ and $\lambda_{3,t+1}$ and substituting their expressions as well as $\lambda_{1,t+1} = 0$, $\lambda_{2,t+1} = -\frac{\alpha}{\kappa}x_{t+1}$, $\gamma_{t+1} = \frac{1}{t+1}$ and $\gamma_{t+2} = \frac{1}{t+2}$ into (21), we get :

$$(1+\tau)(t+1)\pi_t + \frac{\alpha(t+1)}{\kappa}x_t = E_t\left[\frac{\alpha\beta^2}{\kappa}x_{t+1} + \beta(1+\tau)(t+1)\pi_{t+1} + \beta\frac{\alpha(t+1)}{\kappa}x_{t+1}\right] \quad \text{(A.21)}$$

Using (2), (5) and $\gamma_{t+1} = \frac{1}{t+1}$, we write

$$x_t = \frac{1}{\kappa}\pi_t - \frac{\beta}{\kappa}a_t - \frac{1}{\kappa}e_t \quad \text{(A.22)}$$

$$x_{t+1} = \frac{1}{\kappa}\pi_{t+1} - \frac{\beta}{\kappa}\left[a_t + \frac{1}{t+1}(\pi_t - a_t)\right] - \frac{1}{\kappa}e_{t+1}. \quad \text{(A.23)}$$

Substituting x_t and x_{t+1} respectively given by (A.22) and (A.23) into (A.21) and arranging the terms lead to

$$E_t\pi_{t+1} = A_{11,t}\pi_t + A_{12,t}a_t + P_{1,t}e_t \tag{A.24}$$

where

$$A_{11,t} \equiv \frac{\alpha + \kappa^2(1+\tau) + \frac{1}{t+1}\alpha\beta^2(1+\frac{1}{t+1}\beta)}{\alpha\beta(1+\frac{1}{t+1}\beta) + \beta\kappa^2(1+\tau)}, \tag{A.25}$$

$$A_{12,t} \equiv \frac{-\alpha\beta + \alpha\beta^2(1-\frac{1}{t+1})(1+\frac{1}{t+1}\beta)}{\alpha\beta(1+\frac{1}{t+1}\beta) + \beta\kappa^2(1+\tau)}, \tag{A.26}$$

$$P_{1,t} \equiv -\frac{\alpha}{\alpha\beta(1+\frac{1}{t+1}\beta) + \beta\kappa^2(1+\tau)}. \tag{A.27}$$

The solution of the ALM of inflation takes the following form:

$$\pi_t = c^{dg}_{\pi,t}a_t + d^{dg}_{\pi,t}e_t. \tag{A.28}$$

Using (5) and (A.28), we obtain:

$$E_t\pi_{t+1} = c^{dg}_{\pi,t+1}\left[(1-\gamma_{t+1})a_t + \gamma_{t+1}\pi_t\right] \tag{A.29}$$

Using equations (A.24) and (A.29) to eliminate $E_t\pi_{t+1}$ and arranging the terms yield:

$$\pi_t = \frac{A_{12,t} - (1-\frac{1}{t+1})c^{dg}_{\pi,t+1}}{\frac{1}{t+1}c^{dg}_{\pi,t+1} - A_{11,t}}a_t + \frac{P_{1,t}}{\frac{1}{t+1}c^{dg}_{\pi,t+1} - A_{11,t}}e_t. \tag{A.30}$$

This implies that:

$$c^{dg}_{\pi,t} = \frac{A_{12,t} - (1-\frac{1}{t+1})c^{dg}_{\pi,t+1}}{\frac{1}{t+1}c^{dg}_{\pi,t+1} - A_{11,t}} \tag{A.31}$$

and

$$d^{dg}_{\pi,t} = \frac{P_{1,t}}{\frac{1}{t+1}c^{dg}_{\pi,t+1} - A_{11,t}}. \tag{A.32}$$

We gather equations (5), (6) and (A.24) while using (A.22) to substitute x_t to obtain the system of three equations:

$$E_t y_{t+1} = A_t y_t + P_t e_t,$$

where

$$y_t \equiv [\pi_t,\, a_t,\, b_t]\,,\; A_t \equiv \begin{bmatrix} A_{11} & A_{12} & 0 \\ \frac{1}{t+1} & \frac{t}{t+1} & 0 \\ \frac{1}{\kappa(t+1)} & \frac{-\beta}{\kappa(t+1)} & \frac{t}{t+1} \end{bmatrix},\text{ and } P_t \equiv \begin{bmatrix} P_{1,t} \\ 0 \\ -\frac{1}{\kappa(t+1)} \end{bmatrix}.$$

The above system is subject to three boundary conditions: a_0, b_0, and $\lim_{s\to\infty} |E_t \pi_{t+s}| < \infty$. The eigenvalues of A_t are given by $\frac{t}{t+1}$ and by the two eigenvalues of $A_{1,t}$:

$$A_{1,t} = \begin{bmatrix} A_{11} & A_{12} \\ \frac{1}{t+1} & \frac{t}{t+1} \end{bmatrix}. \tag{A.33}$$

We can show that $A_{1,t}$ has a real eigenvalue inside and one outside the unit circle.□

A.5 The properties of the single stable solution under decreasing-gain learning

Among infinite stochastic sequences satisfying equation (A.31), we focus on a non-explosive solution. To characterize the properties of this solution, we consider the value of $c^{dg}_{\pi,t}$ when $t \to +\infty$. Using the boundary conditions $\lim_{t\to+\infty} A_{11,t} = \frac{1}{\beta}$ and $\lim_{t\to+\infty} A_{12,t} = \frac{-\alpha(1-\beta)}{\alpha+\kappa^2(1+\tau)}$, we find that in the limit, $c^{dg}_{\pi,t}$ evolves according to:

$$\lim_{t\to+\infty} c^{dg}_{\pi,t} = \beta \lim_{t\to+\infty} c^{dg}_{\pi,t+1} + (1-\beta)\frac{\alpha\beta}{\alpha+\kappa^2(1+\tau)}. \tag{A.34}$$

The boundary condition $\lim_{n\to\infty} |\pi_{t+n}| < \infty$ implies that $\lim_{n\to+\infty} \beta^n c^{dg}_{\pi,t+n} = 0$. Using this condition and solving (A.34) forward yield one and only one bounded solution of $c^{dg}_{\pi,t}$:

$$\lim_{t\to+\infty} c^{dg}_{\pi,t} = \frac{\alpha\beta}{\alpha+\kappa^2(1+\tau)}.$$

Furthermore, it follows from (A.34) that

$$\lim_{t\to+\infty} c^{dg}_{\pi,t+1} > \lim_{t\to+\infty} c^{dg}_{\pi,t}, \tag{A.35}$$

implying that when $t \to +\infty$, we have $c^{dg}_{\pi,t} < \dfrac{\alpha\beta}{\alpha + \kappa^2(1+\tau)}$.

Assuming that for $t = n+1$, we have $c^{dg}_{\pi,n+1} < \dfrac{\alpha\beta}{\alpha + \kappa^2(1+\tau)}$. It follows from (A.31) that

$$c^{dg}_{\pi,n} = \frac{A_{12,n} - \frac{n}{n+1} c^{dg}_{\pi,n+1}}{\frac{1}{n+1} c^{dg}_{\pi,n+1} - A_{11,n}} \Rightarrow$$

$$c^{dg}_{\pi,n+1} = \frac{A_{12,n} + c^{dg}_{\pi,n} A_{11,n}}{c^{dg}_{\pi,n} \frac{1}{n+1} + \frac{n}{n+1}} < \frac{\alpha\beta}{\alpha + \kappa^2(1+\tau)} \Rightarrow$$

$$\left\{ \left[\alpha + \kappa^2(1+\tau)\right] A_{11,n} - \frac{\alpha\beta}{n+1} \right\} c^{dg}_{\pi,n} < \frac{\alpha\beta n}{n+1} - A_{12,n} \left[\alpha + \kappa^2(1+\tau)\right]. \tag{A.36}$$

Substituting $A_{12,n}$ and $A_{12,n}$ by their respective expression given by (A.25)-(A.26), we obtain:

$$c^{dg}_{\pi,n} < \frac{\alpha\beta}{\left[\alpha + \kappa^2(1+\tau)\right]} \frac{\left[\alpha + \kappa^2(1+\tau)\right] - \frac{n}{(n+1)^2}\beta^2\kappa^2(1+\tau)}{\left[\alpha + \kappa^2(1+\tau)\right] + \frac{1}{(n+1)^2}\alpha\beta^3 \frac{\kappa^2(1+\tau)}{\left[\alpha+\kappa^2(1+\tau)\right]}} \ll \frac{\alpha\beta}{\alpha + \kappa^2(1+\tau)}$$

Thus, we have by recurrence that $c^{dg}_{\pi,t}$ is increasing with time and

$$c^{dg}_{\pi,t} < \frac{\alpha\beta}{\alpha + \kappa^2(1+\tau)}, \ \forall t \in [1, +\infty[.$$

Using the definition of $c^{cg}_{x,t}$, $d^{cg}_{x,t}$, $c^{cg}_{r,t}$ and $d^{cg}_{r,t}$, it is easy to show their bound and their evolution over time.

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