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Stochastic Economic Growth and Volatile Population Dynamics: Past Imperfect and Future Tense*

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Abstract

We build an analytical model to understand dynamic interlinkage between volatility in economic growth and stochastic demographic dynamics. The time series properties of the model are exploited to offer introspective understanding of the existence and persistence of endogenous and exogenous growth dynamics within our analytical setting. Our research shows that if the economy faces high degree of interdependence between its volatility and stochastic demographic growth in the past with the possibility of slow dissipation of shocks at present, then future economic growth will experience chaotic dynamics. We investigate two possibilities: a process with persistent shocks that can slowly wither away in future, and a jump process that would characterize how economic growth would respond to the arrival of sudden change in demographic system.

JEL Classification: O13, O47, C14, C22, J11, D1.
Key words: Stochastic population, Volatility in economic growth, long-memory, Past dependence, Jump process.

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1 Introduction

The fact that a stochastic demographic system can determine the pattern of future economic growth is something that has been repeatedly investigated in the recent literature (see for instance, Azomahou, Mishra, and Parhi [in press]; Birdsall et al. (2001); Lindh and Malmberg (2007); Parhi and Mishra (2009), and others). The proposed channels have been varied; for instance, while Azomahou, Mishra, and Parhi [in press] propose the instrumental role of environment within the dual system of economic growth and demography, Parhi and Mishra (2009) underline the importance of human capital within this framework. The common points in these and the extant theoretical and empirical research are as follows. First, the theoretical models often assume stationary population growth. Wherever non-stationarity has been introduced, its effects have been assumed to be either 'one-off' type or a continuous dissipation before long where the shocks die out completely. From empirical perspective, long-memory mechanisms have been introduced to lend flexible interdependence dynamics between economic growth and demographic system, in particular, population growth. Despite appreciable progress of the existing literature towards a comprehensive understanding of the dynamic association between these systems, further research is warranted to characterize (i) if stochastic population growth contributes towards the existence and persistence of exogenous/endogenous economic growth in various economies, (ii) if demographic system experiences a sudden jump process like character, how would that govern the volatility of the economic system. This paper aims to undertake a serious study of these perspectives.

Extant literature in demography and economic growth for the past three centuries since Malthus (1798) have impelled us to believe that despite having distinct evolutionary characters, perturbations in demographic system is very likely to induce instability in economic growth in the long run. However, the conventional practice in (empirical) economic growth models has been to treat population growth as stationary implying that stochastic shocks to the population series would completely disappear in the long-run and thus would exert no measurable impact on its long-run mean and variance. Exception being Azomahou, Mishra and Parhi [in press]. Statistically, a stationary series may still accommodate long-memory features, however the shock convergence patterns of short-memory and long-memory stationary processes are vastly different (e.g., Bailey, 1986). From economic theoretic and policy perspectives such differences are interesting as they determine, among other things economies' speed and pace of growth over time as well as could provide important information about the existence of exogenous or endogenous growth mechanism. Although recent research (e.g., Boucekkine et al., 2002; Azomahou et al., 2009) have rendered some observations on the effects of stochastic shocks using dynamic overlapping generations and spatial vector autoregressive models, they have remained silent on the plausibility of stationary population growth assumption, and how, the exception (that is, non-stationary population growth) could characterize exogenous and endogenous growth mechanisms. Lau (1999, 2003) provide some directions of research on the implications of unit root on endogenous/exogenous growth, but not how such characterisations in case of demographic system would reveal the economic growth dynamics.

Empirical growth literature often presents a gaussian distribution for economic growth implying that the growth rate of output consists of small shocks that satisfy the conditions of the standard central limit theorem (CLT). That is, the output growth is considered as the consequence of an accumulation of many small shocks. By the central limit theorem, then one can predict that the distribution of growth rates becomes Gaussian. However, the distribution of output growth does not have to be Gaussian; rather it can closely follow Laplace distribution, which has a fatter tail than Gaussian. The implication is that there is a relatively higher probability that an economy experiences more extreme events than predicted by a Gaussian distribution. The non-normality of growth rates challenges the existing models because it indicates the underlying mechanism of economic growth is different from the one suggested by many empirical research. Thus, one may assume that the growth rate consists of independent random shocks but does not satisfy the conditions of the standard central limit theorem. If we look at the extant research on the determinants of economic growth (Sala-i-Martin, 1997), it is quite obvious that an economy can growth for all sorts of reasons, for example, as in our case, demographic changes, environmental dynamics, financial and political growth, etc. Some of them can have positive and some negative effects on long-run growth. The shocks these variables would impart on economic growth can thus be varied, of heterogenous magnitudes, and can have a disproportionate impact on the output. Therefore, we need a broad generalisation of the standard central limit theorem.

Khintchine (1937) provides description of the fundamental limit theorem on sums of independent random variables, which state that if the distribution of sums converges, its limit belongs to infinitely divisible distributions. This family of distributions includes the Laplace distribution, as well as the Gaussian. Moreover, if an infinitely divisible distribution is given, there exists a corresponding Levy process. The Levy process corresponding to the Laplace distribution is called variance gamma process. Therefore, the sample path properties of this process represent how an economy grows. One can find that this process is a pure jump process, that is, its value increases or decreases by jumps. This process, which is used in our paper, would be of immense interest to understanding growth dynamics. There are some interesting research though; Moreno et al. (2011), for instance, discuss the usefulness of jump process to reflect how economic variables respond to the arrival of sudden information.

The rest of the paper is structured as follows. In section 2, we discuss implications from literature covering deterministic and stochastic models. Section 3 present some discussions about the source of stochasticity and provide a description of basic framework of persistence and stochasticity. Section 4 presents and discusses the implications of the results for exogenous and endogenous growth. Section 5 introduces a jump process in population growth and presents analytical results. Empirical results are presented in section 6. Results of simulation exercise are discussed in section 7. Finally, section 8 concludes with some notes on the main findings.

2 Literature

The rapid advancement of modern science has impelled us to believe that despite existence of distinct evolutionary mechanisms of populations (human and/or animals)[1], economy and the environment, the individual systems are getting increasingly interdependent over time. A shock to one of these systems, therefore, is bound to generate ripple effect in another, albeit in smaller magnitudes. Given the intricacies of interaction and complex mapping of shock profile for the interacting systems, it is always difficult - though not impossible - to design course of actions limiting further proliferation of shocks and with an intent to bring stability to the systems at large. If two systems are similar, working out the shock profiles for individual and interactions may be easy. However if the systems possess different evolutionary characters (e.g., human population and economic system), understanding their interactions in the face of an endogenous or exogenous shocks are not always straightforward. Indeed, the uniqueness of demographic evolution lies in the fact that it depends invariably on the (inter) actions of the economy and the environment. Research over the past two centuries (since Euler, 1760; Malthus, 1798) have not gone in vain: demographers, economists and environmental scientists recently have been constantly offering rich theoretical and empirical analysis in an attempt to

[1] Throughout the paper we will imply human population growth as the broad term for population growth. Evolutionary process of animal population - especially some small species - are independent of human actions and therefore the implied stochasticity in economic growth in this paper does not result from stochasticity in animal population population growth. We are only interested in the causes and consequences of stochasticities in human population.

understand the underlying dynamics of evolutions of these systems with and without interaction. Easterlin (1966, 1968), Dasgupta and Heal (1980), Dasgupta (1995), Kelley and Schmidt (2001), Birdsall et al. (2001), Jaimovich and Siu (2009) are some of the fine examples in this regard.

Fortunately, inconclusive results in the past demonstrating the impact of demographic fluctuations on economic growth (see Kelley and Schmidt, 2001) did not dissuade researchers to further investigate the interactions from new perspective. In fact, the endogenous economic growth theory's (e.g., Lucas, 1988; Romer, 1990; Rebelo, 1991) enormous success and intuitive appeal in recent times has only meant that the centrality of age-specific population growth in economic fluctuations is irrefutable and irreconcilable (Boucekkine et al., 2002; Mishra, 2006; Jaimovich and Siu, 2009; Mishra and Diebolt, 2010). This implies then a shock (either endogenous or exogenous) in one of the systems would induce persistence effect in the other and that continuous interactions in the presence of perturbations may lead to chaotical growth dynamics with monotonic volatility over time. Prskawetz and Feichtinger (1995) show that the underlying mechanism describing the demographic process is exceedingly complex, characteristically non-linear and may result in a pattern which exhibits chaotical dynamics. This is not surprising given that demographic process experiences many endogenous shifts over time due to the interaction between demography and the economy and the feedback effect following them. These typical features are however infrequently studied in the theoretical literature where temporal variation of the demographic variables come into the prominence.

Relatedly, stochastic models of population growth has been examined in theoretical biology where stochasticity was shown to arise from multistate Markov transition. Volatility in economic growth in economic-demographic process then would mean that conditional on the state of economic system, demographic response (in terms of transition probability) may generate stochastic behavior. However, the extant models banked upon a memory-less property of demography and economic system in the sense that dependence on shock in course of history did not matter for current growth. All past information on correlatedness of shocks was only appended in a single past information set - reflecting irrelevance of the long past events while using Markov models. But then, a stochastic shock in demographic and economic system could also be generated with a more realistic approximation, viz., a nonstationary Markovian transition process or a long-memory process. However, research considering the former is new and has not been rigorously carried, most possibly due to the problem of characterization of transition matrix of the demographic process in a non-stationary domain. The research in case of the latter is also underdeveloped despite its intuitive appeal for modeling evolution of stochastic shocks.

While the use of Markov process provides theoretically tractable results with reasonably easy interpretation and inferences on stability and invertibility of the system, it does not approximate reality. Indeed, recent empirical research (viz., Gil-Alana, 2004; Mishra et al. (2009) on population and scores of research on output), thanks to the pioneering work on nonstationary time series in the past four decades since Dickey and Fuller (1971), have demonstrated that population and economic growth are characterized by 'long-memory' mechanism in the sense that a shock to these systems take very long time to converge to the long-run mean. In the context of foregoing discussion, it implies then that 'systems with memory rather than without memory' of shocks approximates better the complex interaction of demography and economic growth in the presence of stochasticity.[2].

[2] Tuljapurkar and Haridas, 2006 mentions about the use of long-memory in discerning the autocorrelation between environment and population growth

3 Stochasticity: source and characterisation of persistence

This section begins with explaining why stochasticity in demographic system is plausible. We describe various channels through which this may arise. We then present an analytical model where a stochastic version of Solow-Swan model is examined both within a long-memory mechanism and with a jump process of stochastic shocks.

3.1 Source of stochasticity

Among several reasons investigating why demographic system might exhibit stochasticity, Shaffer's (1987) proposition is of interest. Arguably, what has, by and large, become the standard classification of stochasticity goes back to the 1978 dissertation by Shaffer, which was later published in 1987 by M. E. Soul in "Viable Populations for Conservation", Cambridge University Press. Shaffer (1987) argues that demographic stochasticity is caused by chance realizations of individual probabilities of death and reproduction in a (finite) population. Shaffer distinguished four sources of uncertainty that can contribute to random extinction of population:

- demographic stochasticity which arises from the chance events in the survival and reproductive success of a finite number of individuals.
- environmental stochasticity due to temporal variation of birth and death rates, carrying capacity, and the population of competitors, predators, parasites, and diseases.
- natural catastrophes such as floods, fires, droughts, etc.
- genetic stochasticity resulting from changes in gene frequencies due to founder effect, random fixation, or inbreeding.

Shaffer went on to argue that all these factors increase in importance as the population size decreases. He defined a minimum viable population (MVP): 'A minimum viable population for any given species in any given habitat is the smallest population having at least a 95% chance of remaining extant for 100 years despite the foreseeable effects of demographic, environmental, and genetic stochasticity, and natural catastrophes. Lande et al. (2003) argue that demographic stochasticity "refers to chance events of individual mortality and reproduction, which are usually conceived of as being independent among individuals" whereas environmental stochasticity "refers to temporal fluctuations in the probability of mortality and the reproductive rate of all individuals in a population in the same or similar fashion. The impact of environmental stochasticity is roughly the same for small and large populations." This is further elaborated: "Random variation in the expected fitness that is independent of population density constitutes environmental stochasticity. Random variation in individual fitness, coupled with sampling effects in a finite population, produces demographic stochasticity."

Nevertheless, environmental fluctuations or even random catastrophes affect the size of a population only insofar as they affect reproduction and death rates, that is, by creating demographic fluctuations. Moreover, Lande and others regard random catastrophes as extreme cases of environmental stochasticity. Usually, models of demographic stochasticity are distinguished from models of environmental stochasticity using as a criterion whether the stochastic factor explicitly depends on the population size as a parameter. If it does, the model in question is one of demographic stochasticity; if it does not, it is one of environmental stochasticity. This choice captures the intuition mentioned earlier that the effect of the former depends on the population size whereas the effect of the latter does not.[3] Whether in the form of catastrophes

[3]The mathematical analysis of these models is non-trivial. The most general and uncontroversial theoretical result to date is that progressively larger populations are required for safety in the face of demographic, environmental, and random catastrophic stochasticity. Moreover, because of the structural uncertainly of these models, apparently slight differences in assumptions and techniques routinely lead to widely divergent predictions.

(i.e., exogenous shocks) or stochasticities from endogenous mechanisms (e.g., from interaction mechanism with environment), prediction in the presence of stochasticity of an accurate demographic pattern long time in the future is always a formidable task. Existence of randomness is nevertheless a natural phenomena, entities facing uncertainties tend to innovate faster for survival, and chance events are what offers hope for a new life. Afterall, stochasticities rule. The crux is to understand its dynamics for a better living.

- *Environmental stochasticity as a cause of demographic stochasticity*

Demographic stochasticity is caused by environmental stochasticity from a nearly continuous series of small or moderate perturbations that similarly affect the birth rates of individuals within each age in a population. Additionally, catastrophes are large environmental perturbations that produce sudden major reductions in population size. Whether in terms of simple environmental or behavioral changes or large scale environmental changes (in terms of catastrophes), demographic system by and large is often subject to continuous perturbations. Sometimes the effect of these perturbations go unnoticed, however, as it happens, these small disturbances contribute to large scale environmental and demographic changes in the long-run.

Demographic system does not evolve independent of economic and environmental systems. The continuous interaction between demographic, economic and environmental factors thrives on continuous feedback effects from one system to the other and that renders the relation highly non-linear (Azomahou and Mishra, 2008). Additionally, it is well-understood that the evolutionary mechanisms of economic and demographic systems are different. To a reasonable extent it can be said that unless certain demographic standards are met (say minimum population with standard replacement rate), the internal dynamics of economic system will be severely upset. For instance, given the speed of demographic growth, specifically in terms of age-structured population growth, declining fertility and mortality, increase in educated mass, and faster population aging, economic functioning must take recourse to cognizable policy changes so as to restore balance for a 'sustainable demography-economic' growth in the longer run. The natural occurrence of demographic and economic system interaction provides reason to stress that any endogenous shift occurring in one system would have long-term consequences for the other. This convention has been stressed, nonetheless, in most of the population literature.

Interestingly, in spite of differences in evolutionary structure of various systems, most of them share common properties. For instance, initial modeling strategies of macroeconomic/financial time series were in line with random walk. That is, they are cyclic and non-periodic. However, recent strategy stresses that most macroeconomic time series resemble neither random walk nor white noise, suggesting that a hybrid between the random walk and its integral may be useful. In a similar vein, for over centuries we have observed cyclical behavioral pattern of demographic system, captured in terms of demographic transition. It has also been observed that the demographic states are repetitive after long years, so that gives rise to a kind of long-swing behavior with past dependence property of the system. Intuitively, this implies that a particular kind of demographic state tend to settle for some period of time due to a specific interaction nature of demography-economic state. After saturation, a new demographic state emerges which owes its course due to innovation and development. Once that also gets saturated, the demography-economic system tend to return to old equilibrium path. Typically, this has been summarized in terms of demographic transition and multi-transition demographic states in demographic literature (e.g., Cohen, 1979; Tuljapurkar and Haridas, 2006).

4 Demographic stochasticity and persistence

An important aspect of studying stochasticity of a system is to understand its persistence character. That is, under stochastic setting, how a shock to the system persists over time. Does it converge fast enough to make the system stable or it drifts away forever without any possibility of returning to the mean value? Such inquiries have formed the basis of stochastic economic growth's application to real world data. Despite its centrality in gauging the long-term consequences on economic growth, study of persistence properties for stochastic population growth is rather sparse. Allen and Allen (2003) in an exceptional research compare three different stochastic population models with regard to persistence time. The authors study discrete and continuous time Markov chains and stochastic differential equations to model the random nature of individual birth and death processes and provide mechanism to understand the persistence effect in these models. An alternative approach would be to study the accumulation dynamics of stochastic shocks in population series over time and estimate magnitude of persistence. The current paper studies the accumulation dynamics of shocks in population series over time. To lend appropriate comparison, we first briefly summarize Allen and Allen's (2003) three stochastic population models, which are basically memory-less models. Next, we present the long-memory persistence approach to population dynamics using a conventional ARFIMA(p,d,q) and duration dependence model with long memory character (Parke, 1999).

4.1 Memory-less model: Markov chain and demographic stochasticity

At time τ, denote the state of the demographic system as D_τ. The elements of D_τ comprises of points (assuming it to be infinite) p_i described at each point in time, t. We have in mind that τ consists of broader time span where t forms the space of τ. During the transition of demographic state, D_τ from say D_{τ_1} to D_{τ_2}, continuous perturbations might have occurred at different t interpoints. Over the span τ_1 to τ_2, the perturbations accumulate and over time the sum of perturbations are likely to produce a non-mean convergent distribution of the system. While this is a natural possibility, extant growth and demography theories assumed this to be stationary, in the sense that shocks get smoothed out and summed perturbations always tend to converge in the long-run. Although most theoretical analysis on stochastic demographic system utilize stationary Markov mechanism, it is prudent to assume that the probability distribution of the state of the demographic system at a given point may depend on the system's state at the previous stage. The use of stationary Markov chain - assuming that the transition probabilities are time-homogenous - is probably to easily characterize the long-run or steady state behavior of the demographic and economic system. However, real world demographic system evince character which are inherently non-stationary and possesses a persistent shock which takes long time to converge. That makes the transition probability time-dependent (instead of time-homogenous).

Anily and Federgruen (1986) provided tools for characterizing the long-run behavior of finite, nonstationary Markov chains in which the time-dependent transition probabilities converge to a limiting matrix. Our purpose in this section is not to describe a non-stationary Markov chain model, rather we present intuitions from stationary Markov process under which persistence properties are defined. For expositional purpose, recall that for the state of transition of demographic system, D_τ, we can define a common state space $1, ..., n$. In period τ, the system moves from state i to state j with probability $p(\tau)_{ij}$. For stationary Markov chain with transition matrix P, it is ergodic if $\lim_{n\to\infty}(P_{ij}^n - P_{lj}^n) = 0$ for all $i, j, l \in \{1, ..., N\}$, i.e., the effect of starting state vanishes as time progresses. A stationary chain[4] is ergodic if and only if it is *aperiodic* and has a single subchain, in which case it satisfies a stronger convergence result: a unique steady state distribution π exists with $\lim_{n\to\infty}(P_{ij}^n) = \pi_j$ for all $i, j \in \{1, ..., N\}$.

Allen and Allen (2003) describe persistence time in Markov chain (MC) and stochastic differential equation models for birth $b(N)$ and death $d(N)$ rates satisfying conditions in $C1 - C4$.

[4] A finite nonstationary Markov chain would mean that a sequence of transition matrices $\{P(\tau)\}_{\tau=1}^{\infty}$ defined on a common state space $1, ..., n$. In period τ, the system moves from state i to state j with probability $p(\tau)_{ij}$. For nonstationary Markov chains, the effect of the starting demographic state may vanish, while the products $\{P(1)...P(\tau)\}_{\tau=1}^{\infty}$ fail to converge.

The discrete time MC assumes that both time and population size are discrete valued. Assuming that Δt is a fixed time interval and $t \in \{0, \Delta t, 2\Delta t, ...\}$ and that Δt is sufficiently small such that at most one change occurs during the time interval Δt. Given the population size N, birth and death occurs with probabilities, $b(N)\Delta t$ and $d(N)\Delta t$ respectively. Denote the probabilities associated with $N(t)$ as $p_N(t) = ProbN(t) = V, V = 0, 1, ..., M$, and $p(t) = (p_0(t), p_1(t), ..., p_M(t))^T$. The transition probabilities is $p_{Nx}(\Delta) = Prob(N(t + \Delta t))$, where $p_{Nx}(\Delta t)$

$$
\begin{aligned}
b(N)\Delta t, x = N - 1, N \in 1, ..., M, \\
d(N)\Delta t, x = N + 1, N \in 0, 1, ..., M - 1, \\
1 - [b(N) + d(N)]\Delta t, x = N, N \in 0, 1, ..., N \quad (1) \\
0, otherwise. \quad (2)
\end{aligned}
$$

Then, $p_N(t + \Delta t)$ satisfies the following difference equations:

$$
\begin{aligned}
p_N(t + \Delta t) = b(N - 1)\Delta t p_{N-1}(t) + d(N + 1)\Delta t p_{N+1}(t) \\
+(1 - [b(N) + d(N)]\Delta t)p_N(t) \quad (3)
\end{aligned}
$$

for $N = 1, 2, ..., M-1$. For $N = 0$ and $N = M$, $p_0(t+\Delta t) = p_0(t)+d(1)\Delta t p_1(t)$ and $p_M(t+\Delta t) = b(M - 1)\Delta t p_{M-1}(t) + (1 - d(N)\Delta t)p_M(t)$. The difference equations project forward in time and in matrix form they can be expressed as:

$$p(t + \Delta t) = P_p(t) \quad (4)$$

where P is the transition matrix.

In continuous MC model, $t \in [0, \infty)$. For Δt sufficiently small, the infinitesimal transition probabilities $p_{Nx}(\Delta t)$ are similar to those given in (1). As $\Delta t \to \infty$, a system of differential equations for the probabilities $p_N(t)$ satisfy the forward Kolmogorov differential equations:

$$
\begin{aligned}
\frac{dp_N(t)}{dt} = b(N - 1)p_{N-1}(t) - [b(N) + d(N)]p_N(t) \\
+d(N + 1)p_{N+1}(t), N \in \{1, ..., M\} \quad (5)
\end{aligned}
$$

and $\frac{dp_0(t)}{dt} = d(1)p_1(t)$. In matrix form,

$$\frac{dp}{dt} = QP, p_{N0} = 1 \quad (6)$$

where matrix $Q = (q_{ij})$ is the infinitesimal generator matrix (for details see Allen and Allen, 2003).

Finally, for stochastic differential equation case, both time and state are continuous variable. Denoting $p(N, t)$ as the probability density function and assuming that birth and death processes satisfy conditions $C1 - C4$, Allen and Allen (2003) state the Kolmogorov differential equation:

$$
\begin{aligned}
\frac{\partial p(N, t)}{\partial t} = -\frac{\partial([b(N) - d(N)]p(N, t))}{\partial t} \\
+\frac{1}{2}\frac{\partial^2([b(N) + d(N)]p(N, t))}{\partial N^2} \quad (7)
\end{aligned}
$$

for $N \in (0, M), t \in (0, \infty)$. From the forward Kolmogorov equation the sample paths N(t) of the stochastic process satisfy Ito stochastic integral which is often expressed as stochastic differential equation:

$$
\begin{aligned}
\frac{dN(t)}{dt} = b(N(t)) - d(N(t)) \\
+\sqrt{b(N(t)) + d(N(T))}\frac{dW(t)}{dt}, N(0) > 0. \quad (8)
\end{aligned}
$$

The persistence time in each of the three models are defined by the difference (discrete time MC) and differential equations (continuous time and stochastic differential equations). Allen and Allen (2003) derive the the explicit formula for mean persistence time for both discrete and continuous time models assuming the initial population size N_0.

4.2 Model with memory: Time dependence and long memory mechanism

The Markov models described above are memory-less models and stochastic shocks in $N(t)$ cannot be sustained for long. The system has limited ability to carry past shocks to the future. However, demography - like any other physical and non-physical system - possesses tendency to remember past shocks and which affect the current as well as future growth trajectory of the system. This implies that Markovian models are not sufficient to describe true persistence property of shocks, which are otherwise characterized by time series processes. Cumberland and Sykes (1982), for instance, examined the crude vital rate of Sweden and supported the view that a natural starting point in modeling the crude vital rate of a human population is a first-order autoregressive (AR) process. Gil-Alana (2003) and Azomahou and Mishra (2009) examined the stochastic nature of population growth for OECD and non-OECD countries using a fractionally integrated autoregressive moving average (ARFIMA) (to be defined shortly) and found that population growth in most of these countries are characterized by stochastic long-memory persistence, in the sense that a shock to the population growth takes very long time to converge - which is contrary to the conventional modeling of the variable in a stationary domain.

In this paper, we draw on Mishra (2006), Mishra et al. (2009) and Azomahou and Mishra (2009) to model demographic and economic growth system with a long memory framework (to be described shortly). Economic historians may refer this to hysteresis effect. However, our point of departure lies in the basic distinction between long memory process and hysteresis process. In the literature, hysteresis effect is often confused with long memory series, since the hysteresis effect is a persistence in the series like the long memory effect. But the long term behavior of the hysteretic series is very different from the long term behavior of the long memory series: the hysteric series are not mean reverting whereas the long memory series are (if correctly differenced). Since the mean reverting property is crucial for many economic models for checking the stability of the equilibria, distinguishing between long memory and hysteresis effect is important. This difference is due to the fact that hysteresis models have in fact a short memory, since the dominant shocks erase the memory of the series, and the persistence is due to the permanent and non-reverting state changes at a microstructure level.

To further elucidate, recall that population growth is denoted by n_t at time t and total population size as N. A long-memory in n_t can be defined as follows:

Definition 1 *Denote d as the integration parameter lying on the real line, k as the lag length. Now, suppose that n_t is a process with autocovariance function $\gamma(k) \sim C(k)k^{2d-1}$ as $k \to \infty$, $C(k) \neq 0$, where k defines the lag between current and distant observations. Then n_t is a long-memory process if the autocovariance function decays slowly to the mean value over time.*

Now let's define the fractional integrated autoregressive (AR) moving average (MA) process (ARFIMA (p,d,q)) for p AR order and q MA order along with fractional d for n_t with/without feedback effect from the economy. Two cases are distinguished. In the pure demographic model, dynamics of population growth (n_t) is determined by its autoregressive and moving average structure such that n_t at time t is led by its own evolutionary characteristics and by the evolution of some stochastic shocks. In this setting, no feedback effect accrues from economy to demography and the converse, but the dynamics is governed by exogenous growth generating mechanism. Interaction model (as described below), however, may contain terms which explain structural dynamics of n_t even while being explained by ARMA features.

We describe two demographic models (viz., pure and interaction) using an autoregressive fractionally integrated moving average (ARFIMA. For details see, Bailley, 1996 for excellent survey on these models).

- *Pure demographic model*:

$$(1-L)^d\Phi(L)(n_t-\mu_0)=\mu_1+\theta(L)u_t \tag{9}$$

In this case, population growth is dependent on past and stochasticity in population growth is modeled solely in terms of past shocks in the system.

- *Interaction model*:

In the model described below stochasticity in population growth depends not only on its own evolutionary effect, it also thrives on the evolutionary consequences of economic system as well.

$$(1-L)^d\Phi(L)(n_t-\mu_0)=\mu_1+\beta x_t+\theta(L)u_t \tag{10}$$

where $u_t\sim iid(0,\sigma_u^2)$; $\Phi(L)=(1-\phi_1L-\cdots-\phi_pL^p)$: AR(p); $\theta(L)=(1+\theta_1L+\cdots+\theta_qL^q)$: MA(q). Furthermore, μ_0 and μ_1 are intercepts which affect the demographic system differently (due to the way they enter the system). x_t is the vector of explanatory variables which may include lagged dependent variable and others with a possibly distributed lag structure. Formally, $(1-L)^d$ can be described by power series expansion mechanism:

$$(1-L)^d=\sum_{j=0}^{\infty}(-1)^j\left(\frac{d(d-1)(d-2)\dots(d-j+1}{j!}\right) \tag{11}$$

where $\frac{d(d-1)(d-2)\dots(d-j+1}{j!}$ is the binomial coefficient which is defined for any real number d and non-negative integer j. The most intuitive exposition of $(1-L)^d$ for a time series is via their infinite order moving average (MA) or autoregressive (AR) representations. In this instance, expressing $MA(\infty)$ of $(1-L)^d$ for the time series would mean that we have an expression: $\sum_{j=0}^{\infty}h_jL^j$, where $h_0=1$ and

$$h_j=\frac{-d\Gamma(j-d)}{\Gamma(1-d)\Gamma(j+1)}=\frac{j-d-1}{j}h_{j-1},j\geq 1. \tag{12}$$

It may be noted that the AR and MA representations of fractionally differenced series illustrate the central properties of fractional process, particularly long-range dependence, which is the focus of this paper. Allowing d to lie on the real line renders a flexible mechanism to display varied shock convergence properties. For instance, with $d=0$ in $(1-L)^d=\epsilon_t$ without AR and MA components, that is with a fractional Gaussian process, the system exhibits 'short memory' because the autocorrelations in this case is summable and decay fairly rapidly so that a shock has only a temporary effect completely disappearing in the long run. Long memory and persistence is observed for $d>0$. In this case, the shock affects the historical trajectory of the series. However, greater is the magnitude of d, stronger is the memory and greater is shock persistence. For $d\in(0,0.5)$, the series is covariance stationary and the autocorrelations take much longer time to taper-off. When $d\in[0.5,1)$, the series is a mean reverting long-memory and non-stationary process. This implies even though remote shocks affect the present value of the series, this will tend to the value of its mean in the long run. For $-1/2<d<0$ the process is known to be fractionally over-differenced. In this case, there is still short memory with summable autocovariances, but the autocovariance sequence sums to 0 over ($-\infty$, $+\infty$).

For $d < -1/2$ the series is covariance stationary but not invertible. And finally, when $d \geq 1$ the series is nonstationary and exhibits 'perfect memory' or 'infinite memory'. There is no unconditional mean defined for the series in this case. The process defined by this value of d is non-stationary and non-mean reverting. In this case, the mean of the series has no measured impact on the future values of the process. Important to note that for $0.5 \leq d < 1$, there is no variance, so the existence of the mean would need to be established in each case. There is a median, however. So this case may be described by 'median reversion'. The results are summarized in Table 1.

Table 1: Fractional components and their interpretation

d	Interpretation
0	: Short-memory population growth, log population is $I(1)$
1	: Non-stationary population growth, log population is $I(2)$
$< 0, 0.5 >$	: Long-memory population growth, log population is I(d+1)

But what intuitive explanation long-memory system offers for understanding demographic and economic growth stochasticity? Silververg and Verspagen (1999) explain that long-memory is intermediate between a relatively unstructured stochastic world in which the present is just the summation of unrelated random events in the past (a random walk), and a rigidly predictable cycle or trend with relatively negligible, mean reverting stochastic disturbances. It is indeed so, as it preserves the notion of even the distant past continuing to influence the present in a somewhat law like fashion, allowing the future to be structured while remaining shrouded in a haze of uncertainty.

4.3 Duration dependence

Stochasticity in population growth can also be defined by duration dependence, which assumes similarity with Markov mechanism in that a shock must survive some periods in order it to be defined it as persistence or long memory. The survival probability indicates the length of persistence. Technically, if we describe population growth by $(1-L)^d n_t = \epsilon_t$, the basic question is what sort of process might generate such data. For finite order autoregressive and moving-average approximations for fractionally integrated processes require extremely long lags to achieve any kind of accuracy. For relatively small samples, it is possible still to identify the source of long-memory in population growth with greater degree of accuracy using an error-duration representation (Parke, 1999). The underlying idea is to describe a process where a shock survives some periods giving rise to persistence characteristics. Assuming $\epsilon_t, t = 1, 2, ...$ as a series of i.i.d. shocks with mean zero and finite variance, σ^2, the error can be described to possess stochastic duration $\eta_s \geq 0$, surviving from period s until period $s + \eta_s$. Let $I_{s,t}$ be an indicator function for the event that error ϵ_s survives to period t such that $I_{s,t} = 1$ for $t \leq s + \eta_s$ and $I_{s,t} = 0$ for $t \geq s + \eta_s$. If p_k is the probability that ϵ_s survives until period $s + k$, i.e., $p_k = I_{s,s+k} = 1$, then a realization z_t can be described by the sum of all errors $\epsilon_{t-i}, i = 0, 1, 2, ...$ that survive until period t: $z_t = \sum_{s=-\infty}^{t} I_{s,t}\epsilon_s$. The survival probabilities $p_0, p_1, p_2, ...$ are the fundamental parameters of the error duration representation of z_t

Irrespective of the mode of definition of stochasticity (i.e., whether duration dependence or simple temporal dependence), we inevitably arrive at the same properties of population growth concerning shock convergence property. In the ensuing sections we utilize the temporal behavior of population and economic growth to lend an intuitive explanation to the possible effect of stochastic long-memory effect of demographic system on economic growth.

5 Model

5.1 Some basic properties

To know how stochastic demographic system may induce volatility in economic growth, we would like to show that the conditional mean and variance of $k-$period aggregate output is a function of stochastic memory of demographic system.

To show this, assume a simple economic-demography growth model (EDM):

$$y_t = \gamma n_{t-1} + \eta_t \tag{13}$$

where $\eta_t \sim (0, \sigma^2_\eta)$. This model provides us with simple explanation that past population growth impacts output at time t because it takes time for the economy to feel the effect of population rise (which is presented in terms of net resource users). Similarly, by adding 1 on each side of this equation, we get a relation that implies output at period $t+1$ depends on population growth at period t. Thus, there is a feedback effect from economy to the demographic system and the converse.

Proposition 1 *Under the assumption of feedback effect between economy and demographic system in the EDM model (described above), long memory in output growth, y_t, can be represented by the long memory in the demographic system.*

Proof of proposition 1

Let n_t in EDM model (Equation 13) follow an ARFIMA(p,d,q) process:

$$\begin{aligned}&(1-\phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p)(1-L)^d n_t \\ &= (1+\theta_1 L + \theta_2 L^2 + \cdots + \theta_q L^q)\epsilon_t\end{aligned} \tag{14}$$

with usual definitions: $\mathbb{E}[\eta_t \epsilon_s] = \sigma^2_{\tau\epsilon}$ if $t = s$, 0, otherwise. We assume $\phi(L) \neq 0$ for $z \leq 1$. Re-write (14) as: $\phi(L)^{-1}(1-L)^{-d}\theta(L)\epsilon_t$. Now, denote $\omega(L) = \phi(L)^{-1}$, where $\omega(L) = \sum_{i=0}^{\infty}\omega_i L^i$ and use the identity $\omega(L)\phi(L) = 1$ to find the unknown coefficients recursively:

$\omega_0 = 1,$
$\omega_1 = \phi_1\omega_0,$
$\omega_2 = \phi_1\omega_1 + \phi_2\omega_0$ and so,
$\omega_i = \phi_1\omega_{i-1} + \cdots + \phi_p\omega_{i-p}$ for $i = p, p+1, \cdots$.

Further, using Binomial expansion of $(1-L)^d$, we have $(1-L)^{-d} = \sum_{i=0}^{\infty}\frac{(d+j-1)\cdots(d+1)d}{i!}L^i$. Multiplying $(1-L)^{-d}$ and $\phi(L)^{-1}$, we get

$$(1-L)^{-d}\phi(L)^{-1} = \sum_{j=0}^{\infty} z_j L^j \tag{15}$$

where
$z_j = 1$ if $j = 0,$
$z_j = \omega_0\frac{(d+j-1)\cdots(d+1)d}{j!} +$
$\omega_1\frac{(d+j-2)\cdots(d+1)d}{(j-1)!} + \cdots + \omega_{j-1}d + \omega_j$, otherwise.

And finally, for $j \geq 0$, describe
$\psi_j = z_j + z_{j-1}\theta_1 + \cdots z_{j-q}\theta_q$
with $z_{-1} = \cdots = z_{-q} = 0.$

Denote the cumulative $k-$period output, y_t at time t as $Y_t^{(k)}$ and the $MA(\infty)$ representation of y_t as

$$y_t = \sum_{j=0}^{\infty} \psi_j \epsilon_{t-j}. \tag{16}$$

To know the effect of stochastic demographic shocks on aggregate output, we utilize EDM and $MA(\infty)$ representations such that:

$Y_t^{(k)} = \sum_{l=1}^{k} y_{t+l} = \gamma. \sum_{l=1}^{k} \sum_{j=0}^{\infty} \psi_j \epsilon_{t-1-j+l} + \sum_{l=1}^{k} \eta_{t+l}$
Denoting $\zeta_i^{(k)} \equiv \psi_i + \psi_{i-1} + \cdots + \psi_{i-(k-1)}$,
we can write
$Y_t^{(k)} = \gamma. \sum_{i=0}^{\infty} \zeta_i^{(k)} \epsilon_{t-1-j+l} + \sum_{l=1}^{k} \eta_{t+l}$

The *conditional expectation* of $Y_t^{(k)}$ then equals:

$$\mathbb{E}\left[Y_t^{(k)}\right] = \gamma. \sum_{i=0}^{\infty} \zeta_i^{(k)} \epsilon_{t-1-j+l} \tag{17}$$

and the *conditional variance* of $k-$period cumulative output is:

$$\begin{aligned} &Var_t\left(Y_t^{(k)} - \mathbb{E}\left[Y_t^{(k)}\right]\right) \\ &= \gamma^2. \sum_{l=1}^{k} \left(\zeta_{k-l}^{(k)}\right)^2 \sigma_\epsilon^2 + \gamma. \sum_{k-l}^{(k)} \sigma_{\epsilon\eta} + \sigma_\eta^2 \end{aligned} \tag{18}$$

Expressed in terms of ζ, aggregate output is a function of stochastic memory component both in mean and variance, thus completing the proof of long memory in output due to long-memory in aggregate population. □

5.2 Stochastic Solow Model with Long-Memory

In this section we provide a theoretical construct expounding the relation between long memory and output growth. We use a stochastic version of Solow-Swan model where population growth in the model, instead of being constant, is assumed to be stochastic so that dynamics of population growth can determine the dynamics of output in the economy. Drawing on the intuition and construct of long-memory population growth described in the preceding section, we allow population in Solow-Swan model to follow a long-memory data generation process (DGP). The economy is assumed to be closed. The production function of the representative agent is given a Cobb-Douglas type:

$$Y_t = AK_t^{\alpha} N_t^{1-\alpha} \tag{19}$$

where $0 < \alpha < 1$, Y_t is output at time t, K_t is capital input at t. Labor input, N_t governed by the growth of population, n_t so that

$$N_t = (1 + n_t) N_{t-1} \tag{20}$$

where population growth, n_t, in our system is assumed to follow a long-memory data generating process which evolves as

$$(1 - L)^d \Phi(L) n_t = \Theta(L) \epsilon_t \tag{21}$$

L is the lag operator as defined before and

$$(1-L)^d = \sum_{j=0}^{\infty} \frac{\Gamma(j-d)}{\Gamma(j+1)\Gamma(-d)} L^j \tag{22}$$

$\Phi(L) = (1 + \phi_1 L + ... + \phi_p L^p)$ and $\Theta(L) = (1 - \theta_1 L - ... - \theta_q L^q)$ are AR and MA polynomials respectively. Moreover, the investment, I_t and capital stock equations are described as

$$K_{t+1} = (1-\delta)K_t + I_t \tag{23}$$

In the above equation, capital stock is assumed to decline at a constant rate of δ $(0 < \delta < 1)$ per period. Given that s is the fraction of Y to be invested, then

$$I_t = sY_t \tag{24}$$

Consumption is defined according to

$$C_t = (1-s)Y_t \tag{25}$$

Proposition 2 *Given a production function of Solow-Swan type where population growth follows a long-memory data generating process (DGP) (Equations 8 - 12), the output growth in the economy will also follow a long-memory DGP. Long-run convergence of output will be determined depending on the 'degree' of memory of stochastic population shocks.*

Proof of proposition 2

The immediate effect of long-memory population growth on economy's long-term output, consumption and investment growth can be observed by plugging the long-memory DGP of n_t in the production, capital, and consumption equations. Assuming that $\psi(L) = \frac{(1-\theta_1 L - ... - \theta_q L^q)}{(1+\phi_1 L + ... + \phi_p L^p)} \equiv 1$[5] in equation 21 and substituting it in equation 20 and then in equation 19, we obtain

$$Y_t = AK_t^{\alpha}[(1 + (1-L)^{-d}\psi(L)\epsilon_t)N_{t-1})]^{1-\alpha} \tag{26}$$

The output per capita, $y_t = (Y_t/N_t)$ in this case is a function of sequence of shocks, thus regulating the 'efficiency unit of output' by the stability of shocks. Moreover, since $(1-L)^d$ can be represented by impulse-response mechanism, viz., $\sum_{j=0}^{\infty}(j+1)^{d-1}$, inducting this in equation 26 then depicts

$$Y_t = AK_t^{\alpha}[(1 + \sum_{j=0}^{\infty}(j+1)^{1-d}\psi(L)\epsilon_t)N_{t-1})]^{1-\alpha} \tag{27}$$

Assuming the effect of technology, A, to be constant on Y_t, or by assuming that growth in A is caused by population pressure, a unit shock in n_t in equation 27 can exhibit how Y_t responds to it. Nevertheless, it is clear that depending on the magnitude of d, the behaviour of N_t can determine the nature of output growth in the economy. Now, since consumption and investment are a function of output, the persistence of shocks in output, consumption and investment growth in the economy. Denoting, aggregate output and aggregate consumption at T as Q_T and C_T it can be shown that $\sum_{t=1}^{T} Y_t = f(K, n(d))$, and $\sum_{t=1}^{T} C_t = f(Y, n(d))$ where $n(d)$ denotes long-memory population growth.

[5]This assumption is not binding but assumed for simplicity.

5.3 Stochastic Solow Model with Brownian Motion

We now have the necessary tools to analyze the behavior of the stochastic model over time. We first introduce the main structure of the Stochastic Solow Model with the Brownian Motion, and then consider the long run or steady state.

The construct

As before, the Cobb-Douglas production function is

$$F(K_t, L_t) = AK_t^{\alpha} L_t^{\beta},$$

where $A > 0$ is the level of technology, $\alpha \in (0, 1)$, $\beta = 1 - \alpha$.

In the deterministic model, the net increase in the stock of physical capital at a point in time equals gross investment less depreciation:

$$\dot{K}_t = sY_t - \delta K_t, \quad L_t = nL_t \tag{28}$$

where $\dot{K}_t = dK_t/dt$ denote the net increase, $s \in (0, 1)$ is a constant saving rate, $\delta \in [0, 1]$ is the depreciation, the population grows at a constant, exogenous rate $n \geq 0$. The fundamental differential equation of the Solow-Swan model is

$$\dot{k}_t = sf(k_t) - (n + \delta)k_t. \tag{29}$$

Assume that the growth rate of K_t and L_t are affected by some random disturbance, (29) can be written as:

$$dK_t = (sY_t - \delta K_t)dt + K_t dB_t^K \tag{30}$$

and

$$dL_t = nL_t dt + L_t dB_t^L \tag{31}$$

where B_t^K and B_t^L are given Brownian Motions. The perturbations of K_t and L_t are results from independent effects of large number of small factors, so we can use Brownian Motions to describe disturbances in (30) and (31). We denote the variance of B_t^K by $\sigma_K dt$, B_t^L by $\sigma_L dt$, and $cov(dB_t^K, dB_t^L) := \sigma_{KL} dt$.

By the Itô lemma, we have

$$dk_t = (sf(k_t) - \mu k_t)dt + k_t dB_t, \tag{32}$$

where

$$\mu = n + \delta + \sigma_{KL} - \sigma_L^2$$

and

$$dB_t = dB_t^K - dB_t^L.$$

(32) describes stochastic Solow-Swan Model with Brownian motion. From (32), we can easily address:

1. Per capita capital k_t is a homogenous diffusion process which the drift and the diffusion coefficients are $sf(k_t) - \mu k_t$ and $k_t^2(\sigma_K^2 - 2\sigma_{KL} + \sigma_L^2)$ respectively;

2. The Markov property of k_t shows that: the economic status at the presents can forecast the trend for the future. Namely, if $s > t$, $k_t = k$, then the probability density of k_s is determined by $k_t = k$. Meanwhile, by using Kolmogorov equation, the transition function $p(t, k, k_1)$ can be obtained. If $s > t$, $\forall (a, b)$,

$$\mathbb{P}\big(a < k_s < b | k_t = k\big) = \int_a^b p(s - t, k, y)dy;$$

3. From (32), the expected growth rate of k_t is

$$\begin{aligned}\psi_k := \mathbb{E}(\frac{dk_t}{k}) &= \frac{sf(k_t)}{k} - \mu \\ &= sf(k_t) - (n + \delta + \sigma_{KL} - \sigma_L^2).\end{aligned} \tag{33}$$

In the absence of stochastic disturbance, we have the deterministic growth rate g_k :

$$g_k = \frac{\dot{k}_t}{k} = \frac{sf(k_t)}{k_t} - (n + \delta). \tag{34}$$

From (33) and (34),

$$\psi_k - g_k = \sigma_{KL} - \sigma_L^2.$$

This means when $\sigma_{KL} > \sigma_L^2$, the existence of the stochastic disturbance can raise economic growth. On the contrary, $\sigma_{KL} < \sigma_L^2$, the existence of the stochastic disturbance can reduce economic growth. Precisely, if the magnitude of L_t is bigger then K_t, then $\sigma_{KL} < \sigma_K\sigma_L < \sigma_L^2$, $\psi_t < g_t$, therefore the growth rate will decrease. In exceptional circumstances, L_t is disturbed but K_t does not, that is $\sigma_L^2 > 0$, $\sigma_K^2 = 0$, in this case the growth rate will decline. Contrarily, if K_t is affected by stochastic shocks but L_t does not, $\sigma_L^2 = 0$, $\sigma_K^2 > 0$, then the growth rate will remain the same. It can be clearly seen that, the perturbation of K_t and L_t has different influences for the economic growth. L_t has relatively bigger impact than K_t.

The Stability Analysis

We define a steady state as a situation in which the various quantities grow at constant rate. In the deterministic Solow Model, the steady state corresponds to $\dot{k}_t = 0$, the corresponding value of k_t is denoted by k_t^*, that is $sf(k_t^*) = (n + \delta)k_t^*$, $t \to 0$, $k_t \to k^*$. We say $k_t \equiv k_t^*$ is asymptotically stable in $(0, \infty)$ globally.

The stability of the system will change along with the appearance of the stochastic disturbance. Firstly, (32) does not have the steady state apart from $k_t = 0$ since k_t^* is eliminated by the stochastic disturbance terms. Secondly $k_t = 0$ in (32) is not obvious, apparently $k_t = 0$ in (29) is not a steady state, so it is not comparable. We will consider the exponential stability of the system in this section. As almost surely exponentially stable can easily apply asymptotically stable in global, so we will focus on this kind of stable on the stochastic system. A lyapunov funcion can be used to analyze the stability of (32).

Suppose that $D = \mathbb{R}_+$, $V(k_t) = k_t^2$, we have

$$\sup_{k>0} \Big[\frac{sf(k_t)}{k} + \sigma^2 - 2\mu\Big] < 2\sigma^2$$

and

$$\inf_{k>0} \Big[\frac{sf(k_t)}{k} + \sigma^2 - 2\mu\Big] > 2\sigma^2.$$

They implies:

$$\sup_{k>0} \frac{sf(k_t)}{k} < \frac{\sigma^2 + 2\mu}{2s} \tag{35}$$

and

$$\inf_{k>0} \frac{sf(k_t)}{k} > \frac{\sigma^2 + 2\mu}{2s}. \tag{36}$$

Let $\varphi(k_t) := \frac{f(k_t)}{k}$. Then one can obtain

$$k_t^2 \varphi'(k_t) = k_t f'(k_t) - f(k_t), \quad k_t > 0.$$

Thus

$$\lim_{k \to 0} \varphi(k_t) = \lim_{k \to 0} f'(k_t) = \infty,$$

and

$$\lim_{k \to \infty} \varphi(k_t) = \lim_{k \to \infty} f'(k_t) = 0.$$

Therefore we have

$$\sup_{k>0} \varphi(k_t) = \varphi(0) = \infty, \quad \inf_{k>0} \varphi(k_t) = \varphi(\infty) = 0.$$

The discriminant condition (35) and (36) can be re-written as

$$\sigma^2 + 2\mu > \infty \tag{37}$$

$$\sigma^2 + 2\mu < 0. \tag{38}$$

Obviously, (37) is not possible. From (38) we can obtain

$$2(n + \delta) + \sigma_K^2 < \sigma_L^2. \tag{39}$$

It means: if $\sigma_L^2 < 2(n+\delta) + \sigma_K^2$, (32)'s solution is not exponentially stable in $[0, \infty]$ a.s.. In other words, per capita capital exponential grows from the initial point, the trajectory of the solution that starts from $D = [0, \infty)$ but also eventually coverage in D. Therefore, (39) is the criterion of the exponential instability of the zero solution of (32).

5.4 Stochastic Solow Model and the Jump Process

So far, we dealt with a continuous time stochastic Solow-Swan model. In this section, we relax this assumption and rather investigate what would happen if the stochastic shocks appear as jump process. The underlying idea behind this is that demographic systems often experience sudden jump in their dynamic behavior - due either to sudden perturbations from the nature or from the economy because of excessive human interventions.

To understand the nature of jump process, we will use Jump-type Lévy processes to represent the stochastic disturbances in the Solow-Swan Model. The most well known examples of Lévy processes are Brownian motion and the Poisson process.

Construct

Let $\{N_t^K\}_{t\geq 0}$ and $\{N_t^L\}_{t\geq 0}$ two poisson processes with intensity measures λ_1, λ_2, we further assume that $\langle N_t^K, N_t^L \rangle = \lambda_1 \lambda_2 t$ where $\langle N_t^K, N_t^L \rangle$ stands for the quadratic process of N_t^K and N_t^L. The Poisson distribution with associated parameter λ is:

$$\mathbb{P}(N_t^K(\omega) = n) = e^{-\lambda_1 t} \frac{(\lambda_1 t)^n}{n!}, \quad n = 1, 2, \ldots,$$

and

$$\mathbb{P}(N_t^L(\omega) = n) = e^{-\lambda_2 t} \frac{(\lambda_2 t)^n}{n!}, \quad n = 1, 2, \ldots.$$

Consider

$$dK_t = (sY_t - \delta K_t)dt + K_t dN_t^K \tag{40}$$

and

$$dL_t = nL_t dt + L_t dN_t^L \tag{41}$$

Due to the Itô formula, we obtain from (41)

$$d(\frac{1}{L_t}) = -\frac{n}{K_t}dt - \frac{1}{2L_t}dN_t^L. \tag{42}$$

We also have

$$d\langle K_t, \frac{1}{L_t}\rangle = \frac{K_t}{2L_t}\lambda_1\lambda_2 dt \tag{43}$$

Also by the Itô formula, we deduce that

$$dk_t = d(\frac{K_t}{L_t}) = \frac{1}{L_t}dK_t + K_t d(\frac{1}{L_t}) + d\langle K_t, \frac{1}{L_t}\rangle. \tag{44}$$

Substitute (40), (42),(43) into (44), we can get

$$dk_t = [sf(k_t) - \delta k_t - nk_t - \frac{k_t}{2}\lambda_1\lambda_2]dt + k_t dN_t^K - \frac{k_t}{2}dN_t^L. \tag{45}$$

Therefor (45) is a stochastic Solow-Swan Model with jumps. From (45), the expected growth rate of k_t is

$$\phi_k = \mathbb{E}(\frac{dk_t}{k}) = \frac{sf(k_t)}{k_t} - \delta - n - \frac{1}{2}\lambda_1\lambda_2 + \lambda_1 - \frac{1}{2}\lambda_2 \tag{46}$$

Recall in the deterministic model, the growth rate g_k :

$$g_k = \frac{\dot{k}_t}{k} = \frac{sf(k_t)}{k_t} - (n+\delta).$$

From (34) and (46), we have

$$\phi_k - g_k = -\frac{1}{2}\lambda_1\lambda_2 + \lambda_1 - \frac{1}{2}\lambda_2.$$

This means when $2\lambda_1 - \lambda_1\lambda_2 - \lambda_2 > 0$ the exitances of the stochastic disturbance driven by jumps can raise the growth. On the contrary, $2\lambda_1 - \lambda_1\lambda_2 - \lambda_2 < 0$ the growth will be reduced. If $\lambda_1 = \lambda_2 = 0$ the stochastic disturbance disappears, then $\phi_k = g_k$.

Stability Analysis

As the same deification in Section 2, the steady state level of capital stock is the stock of capital at which investment and depreciation just offset each other, that $\dot{K}_t = 0$.

In order to analyze the stability of (45). we will employ lyapunov function. Assume $D = \mathbb{R}_+$, $V(k_t) = k_t^2$, we have

$$\begin{aligned} dk_t^2 = k_t^2[&\frac{sf(k_t)}{k_t} - 2\delta - 2n - 2\lambda_1\lambda_2 + 3\lambda_1 - \frac{3}{4}\lambda_2] \\ &+ \text{ Martingale.} \end{aligned}$$

By the definition of lyapunov function, if

$$\sup_{k>0}\left(\frac{sf(k_t)}{k_t} - 2\delta - 2n - 2\lambda_1\lambda_2 + 3\lambda_1 - \frac{3}{4}\lambda_2\right) < 0,$$

The above condition can guarantee the system is exponentially stable, otherwise

$$\inf_{k>0}\left(\frac{sf(k_t)}{k_t}-2\delta-2n-2\lambda_1\lambda_2+3\lambda_1-\frac{3}{4}\lambda_2\right)>0,$$

the system is not exponentially stable. They implies:

$$\sup_{k>0}\frac{sf(k_t)}{k_t}<\frac{2\delta+2n+2\lambda_1\lambda_2-3\lambda_1+\frac{3}{4}\lambda_2}{2s} \tag{47}$$

and

$$\inf_{k>0}\frac{sf(k_t)}{k_t}>\frac{2\delta+2n+2\lambda_1\lambda_2-3\lambda_1+\frac{3}{4}\lambda_2}{2s} \tag{48}$$

Let $\varphi(k_t):=\frac{f(k_t)}{k}$, then, we can get

$$k_t^2\varphi'(k_t)=k_tf'(k_t)-f(k_t),\quad k_t>0.$$

Thus

$$\lim_{k\to 0}\varphi(k_t)=\lim_{k\to 0}f'(k_t)=\infty,$$

and

$$\lim_{k\to\infty}\varphi(k_t)=\lim_{k\to\infty}f'(k_t)=0.$$

Therefore we have

$$\sup_{k>0}\varphi(k_t)=\varphi(0)=\infty,\quad \inf_{k>0}\varphi(k_t)=\varphi(\infty)=0.$$

The discriminant condition (47) and (48) can be re-written as

$$2\delta+2n+2\lambda_1\lambda_2-3\lambda_1+\frac{3}{4}\lambda_2>\infty \tag{49}$$

and

$$2\delta+2n+2\lambda_1\lambda_2-3\lambda_1+\frac{3}{4}\lambda_2<0. \tag{50}$$

Apparently, (49) does not hold. Under the criterion of (50), that

$$2(\delta+n)<3\lambda_1-2\lambda_2\lambda_2-\frac{3}{4}\lambda_2$$

The stochastic system is exponentially unstable.

The stationary distribution of k_t

In the deterministic Solow-Swan model, k_t finally goes to steady state k_t^*. In a similar way, in the stochastic model, k_t will go to a non-zero random variable as $t\to\infty$. If this random variable is continuous, then we can apply apply it's probability density function (PDF) denoted by $\pi(\cdot)$. Here $\pi(\cdot)$ is named the stationary distribution of k_t.

Merton (1975) and Bourguigono (1974) first proposed to use a definition of the stationary distribution in the theory of economic growth. In this Section, based on Merton's model, we provide some extensions for computing the stationary distribution of k_t in the stochastic Solow Model.

Now let us discuss $\pi(\cdot)$. Recall k_t is from (32), the production function is k_t^α $(0<\alpha<1)$, by the Kolmogorov forward equation, we have

$$\varphi(k_t)=\frac{s(1-k_t^{-\alpha'})}{\alpha'\sigma^2}-\frac{\mu}{\sigma^2}\ln k_t,$$

where $\alpha' = 1 - \alpha$. Then

$$\begin{aligned}\frac{1}{\pi(k_t)} &= k_t^2 e^{2\varphi(k_t)} \int_0^\infty x^{-2-\frac{2\mu}{\sigma^2}} e^{-\beta k_t^{-\alpha'}} dx \\ &= \frac{\Gamma(\omega)}{\alpha'} \beta^{-\omega} k_t^{2+2\mu/\sigma^2} \exp(-\beta k_t^{-\alpha'}),\end{aligned}$$

where $\beta = \frac{2s}{\alpha'\alpha^2}$, $\omega = \frac{2\mu+\sigma^2}{\sigma^2\alpha'}$.

Therefore,

$$\pi(k_t) = \frac{\alpha'}{\Gamma(\omega)} \beta^{\omega} k_t^{-2-2\mu/\sigma^2} \exp(-\beta k_t^{-\alpha'}).$$

We need $\omega > 0$, which implies the following condition

$$2(n+\delta) + \sigma_K^2 > \sigma_L^2. \tag{51}$$

Here (51) is the condition which can make k_t converges to stationary distribution.

Now let us assume (51) is satisfied and

$$\omega_\tau = \frac{2\mu + \tau'\sigma^2}{\alpha'\sigma^2} = \omega - \frac{\tau}{\alpha'}.$$

Obviously, $\omega = \omega_0$, suppose k_t has stationary distribution, then

$$\begin{aligned}\mathbb{E}(k_t^\tau) &= \int_0^\infty k_t^\pi \pi(k_t) dk_t \\ &= \frac{\alpha'}{\Gamma(\omega)} \beta^\omega \int_0^\infty k_t^{\tau-2-2\mu/\sigma^2} \exp(-\beta k_t^{-\alpha'}) dk_t \\ &= \beta^{\frac{\tau}{\alpha'}} \frac{\Gamma(\omega_\tau)}{\Gamma(\omega)}.\end{aligned}$$

Particularly, let $\tau = 1$, $\tau = \alpha$, $\tau = -\alpha'$, we have

$$\begin{aligned}&\bar{k}_t = \mathbb{E}(k_t) = \beta^{\frac{1}{\alpha'}} \Gamma(\omega_1)/\Gamma(\omega), \\ &\bar{y}_t = \mathbb{E}(k_t^\alpha) = \beta^{\frac{\alpha}{\alpha'}} \frac{\Gamma(\omega_\alpha)}{\Gamma(\omega)} = \beta^{\frac{\alpha}{\alpha'}} \frac{\Gamma(\omega_1+1)}{\Gamma(\omega)} = \beta^{\frac{\alpha}{\alpha'}} \frac{\omega_1\Gamma(\omega_1)}{\Gamma(\omega)} = \frac{\mu}{s}\bar{k}_t, \\ &\mathbb{E}(\frac{y_t}{k_t}) = \mathbb{E}(k_t^{-\alpha'}) = \beta^{-1} \frac{\Gamma(\omega_{-\alpha'})}{\Gamma(\omega)} = \beta^{-1} \frac{\Gamma(\omega+1)}{\Gamma(\omega)} = \beta^{-1}\omega = \frac{2\mu+\sigma^2}{2s}.\end{aligned}$$

In the deterministic system, due to $f(k_t) = k^\alpha$, the steady state of k_t is:

$$k_t^* = \Big(\frac{s}{n+\delta}\Big)^{\frac{1}{\alpha'}}.$$

Assume $n + \delta > 0$, now we can compare $\bar{k}_t$ and k_t^*. Set $\varepsilon = \frac{1}{\alpha'} = \frac{1}{1-\alpha} > 1$ and $x = \frac{2\mu}{\alpha'\sigma^2} = \omega_1$, then we have

$$\begin{aligned}\frac{\bar{k}_t}{k_t^*} &= \Big(\frac{2n+2\delta}{\alpha'\sigma^2}\Big)^\varepsilon \frac{\Gamma(x)}{\Gamma(x+\varepsilon)} \\ &= (\frac{n+\delta}{\mu})^\varepsilon \frac{x^\varepsilon \Gamma(x)}{\Gamma(x+\varepsilon)}.\end{aligned}$$

When $\sigma^2 \to 0$, $(\frac{n+\delta}{\mu})^\varepsilon \to 1$; By Stirling Approximation, we have $\frac{x^\varepsilon\Gamma(x)}{\Gamma(x+\varepsilon)} \to 1$ as long as $x \to \infty$. Therefor we can say when $\sigma^2 \to 0$, $\bar{k}_t \to k_t^*$.

This conclusion can prove that when the stochastic terms disappears, $\bar{k}_t = k_t^*$, the steady state of k_t^* (in deterministic model) is equivalent to $\bar{k}_t$ (in stochastic model).

Thus, the above exercise showed that under both Brownian motion and Jump process, stochastic population growth would exert differential dynamic effects on volatility in economic growth. A permanent type of effect is observed when demographic growth follows a Jump process. These theoretical findings need to be qualified. In the next section, we perform a small empirical exercise.

6 Empirical analysis

We present here estimates of long memory for output and population growth for a set of OECD and non-OECD countries. In Tables 2 and 3 below we report the estimates of the long memory parameter, d, the magnitudes of which indicate the relative rates of convergence of shocks to the long-run mean-values over time. The estimation has been performed using Kim and Phillips' (2000) modified log-periodogram regression method (MLPR). The MLPR method is a modified version of the following Geweke and Porter-Hudak (GPH, 1983) log periodogram regression:

$$ln[I_n(\lambda_\zeta)] = -2dln|1 - e^{i\lambda_\zeta}| + ln(f_u(\lambda_\zeta)) + \eta_j \tag{52}$$

where the periodogram ordinates of population growth (left hand side of the equation) are regressed over the spectral representation of the error term and the transformation of $(1 - L)^d$ in the frequency domain. The ordinates are evaluated at the fundamental frequencies $\zeta = 1, ..., \nu$. Kim and Phillips (2000) note that (52) is a moment condition and not a data generating mechanism. The modified GPH, i.e., the MLPR is given as:

$$ln(I_V(\lambda_\zeta)) = \alpha - dln|1 - e^{i\lambda_\zeta}|^2 + u(\lambda_\zeta) \tag{53}$$

in which the periodogram ordinates, $ln(I_n(\lambda_\zeta))$ are replaced by $ln(I_V(\lambda_\zeta)) = V_n(\lambda_\zeta)V_P(\lambda_\zeta)^*$ with $\alpha = ln(f_u(0))$ and $u(\lambda_\zeta) = ln[I_n(\lambda_\zeta)/f_n(\lambda_\zeta)] + ln(f_u(\lambda_\zeta)/f_u(0))$. Note that $V_n(\lambda_\zeta)V_P(\lambda_\zeta)^*$ is the discrete fourier transform and is to be used in the regression instead of $ln(I_V(\lambda_\zeta))$. Detailed derivations are presented in the appendix.

A practical problem is the choice of ν, the number of periodogram ordinates to be used in the regression. Geweke and Porter-Hudak (GPH, 1983) suggests that the optimal $\nu = T^\alpha$ where $\alpha = 1/2$ and T is the sample size. The choice involves a tradeoff that may be described as follows. The smaller the bandwidth, the less likely the estimate of d is contaminated by higher frequency dynamics, i.e., the short-memory. However, at the same time smaller bandwidth leads to smaller sample size and less reliable estimates. As in the case of GPH method, the smaller value of α (as in $\nu = T^\alpha$) implies the smaller number of harmonic ordinates (i.e., the smaller bandwidth) will be used for the estimation of d. Generally, in empirical analysis, preference is given to increasing the value of α to check for the consistency of the estimate of d although simulation experiments can confirm the validity of the selection. For our purpose, we have used $\alpha = 0.60$ through $\alpha = 0.80$ to estimate d. We choose $\alpha = 0.7$ based on a Monte Carlo simulation experiment (see table below) where we have minimum bias for that bandwidth.[6]

Tables 2 and 3 we test the null hypothesis of short-memory against the alternative of long-memory. From the d estimates, we find clear evidence of long-memory for non-OECD countries where estimated d are significantly greater than $1/2$. The d estimates are however smaller than 1 indicating the long-memory persistence with the possibility of convergent shocks in the long-run. For OECD countries' income are less persistent than non-OECD countries as we analyze the estimates over different bandwidths. Stationary long-memory features are observed for

[6]Davidson's (2007) TSM software is used to carry out the simulation experiment which is built for the GPH model.

both sets of countries for per capita income and non-stationary long-memory or highly persistent demographic shocks are observed for some non-OECD countries (where estimated d is higher than 0.5).

Table 2: Modified log-periodogram estimation of the long-memory parameter for per capita income

T^{τ}	τ =0.5	τ =0.55	τ =0.60	τ =0.65	τ =0.70	τ =0.75	τ =0.80
Periodogram Ordinates	11	14	18	24	30	39	50
OECD							
USA	0.416 (0.215)	0.269 (0.172)	0.204 (0.151)	0.096 (0.131)	0.203 (0.118)	0.162 (0.112)	0.104 (0.052)
Japan	0.160 (0.066)	0.211 (0.056)	0.331 (0.113)	0.312 (0.094)	0.200 (0.083)	0.211 (0.075)	0.244 (0.064)
UK	0.052 (0.187)	0.101 (0.144)	−0.029 (0.146)	−0.340 (0.147)	−0.461 (0.124)	−0.484 (0.104)	−0.478 (0.085)
France	−0.515 (0.335)	−0.446 (0.250)	−0.397 (0.189)	−0.163 (0.175)	−0.089 (0.142)	−0.119 (0.120)	−0.152 (0.103)
Canada	0.703 (0.130)	0.453 (0.153)	0.324 (0.129)	0.307 (0.106)	0.173 (0.101)	0.251 (0.133)	0.169 (0.113)
Non-OECD							
China	0.706 (0.134)	0.519 (0.154)	0.529 (0.131)	0.514 (0.137)	0.583 (0.126)	0.619 (0.105)	0.625 (0.085)
India	0.387 (0.314)	0.658 (0.298)	0.795 (0.264)	0.503 (0.218)	0.544 (0.179)	0.280 (0.173)	0.241 (0.157)
Brazil	0.373 (0.271)	0.321 (0.215)	0.455 (182)	0.508 (0.138)	0.547 (0.123)	0.538 (0.101)	0.611 (0.125)
South Africa	0.589 (0.296)	0.969 (0.379)	0.751 (0.304)	0.731 (0.235)	0.593 (0.203)	0.404 (0.179)	0.327 (0.145)
Mexico	0.117 (0.294)	0.108 (0.235)	0.267 (0.215)	0.501 (0.179)	0.550 (0.157)	0.476 (0.130)	0.399 (0.107)

6.1 Implications for endogenous/exogenous growth

Lau (1999) used unit root framework to describe the time series properties of endogenous growth models. He showed that integration and cointegration properties arise intrinsically in stochastic endogenous growth models under fairly general conditions. If the unit root is present in the autoregressive polynomial of the variables, it can then characterize the outcome of endogenous growth mechanisms. They can produce steady-state growth in the absence of exogenous-growth generating element. From our long-memory estimates, it is clear that there is significant evidence of long-memory for both per capita income and population. While a difference stationary model (with unit root assumption) can imply the presence of stochastic endogenous mechanism, a fractional integration/long-memory in output growth can imply the existence of stochastic semi-endogenous growth setting. The latter appears to be more powerful as real life economic variables often display slow-convergence pattern of stochastic shocks. One way to understand the impact of stochastic shock in population on per capita income, is to perform a cointegration analysis. This would also enable us to infer if the joint process is governed by an exogenous or endogenous growth mechanism.

For the purpose, we adopt the two-step strategy as in Caporale and Gil-Alana (2004, 2005) and discussed succinctly in Gil-Alana and Hualde (2009). The strategy is to employ fractional integration test in various stages. Accordingly, in the first step, we test for the order of integration of each series, and if they are found to be of the same order, we test, in the second step, the order of integration of the estimated residuals of the cointegration relationship. Let us call ϵ_t, the estimated equilibrium errors between two series, real GDP per capita and aggregate

Table 3: Modified log-periodogram estimation of the long-memory parameter for total population

T^{τ}	τ =0.5	τ =0.55	τ =0.60	τ =0.65	τ =0.70	τ =0.75	τ =0.80
Periodogram Ordinates	11	14	18	24	30	39	50
OECD							
USA	0.962 (0.171)	0.925 (0.131)	0.982 (0.116)	1.027 (0.115)	1.046 (0.117)	0.968 (0.102)	1.003 (0.090)
Japan	0.696 (0.240)	0.720 (0.179)	0.581 (0.145)	0.430 (0.117)	0.574 (0.132)	0.708 (0.110)	0.664 (0.088)
UK	0.504 (0.371)	0.414 (0.276)	0.315 (0.208)	0.320 (0.150)	0.262 (0.122)	0.195 (0.097)	0.137 (0.078)
France	0.125 (0.152)	0.540 (0.266)	0.494 (0.240)	0.494 (0.176)	0.607 (0.152)	0.575 (0.120)	0.526 (0.103)
Canada	1.087 (0.311)	1.077 (0.268)	0.843 (0.223)	0.658 (0.171)	0.788 (0.153)	0.728 (0.127)	0.703 (0.106)
Non-OECD							
China	0.405 (0.267)	0.528 (0.224)	0.534 (0.175)	0.476 (0.127)	0.581 (0.117)	0.670 (0.109)	0.813 (0.108)
India	0.118 (0.072)	0.110 (0.060)	0.119 (0.047)	0.116 (0.035)	0.112 (0.028)	0.089 (0.023)	0.077 (0.019)
Brazil	1.169 (0.213)	1.092 (0.198)	0.978 (0.164)	0.880 (0.131)	0.855 (0.118)	0.773 (0.099)	0.700 (0.081)
South Africa	0.880 (0.239)	0.921 (0.201)	0.984 (0.153)	1.009 (0.115)	1.029 (0.116)	1.016 (0.098)	1.037 (0.085)
Mexico	0.978 (0.509)	0.686 (0.435)	0.707 (0.359)	0.535 (0.264)	0.392 (0.213)	0.466 (0.185)	0.357 (0.148)

Table 4: Monte Carlo simulation for choice of bandwidth

Bandwidth	Estimated bias	Significance	RMSE bias
τ=0.60	0.018	3.03	0.019
τ=0.65	0.021	2.86	0.023
τ=0.70	0.014	2.20	0.015
τ=0.75	0.015	2.47	0.016
τ=0.8	0.017	2.83	0.018

population for each country:

$$\epsilon_t = \ln(Y_t) - \hat{\alpha_1} \ln P_t \tag{54}$$

where Y_t and P_t are real GDP per capita and population respectively. $\hat{\alpha}$ are the OLS estimator of the cointegrating parameter. Let us consider the model:

$$(1-L)^{d+\theta} = u_t \tag{55}$$

where u_t is a $I(0)$ process ; we applied the Robinson (1995)'s testing procedure in order to test the null hypothesis $H_0 : \theta = 0$ against the alternative $H_1 : \theta < 0$. If the null hypothesis is rejected, it implies that the equilibrium error exhibits a smaller degree of integration than the original series: Y_t, P_t and E_t are thus fractionally cointegrated. On the opposite, if the null hypothesis is not rejected, the series are not cointegrated because the order of integration of θ is the same as the order of the original series. As a first step to testing this hypothesis, we have saved residuals from regression of real GDP per capita on total population for each country.[7]

[7]The detailed results have not been reported here but are available with the authors.

Due to the unavailability of real GDP data before 1950 for some countries and for the sake of comparison, the regression has been run for the truncated sample over the period 1950-2003. In the next step, the equilibrium errors $\hat{\epsilon}_i$ where i is indexed for each country, are tested for short or long-memory using Robinson's (1995) semi-parametric log periodogram regression. Table 5 presents results of the d estimates of equilibrium errors for each country. It is observed that at $\tau = 0.9$, the default value as in Robinson (1995), USA, Japan and UK have $d < 0.5$ implying that shocks in the equilibrating mechanism will converge and that there is a stable co-movement among GDP per capita, population and CO_2 emissions in these countries. For others, we find that d values range from 0.572 - 0.959, that is $1 > d > 0.5$. The co-movement of GDP, population and CO_2 emissions in these countries contain non-stationary long-memory with a possibility of mean convergence in the long-run. Among countries with values of d in the range 0.5-0.9, China has highest d (0.959) for equilibrium errors, while South Africa has the lowest d value (0.572). All d values are statistically significant at 5 percent significance level.

Table 5: Robinson's (1995) semi-parametric estimation of d for estimated equilibrium errors (Note: H_0: d = 0. Standard errors are in parentheses)

T^τ	τ =0.70	τ =0.75	τ =0.80	τ =0.85	τ =0.90
Periodogram Ordinates	25	33	41	51	65
OECD					
USA	0.217	0.233	0.287	0.205	0.183
	(0.085)	(0.069)	(0.081)	(0.070)	(0.069)
Japan	0.443	0.449	0.635	0.452	0.426
	(0.314)	(0.252)	(0.175)	(0.166)	(0.115)
UK	1.191	0.920	0.752	0.648	0.448
	(0.331)	(0.334)	(0.249)	(0.209)	(0.162)
France	1.023	1.051	1.190	1.112	0.896
	(0.464)	(0.373)	(0.255)	(0.223)	(0.205)
Canada	1.074	1.046	1.066	1.004	0.711
	(0.330)	(0.265)	(0.214)	(0.171)	(0.169)
Non-OECD					
India	1.065	0.919	0.5915	0.847	0.665
	(0.238)	(0.220)	(0.222)	(0.199)	(0.153)
China	0.394	0.483	0.993	1.467	0.959
	(0.121)	(0.128)	(0.233)	(0.307)	(0.284)
Brazil	1.039	0.978	0.943	0.683	0.680
	(0.262)	(0.214)	(0.218)	(0.217)	(0.152)
South Africa	0.894	0.775	0.725	0.761	0.572
	(0.486)	(0.400)	(0.259)	(0.208)	(0.172)
Mexico	1.488	1.104	1.006	0.813	0.687
	(0.435)	(0.452)	(0.288)	(0.247)	(0.235)

7 Discussion and conclusion

This paper had two broad objectives. First, we attempted to model demographic and economic growth volatility using both continuous time Brownian motion and Levy Jump processes. The idea was to study how economic growth volatility responds to the such types of stochasticity in demographic system. Second, we introduced a temporal characteristics of demographic and economic growth system to understand the cross effects of stochasticities in these processes. Our characterisations were meant to shed light on the existence of endogenous/exogenous growth mechanisms under the built systems. We especially emphasized on the past dependence property of demography and economic growth to understand the exact nature of stochasticity. Estimates of stochastic shocks evinced high degree of persistence. We also built a simple growth theoretic framework in line with Solow-Swan and provided conditions of existence of economic growth stability when both demographic and economic growth stochasticities produce intricate evolutionary behavior.

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