Modelling Behavioral Heterogeneity

Gaël Giraud & Isabelle Maret*

Bureau d'Économie Théorique et Appliquée, CNRS-UMR 7522, Université Louis Pasteur, 61, avenue de la Forêt Noire, 67085 Strasbourg Cedex, France (e-mails: giraud@cournot.u-strasbg.fr, maret@cournot.u-strasbg.fr)

September 2002

Abstract: If one wants to get rid of the paradoxes pointed out by Hildenbrand ([19]) and B. de Villemeur ([4]), one needs to reformulate Grandmont's ([16]) notion of behavioral heterogeneity such as to get *exact* insensitivity of the aggregate budget share function with respect to changes in prices and income, instead of a mere approximate insensitivity. Here, we propose a non parametric set-up such that, if the population is distributed according to some "uniform" measure, the aggregate budget share function is constant. This exact insensitivity is not explained by any insensitivity property at the micro-economic level, but rather by a perfect "balancing effect". We then discuss the economic interpretation of some concrete examples illustrating our theory.

Keywords: Aggregation of demand, behavioral heterogeneity, large economy, Law of Demand, Insensitivity of market budget shares.
JEL Classification Numbers: D11, D12, D30, D41, D50, E1

*We thank two anonymous referees, Reinhard John, John Quah and Etienne B. de Villemeur for helpful discussions, and for having given us access to their unpublished work. We are also grateful to Pierre Dehez, Rodolphe Dos Santos Ferreira, Kurt and Werner Hildenbrand, Alois Kneip, Walter Trockel as well as to participants at the CORE-IRES Workshop on Aggregation of Demand 2001 and at the 56th ESEM/EEA Congress for helpful comments. The usual caveat applies.

1. Introduction

One of the main challenges, both of macro-economics and econometrics, is to provide micro-economic foundations for its analysis of aggregate demand. It is well known however that the sole restrictions induced by individual optimization on aggregate excess demands are essentially continuity, zero-homogeneity, Walras' identity and a boundary condition — this is the celebrated Sonnenschein-Mantel-Debreu theorem, see, e.g., [14]. Therefore, if one abstracts the boundary condition, the vector field induced, say on the unit sphere of normalized prices, by aggregate demand is locally arbitrary. As a consequence, the set of equilibrium prices of an exchange economy is, in general, not unique, and, in fact, essentially arbitrary (cf. [29]). More recently, these results have been extended to excess demand functions in economies with incomplete markets ([6],[12] and [32]) and to demand functions both with complete and incomplete markets ([11] and [7]). The main economic lesson of this is essentially negative: individual optimization does not sufficiently restrict the aggregate demand, at least locally, to get sound properties such as the 'Law of Demand', gross substitutability, and thereby uniqueness and stability. Consequently, so goes the story, general equilibrium theory is often viewed as unable to make any observable, predictive statement while one of its favorite exercises — comparative statics — relies on especially vulnerable grounds. As was suggestively expressed by [25], one has to confess that "the emperor has no clothes".[2]

The decisive contribution of Hildenbrand ([21]) brought some hope by showing that certain restrictions on the distribution of income can induce macro-economic properties such as the "Law of Demand", even if these properties are not satisfied at the micro-economic level. Though, as underlined by the author, these restrictions — in particular the fact that the income distribution was assumed to be downward slopping and the collinearity of initial endowments to generate uniqueness and stability under the Walras' tâtonnement of the price equilibrium in exchange economies — are not very realistic, this contribution induced a shift of viewpoint.[3] Hence, an important issue became: What are the properties of the mean demand induced by a large, heterogeneous population of possibly irrational and/or irregular households ?[4] Grandmont ([16]) gave a first, illuminating answer to this question by proving that, within a parametric model of demand functions, sufficiently dispersed demand functions may generate the diagonal dominance of the Jacobian of market demand. Quah ([33]) extended Grandmont's model of heterogeneity to allow for the possibility of atoms; he also showed how a weaker form of demand heterogeneity, together with other assumptions, could lead to the Law of Demand in exchange and production economies. Finally, Kneip [26] extended this approach to a non-parametric setting. The main common idea behind these various frameworks is to show that enough heterogeneity of behavior can explain the insensitivity of the market budget share function to changes in prices and/or income. For this purpose, one considers some well-defined metric on, say, $(\mathcal{W}, \nu)$, the probability space of household budget share functions, and one analyses some distance-preserving transformations T on the space $\mathcal{W}$. The probability measure ν is then said to satisfy a "high degree" of heterogeneity if the probability of all sets A and $T(A) \subset \mathcal{W}$ is extremely close, whenever T is not too far from the identity mapping. The main piece of good news is then that a "highly heterogeneous" population of consumers effectively admits a market budget share function approximately insensitive to changes in prices and/or income.[4]

The problem which emerges, as pointed out by B. de Villemeur ([3],[4]) and Hildenbrand ([19]), is that the precise nature of "behavioral heterogeneity" captured by the aforementioned formalism is not clear. This point may be illustrated as follows. Suppose you parameterize the space of budget share functions of your population by some number in $\mathbb{R}$. (This will be the case, for instance, if one considers homothetic transformations à la Quah ([33]) acting on demand functions as follows: If f is some generating demand function, each agent in the economy has a demand function f_α, for some $\alpha \in \mathbb{R}$ — where $f_\alpha(p, x) := e^\alpha f(p, e^{-\alpha}x)$, when p is a price vector and x an income level.) Assume, furthermore, that the transformations with respect to which you want to test the 'heterogeneity' of the population you face can be reduced to some collection of translations of the parameter $\alpha \in \mathbb{R}$. Claiming that the population is "highly heterogeneous" amounts to assuming that the distribution tends to be invariant with respect to this collection of translations, which means, that in the limit, the distribution of agents should converge to some 'uniform' probability distribution on $\mathbb{R}$. Since, however, there is no such probability distribution on the real line, this implies that the measure towards which the distribution of characteristics converges is a measure, on the completed real line $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$, whose support reduces to $\{+\infty, -\infty\}$). In particular, any compact subset of $\mathbb{R}$ is asymptotically of measure zero.

However particular and simple-minded this example might be, it shows the essence of what is going on. Due to this concentration phenomenon, the approximate insensitivity of market budget share, obtained for a highly heterogeneous population, can hardly be interpreted: Does it emerge from highly heterogeneous reactions of households which compensate one another or, on the contrary, from the insensitivity of almost all households? In fact the two cases emerge in Grandmont's formalism depending on the boundary behavior of the generating demand function. If this behavior is such that the associated budget share function admits limits on the boundaries of the price-income space, then as brought out by [3] and [19] the "uniform" probability puts all its mass on Cobb-Douglas behaviors. In this case Grandmont's formalism can be related to Hildenbrand and Kneip's ([23]) alternative formalization where it turns out that, at each price system, almost all households have budget shares insensitive to changes in prices and income. Note that, in this latter formalization, the population does not necessarily converge towards a population where everybody is Cobb-Douglas as all households might have a non constant budget share function, even at the limit. In this set-up, behavioral heterogeneity is defined in terms of the price set over which the household is sensitive to a change in prices and income, in such a way that at each price system the measure of the set of households sensitive to changes is almost zero. Note that, as shown by [4], the concentration phenomenon might also emerge in [26], while [27] points out that, in Kneip's formalisation, the support of the limit probability distribution over the set of CES budget share functions is restricted to a subset of Cobb-Douglas functions.

The main message of the present paper is the following : We give some conditions over the space $\mathcal{W}$ of individual budget share functions which ensure two important results. First, without any individual rationality assumption, there exist "uniform" distributions over the space $\mathcal{W}$ such that the aggregate budget share function is exactly constant. In other words, for these populations, the market takes on exact Cobb-Douglas properties, although no individual behaviour satisfies even the weakest form of rationality or regularity. When framed in a general equilibrium setting, this result implies the uniqueness and global stability of the equilibrium price. Second, the insensitivity of the market budget share function is not explained by any (even approximate!) insensitivity property at the microeconomic level but rather by a perfect "complementary" or "balancing effect".[5] More precisely, we show that there exists a distribution which has the following much stronger property: every non-empty, open subset of the family is of positive measure. This insures that our 'uniform' measure cannot take its support in, say, a subset of people who would possess, by chance, budget shares insensitive to changes in prices and income, while neglecting the rest of the population. A first step in this direction was made by [27]. The author introduces a new class of distance preserving transformations that ensures that the concentration phenomenon cannot emerge in any orbit induced by a given budget share function. However, since any orbit might have a measure zero, this result does not prevent the concentration phenomenon over the whole space $\mathcal{W}$.

Of course, to assume that the distribution of characteristics of a given population is uniform (in the precise sense given to this term in this paper) is probably heroic. It should be understood as an "ideal limit-case", like the continuum hypothesis in [1]. Our contention is that it proves that behavioral heterogeneity makes sense, even in the limit. On the other hand, by analogy with the core equivalence, the 'large' case should prove to be the limit of the finite setting. But this is exactly the way our proof goes. Indeed, rather than reducing the problem to a fixed-point theorem, we explicitly construct a sequence of probabilities with finite support converging to the 'uniform' measure, and for which the aggregate budget share function is approximately insensitive.[6] Hence, the insensitivity property holds approximately for a finite population sufficiently close to, but distinct from, the perfectly heterogeneous one (see our Corollary 1). Finally, we provide sufficient conditions under which the 'uniform' distribution is unique.

Then, we look at concrete examples that illustrate our theory, and discuss their economic interpretation. Our theory applies, indeed, to a vast class of transformations of individual budget share functions, with respect to which behavioral heterogeneity can be tested. Within this class, a salient subclass is provided by affine transformations, which have been widely used in the literature. We show by means of our Example 1 (and by means of Proposition 2 that, when restricting oneself to affine transformations, and in order to give an account of the insensitivity property of the market share through the balancing effect, one has to assume that any budget share function in the population admits no limit on the boudaries of the price-income space. An open question is then whether this is an economically sound restriction or not. In a second example, we consider a population generated by the class of rotations (instead of affine transformations) ; again, its interpretation might also be subject to some controversy, whose final point is left to the reader.

In a somewhat similar context, [15][7] prove that aggregation has a smoothing effect on the demand behavior in a fashion that looks very much like ours. Interpreting a price as a linear operator on the commodity space, they define an action of the group of normalized prices on individual preferences; the notion of "price-dispersed preferences" is then defined by requiring that the distribution on the functional space of smooth utilities be absolutely continuous with respect to the Haar measure on the group. By comparison, the framework employed in this paper ensures that the aggregate budget share function is constant ; this trivially implies that the market demand is differentiable, but it also says much more.[8] The price to pay, however, is that we cannot content ourselves with the absolute continuity with respect to some 'uniform' distribution: we need the distribution of households' characteristics itself to be (approximately) 'uniform' in a certain sense.

In the next section, we set the framework and state our results. We shall be careful when relating our hypotheses to the usual understanding of a "large", "dispersed" and "heterogeneous" population. Our two main examples are discussed in section 3. Finally, section 4 contains the proofs.

2. Towards insensitive aggregate budget shares

2.1. The problem

Consider[9] an economy with $L \geq 1$ commodities. Each household is characterized by a demand function f:

$$f : \mathbb{R}^L_{++} \times \mathbb{R}_{++} \to \mathbb{R}^L_{+}, \tag{1}$$

which associates to each pair (p, x) of prices and income, a point in the consumption set. As convincingly argued by [26], it is more convenient to work with the corresponding budget share function $w : \mathbb{R}^L_{++} \times \mathbb{R}_{++} \to [0, \gamma]^L$ where $\gamma > 0$, defined by:

$$\forall (p, x) \quad w(p, x) = \frac{p \quad f(p, x)}{x}. \tag{2}$$

We consider a subpopulation of households with identical income. Households differ in their budget share functions, hence in their characteristics affecting demand independently of prices and income. Let denote by $\mathcal{W}$ the space of budget share functions of the economy at hand, endowed with the sup-norm $\| \cdot \|_{\infty}$. The joint distribution of households' characteristics induces a distribution ν of budget share functions on $\mathcal{W}$. The assumption that all households have the same income is common to all the previous literature, and could be relaxed. Indeed, one easily sees that the properties obtained below for the aggregate budget share of a given subpopulation are preserved through aggregation. Hence, subsequent analysis could apply to suitable sub-economies populated by individuals with identical incomes.

The aggregation problem consists in asking whether there exists certain restrictions on $\mathcal{W}$ and a Borel probability distribution ν such that certain properties (e.g., the Law of Demand) are fulfilled by the aggregate budget share function

$$(p, x) \mapsto W(p, x) := \int_{\mathcal{W}} w(p, x)\nu(dw). \tag{3}$$

In other words, we want to take the space $\mathcal{W}$ itself as given, provided it belongs to a convenient class of functional spaces, and to prove that an adequate choice of the distribution of households' characteristics, which can be interpreted as representing a perfectly heterogeneous population, can induce per se economically sound properties on the macro-economic level. In this sense, we view the approach taken in this paper as quite distinct from the one adopted, e.g., by [5]. There, it is argued, loosely speaking, that, given a budget share function it is always possible to construct a complementary one such that their sum satisfies the Law of Demand.

The celebrated "Law of Demand" can be expressed in terms of the aggregate budget share function:

$$\forall p, q \in \mathbb{R}^L_{++}, \quad (p - q) \cdot \Big(p^{-1} \quad W(p, x) - q^{-1} \quad W(q, x)\Big) \square\, 0. \tag{4}$$

What kind of behavior can be expected from the aggregate budget share function of a large, heterogeneous population ? The most demanding property is certainly the insensitivity of the map W with respect to changes in prices and/or income. This property (which is the Cobb-Douglas functions' benchmark) induces, indeed, most of the properties one could dream of: the Law of Demand (since the above inequality clearly holds when W is constant), but also the gross substitutability property, and eventually the uniqueness and global stability (for the Walrasian tâtonnement) of the equilibrium of a pure exchange economy.

2.2. The results

In order to formally define heterogeneity of households with respect to a 'perturbation' of the price-income vector, the literature has focused so far on the class of affine transformations on the functional space $\mathcal{W}$ (see, for example, [16] and [26]). However, this is just one of many possible classes that can be used. We shall consider a broader class $\mathcal{T}$ of transformations T on $\mathcal{W}$ defined by the two following conditions:

A) The map $w \mapsto T[w]$ is an isometry over $\mathcal{W}$.

B) Every function $W : \mathbb{R}^L_{++} \times \mathbb{R}_{++} \to \mathbb{R}^L_+$, which is invariant with respect to every transformation $T \in \mathcal{T}$, is constant over $\mathbb{R}^{L+1}_{++}$.

The class of <u>affine</u> transformations T_Δ is an example of transformations verifying A) and B). They are defined by:

$$\forall w \in \mathcal{W}, \forall \Delta \in \mathbb{R}^{L+1}_{++}, \forall (p,x) \in \mathbb{R}^L_{++} \times \mathbb{R}_{++} \quad T_\Delta[w](p,x) = w(\Delta \quad (p,x)). \quad (5)$$

Or, one could also use, as in [33], a subclass of the class of affine transformations, namely the class of <u>homothetic</u> transformations. Note, that what makes affine transformations special is their preservation of the possible rationality properties. It is straightforward to check that if a function w defined on $\mathbb{R}^L_{++} \times \mathbb{R}_{++}$ satisfies the weak axiom, then so will its transformation $T_\Delta[w]$, and if w is generated by the utility function $u(\cdot)$, then its transformation is generated by the utility function $u_\Delta = u(\Delta^{-1} \quad \cdot)$. This shows incidentally that it is possible, in the case of affine transformations, to formulate all the assumptions put on the record in this paper on the (more fundamental ?) level of individual preferences, rather than on demand functions.

We shall make the following assumption (which makes precise the space $\mathcal{W}$ on which our result applies):

Assumption 1:

(i) The space $\mathcal{W}$ of admissible budget share functions is a subset of the set of all functions from $\mathbb{R}^L_{++} \times \mathbb{R}_{++}$ to $[0,\gamma]^L$ where $\gamma > 0$.

(ii) The normed subspace $(\mathcal{W}, \|\cdot\|_\infty)$ is compact.

(iii) $\mathcal{W}$ is large enough to satisfy:

$$\forall T \in \mathcal{T}, \forall w \in \mathcal{W}, \quad T[w] \in \mathcal{W}.$$

Compactness is the topological analogue of finiteness, and was already assumed by [13, p. 17], $\mathcal{M}_{\text{com}}$. It can be thought of as arising from the continuity of some mapping that associates to each individual in, say, the real interval $[0,1]$ her budget share function. In other words, in a parametric setting, all we need is that the parameter set describing the set of feasible budget share functions be compact (see examples below). Assumption (iii) corresponds to Assumption 1(2) in [26]. It requires the set of budget share functions to be large enough (unless everybody is Cobb-Douglas) in order to remain stable by perturbations on prices and/or income. In particular, it prevents the set $\mathcal{W}$ from being finite, and we think of it as playing a role similar to the atomless hypothesis for "large" economies (see [20]).

We shall formalize a perfectly heterogeneous population (in terms of households reactions to changes in prices and income) by an invariant measure with respect to every transformation $T \in \mathcal{T}$. The following theorem establishes that this measure exists.

Theorem 1 *Under Assumption 1, there exists a (Borel-) probability measure* λ *on* $\mathcal{W}$ *such that the aggregate budget share function* W *is constant over* $\mathbb{R}^L_{++} \times \mathbb{R}_{++}$.

Note that for a constant function W over $\mathbb{R}^L_{++} \times \mathbb{R}_{++}$, one has: $\forall p, q \in \mathbb{R}^L_{++}$,

$$(p-q) \cdot \left(p^{-1} \quad W(p,x) - q^{-1} \quad W(q,x)\right) = -\sum_{l=1}^{L} \frac{(p_l - q_l)^2}{p_l q_l} W_l(p,x) \square 0. \quad (6)$$

Hence, from (4) we deduce that the Law of Demand holds in the aggregate. It is important to observe that, in contrast with [16] and [26], the Law of Demand is deduced here from the insensitivity property without any strong desirability requirement of any commodity. In particular, for any commodity, nothing requires the market budget share to be strictly positive for all prices.

The following corollary extends Theorem 1 to a finite population not too far away from a perfectly heterogeneous population.

Corollary 1 *Suppose that Assumption 1 is in force. For any* $\varepsilon > 0$*, there exists a probability distribution with finite support* ν *such that, for any* $T \in \mathcal{T}$ *and any* $l \in \{1, \dots, L\}$*:*

$$\left| \int_{\mathcal{W}} T[w_l]((p,x))\nu(dw) - \int_{\mathcal{W}} w_l(p,x)\nu(dw) \right| \square \varepsilon \quad (p,x) \in \mathbb{R}^L_{++} \times \mathbb{R}_{++} \quad (7)$$

It is obvious that, if transformations in $\mathcal{T}$ are such that, given some point (p,x), there is a neighborhood V of (p,x) such that every element $(q,y) \in V$ can be viewed as $(q,y) = T(p,x)$ for some $T \in \mathcal{T}$, then Corollary 1 means that ν is approximately constant. This is the case, for instance, if $\mathcal{T}$ contains the class of affine transformations (as in Example 1) or the class of rotations (as in Example 2 under the no-money-illusion hypothesis). This approximate insensitivity of the market share should not be confused with the approximate results available in the literature. There, indeed, atomless economies are shown to have approximately insensitive market shares, and nothing is said about their finite approximations. Here, every finite economy which is approximately heterogeneous exhibits an approximately constant market share, and converges, as the number of agents grows to infinity, towards a large economy that turns out to be perfectly heterogeneous.

If we further introduce the desirability requirement that for any given compact price set K, $W(p) > 0$, $\forall p \in K$, then we can deduce from Corollary 1 that the Law of Demand holds in K.[10] It is important to observe that, in contrast with

[16] and [26], the latter assumption is not required for all prices but only for prices in K.

Notice that we nowhere assume that the budget share functions are continuous or homogeneous or that each individual budget constraint is satisfied.[11] Nor need the weak axiom of revealed preferences (WARP) be satisfied at any level[12] or the aggregate budget share function W be differentiable. This shows that extreme diversification of possibly extremely irregular and irrational characteristics may, on its own, generate an extremely regular mean outcome.

Is a measure like λ always immune against the criticism addressed by [3] and [4], [19] and [27] to the Grandmont-Quah-Kneip approach ? To make this point, it will suffice to show that any non-empty open subset of $\mathcal{W}$ is non-negligible.[13] For this purpose, the following additional assumption will fit the bill:

Assumption 2

For any pair $(v, w) \in \mathcal{W}^2, \quad \exists\, T \in \mathcal{T} \quad / \quad w = T[v]$.

Assumption 2 means that we restrict ourselves to the type of heterogeneity generated by the transformations T: it is possible to go from one's budget share function to another by composing transformations T. This requirement implies, as in [16], that all the individual budget share functions can be generated from a unique, fundamental one (the generator) by the class of transformations. We stress that this hypothesis is not needed for Theorem 1 to hold, hence to get the insensitivity of the aggregate budget share function. On the other hand, notice that, if the generator is distinct from a Cobb-Douglas function, so are all the individual budget share functions of the economy.

Proposition 1 *Under Assumptions 1 and 2, the measure λ satisfies:*

$$\lambda(O) > 0 \quad \forall O \text{ non-empty, open subset of } \mathcal{W}.$$

By forbidding the concentration of the measure λ over any closed proper subset of $\mathcal{W}$, Assumptions 1 and 2 truly impose the behavioral heterogeneity we are looking for in this paper .

Proposition 1 prompts the question as to whether there is a unique way for the space of budget share functions $\mathcal{W}$ to be perfectly heterogeneously distributed. The next result provides sufficient conditions on $\mathcal{W}$ for the measure λ to be unique. One could view it alternatively as 1) showing that 'behavioral heterogeneity' is defined in a non-ambiguous way; 2) suggesting that being heterogeneous is a rather exceptional property for a population. In this context, it should be noted, however, that most of the micro-economic foundations of macro-economics we have in mind when dealing with the aggregation problem, as well as most of the econometric inquiries, do not need the population to be perfectly heterogeneous. It usually suffices that it be sufficiently close to a 'uniform' distribution such as the one exhibited in the two preceding results. On the other hand, even in the atomless, perfectly heterogeneous case, the measure λ is unique given some class $\mathcal{T}$ of transformations. Changing this class would also change λ, so that the next result only shows a conditional uniqueness. With this in mind, 1) is probably the most relevant standpoint.

Assumption 3

(i) For any pair $T, T' \in \mathcal{T}$, if $T[w] = T'[w]\ \forall w$, then $T = T'$.

(ii) For any sequence (T_n) in $(\mathcal{T})^{\mathbb{N}}$, and any continuous map $f : \mathcal{W} \to \mathcal{W}$, if $T_n[w]$ converges to $f(w)$ uniformly on $\mathcal{W}$, then $\exists\ T \in \mathcal{T}$, such that $f(w) = T[w],\ \forall w \in \mathcal{W}$.

Assumption 3 (i) suggests, roughly, that, for any price-income vector (p, x), whatever being the direction in which it is perturbed, there exist two consumers who react differently to this perturbation. Assumption 3 (ii) is a technical, closedness requirement strengthening the compactness hypothesis 1(ii).

Theorem 2 *Under Assumptions 1 to 3, the 'uniform' probability distribution λ alluded to in Theorem 1 is unique.*

3. Examples and interpretation

In this section, we exhibit examples illustrating the theory outlined in the previous section, and discuss their interpretation.

3.1 Example 1.

In the spirit of Grandmont's ([16]) seminal construction, our population is a collection of functions $\{w_\alpha\}$ where for $\alpha \in \mathbb{R}^{L+1}$, w_α is defined by:

$$w_\alpha(p, x) = T_{e^\alpha}[\overline{w}](p, x) := \overline{w}\left(e^{\alpha_1}p_1, e^{\alpha_2}p_2, ..., e^{\alpha_{L+1}}x\right). \tag{8}$$

for all $(p, x) \in \mathbb{R}^L_{++} \times \mathbb{R}_{++}$, with $\overline{w}$ a continuous, non-constant function over $\mathbb{R}^L_{++} \times \mathbb{R}_{++}$, called the generator. Notice that here, in contrast to [16], the distribution of the parameters α in the population is not assumed to admit a density function and a fortiori a flat density function.

In order to guarantee the compactness of $\mathcal{W}$, we shall "compactify" the parameter set $\mathbb{R}^{L+1}$. This is done by assuming that $\overline{w}$ is completely described by its behavior over some compact subset of $\mathbb{R}^L_{++} \times \mathbb{R}_{++}$. For this purpose, we introduce a second function $\tilde{w}$ from $\mathbb{R}^{L+1}$ into $\mathbb{R}^L_+$, defined by

$$\tilde{w}(t_1, t_2, ..., t_{L+1}) = \overline{w}(e^{t_1}, ..., e^{t_{L+1}}) \tag{9}$$

where $t = (t_1, t_2, ..., t_{L+1}) \in \mathbb{R}^{L+1}$. Let us begin with the simplest case, where heterogeneity of the households' share functions is required only with respect to one argument of the budget share function. This is the case, if for example, following [33], heterogeneity is to be required with respect to changes in income only. It implies that we can focus on the subset of affine transformations such that $\alpha_i = 0$ for i □ L. We are thus concerned with the behavior of $\overline{w}$ over the space, $\mathbb{R}_{++}$, of positive income t_{L+1}. We simply make a periodicity requirement:

$$\tilde{w}(t_{L+1} + nc) = \tilde{w}(t_{L+1}), \tag{10}$$

for some $c > 0$ and all $n \in \mathbb{N}$. Under this assumption, the extreme points k and $k + c$ can be identified since $\tilde{w}$ assumes the same value for both. Hence, $\tilde{w}$ can be equivalently described by its behavior over the (compact) circle $\mathbb{R}/c\mathbb{Z}$ made of equivalence classes $\dot{t}$ of points $t \in \mathbb{R}$ modulo $c\mathbb{Z}$. More precisely, we shall look at the quotient function $\mathbf{w} : \mathbb{R}/c\mathbb{Z} \to [0, \gamma]$ defined by: $\mathbf{w}(\dot{t}) := \tilde{w}(t)$ for every t in the class $\dot{t}$. The set of feasible budget share functions, $\mathcal{W}$, is the uniform closure of the collection of functions $\{w_\alpha\}_{\alpha \in \mathbb{R}/c\mathbb{Z}}$. For any $w_\alpha \in \mathcal{W}$, we denote by $\mathbf{w}_\alpha$ the corresponding quotient map, defined by $\mathbf{w}_\alpha(t_{L+1}) = \mathbf{w}(t_{L+1} + \alpha)$. It follows from (9) and (10) that $\|\mathbf{w}_\alpha - \mathbf{w}_\beta\|_\infty = \|w_\alpha - w_\beta\|_\infty$. In order to prove that $\mathcal{W}$ satisfies our assumptions, it therefore suffices to prove that $\mathbf{W}$, the uniform closure of the set $\{\mathbf{w}_\alpha |\ \alpha \in \mathbb{R}/c\mathbb{Z}\}$, is compact. Since the domain of any function in this latter set is compact, one can apply Ascoli's theorem and prove that it is relatively compact for the uniform topology. Taking its uniform closure yields compactness. Thus, according to Theorem 1, we conclude that $\mathcal{W}$ admits a probability measure with respect to which the market budget share function is constant. In order to see that our Assumption 2 is also satisfied, just observe that the set of translation parameters $\alpha \in \mathbb{R}/c\mathbb{Z}$ itself is compact. Hence, if w belongs to the closure of $\mathcal{W}$, it must be the limit of some sequence $T_{\Delta_n}[\tilde{w}]$, with $\Delta_n = e^{\alpha_n}$. It suffices to take the limit Δ^* of some subsequence of $(\Delta_n)_n$ to see that $w = T_{\Delta^*}[\tilde{w}]$. Hence, Proposition 1 holds. Finally, Assumption 3 is immediately satisfied if one adopts as space of translations the quotiented space $\mathbb{R}/c\mathbb{Z}$. As a consequence, Theorem 2 is verified.

For readers who are not familiar with quotiented spaces, subsection 4.6 gives an elementary analysis of a subcase of Example 1, obtained by taking $\tilde{w}(t_{L+1}) = 1 + \sin t_{L+1}$.

This fairly simple example is analogous to, and can be compared with, the cases 2 and 3 in [35]. There, the population is described by a density function of the parameters α_{L+1} defined on $\mathbb{R}$. In this formalism, a heterogeneous population is described by a flat density function over $\mathbb{R}$. As the density function becomes flatter, however, it has to be spreading to the left or to the right. This means that the values of α_{L+1} that predominate are those that are very small or very high. As already underlined by [19], this means that insensitivity at the aggregate might emerge because it is already satisfied at the individual level. Ruling out this trivial situation implies that for a fixed p, $\overline{w}(p, x)$ must have no limit as income goes to zero or infinity. This is also the case in our example just described. The difference between our approach and the one exemplified in [35], and followed by all the previous literature is that we can assert the existence of a perfectly heterogeneous probability measure, while in the already mentioned previous work, the exhibited distribution only induces some approximately heterogeneous measure. Moreover, thanks to our Corollary 1, we can assert that a perfectly heterogeneous population can be approximated by a finite one, that is almost perfectly heterogeneous.

In order to extend the previous example to more sophisticated situations where $\tilde{w}$ depend upon more than one parameter, it suffices to consider the following periodicity requirement:

$$\tilde{w}(t + nc_i) = \tilde{w}(t) \quad \forall i = 1, ..., L+1, \tag{11}$$

where $n \in \mathbb{Z}$, $c_i = \rho_i e_i \in \mathbb{R}^{L+1}_{++}$ with $\rho_i \in \mathbb{R}_{++}$ and $e_i = (0, ..., 1, 0...)$ denotes the i^{th} element of the canonical basis of $\mathbb{R}^{L+1}$ $\forall i$. Denote by $\mathcal{C}$ the compact set $\{t \in \mathbb{R}^{L+1} \quad / \quad t_i \in [0, \rho_i] \quad \forall i = 1, ..., L+1\}$ and by $\mathcal{K}$ the compact set $\{(p, x) \in \mathbb{R}^L_{++} \times \mathbb{R}_{++} \quad / \quad (p_1, ..., p_L, x) = (e^{t_1}, ..., e^{t_L}, e^x), \quad t \in \mathcal{C}\}$. Again, the function $\tilde{w}$ is entirely described by its behavior over $\mathcal{C}$ where, in addition, pieces of the boundary $\partial\mathcal{C}$ can be identified by taking the suitable quotients (which are topologically equivalent to tori). Details are left to the reader.

It is not difficult to see that one consequence of the periodicity requirement (11) is that the individual budget share functions w_i must have some "erratic" behavior near the boundary of the domain of its budget share function: w wiggles as income or some price tends to zero or infinity. This is, in fact, a general property of affine transformations:[14]

Proposition 2 *Suppose that the space $\mathcal{W}$ satisfies Assumption 1, and that the class $\mathcal{T}$ consists of affine transformations. Then, for every $w \in \mathcal{W}$, $(\overline{p}, \overline{x}) \in \mathbb{R}^{L+1}_{++}$ and $i, j \in \{1, ..., L\}$,*

(i) either $w_j(\overline{p}, \cdot)$ is constant or the limits $\lim_{x \to 0} w_j(\overline{p}, x)$ and $\lim_{x \to \infty} w_j(\overline{p}, x)$ do not exist;

(ii) either $w_j(\overline{p}_1, ..., \overline{p}_{i-1}, p_i, \overline{p}_{i+1}, ..., \overline{x})$ is constant or the limits $\lim_{p_i \to 0} w_j(\overline{p}_1, ..., \overline{p}_{i-1}, p_i, \overline{p}_{i+1}, ..., \overline{x})$ and $\lim_{p_i \to \infty} w_j(\overline{p}_1, ..., \overline{p}_{i-1}, \cdot, \overline{p}_{i+1}, ..., \overline{x})$ do not exist.

The problem even occurs for more general classes of transformations. Suppose Assumption 2 is in force. In addition, assume that $\overline{w}$ is "nice", in the sense that the partial derivatives $\frac{\partial \overline{w}_j}{\partial p_j}$ or $\frac{\partial \overline{w}_j}{\partial x}$ exist and have constant sign. If the transformations in $\mathcal{T}$ have the property of preserving the sign of these derivatives (let us qualify such transformations sign-preserving transformations), the corresponding derivatives of the aggregate budget share function have the same sign. Consequently, they can never be equal to zero as intended by our theory. Following this line of thought, there are strong reasons to believe that the main theory of behavioral heterogeneity rests on vulnerable grounds, as it is in fact hard to interpret it in some economically convincing way.

An alternate standpoint is nevertheless conceivable. As outlined by [35], for many purposes (in particular for uniqueness and global stability of the price equilibrium under the standard boundary condition of individual preferences) we are only interested in the behavior of market demand on a proper compact subset, K, of prices and income. Hence, it is enough to require that the household described by α has a budget share function which coincides with w_α only on K. In this case, the periodicity requirement of Example 1 is not restrictive as long as $\mathcal{K}$ is chosen such that $K \subset \mathcal{K}$. In particular, our example does not require any "periodicity" of the generator over K, but only outside this relevant compact set. Furthermore, every function w^α in the population is of bounded variations over K. More generally, following this second line of thought, the theory of behavioral heterogeneity is satisfactory in the sense that one need not take into account individual behaviors near the boundary of price and/or income domains. (Similarly, this viewpoint insists on saying that individual budget share functions need not be "nice" — in the above sense — in order to make sense, nor the transformations in $\mathcal{T}$ need to be sign-preserving transformations.)

3.2. Example 2.

As shown in the previous subsection, "erratic" individual behavior are needed near the boundary of the price/income domain whenever heterogeneity is checked against sign-preserving transformations. We shall now build an example where behavioral heterogeneity of households with respect to a perturbation of the price-income vector is defined with the use of the (non-sign preserving) class of <u>rotations</u> of the price-income vector. This time, Assumption 1 can be fulfilled without imposing any erratic individual behavior. However, another stumbling block is to be encountered in terms of interpretation, as we now show.

To simplify the presentation we consider an economy with two commodities and we focus on heterogeneity of the households' share functions with respect to the price vector. All households possess the same income level, $x > 0$. Denote again by $\overline{w}$ the generator of the population. Suppose that that we are concern with the behavior of the population on the compact set of prices and income, K. We assume that $\overline{w}$ is a non constant function continuous and homogeneous of degree zero in (p, x) over K. Hence, our main assumption at the household level is the absence of money illusion. We furthermore extend $\overline{w}$ by continuity outside K in such a way that:

$$\overline{w}(0, 1, x) = \overline{w}(1, 0, x). \tag{12}$$

The set of prices (p_1, p_2) can be identified with the non negative orthant of the complex plan:

$$\mathbb{C}_+ := \{z_p = p_1 + ip_2 \in \mathbb{C} \;:\; (p_1, p_2) \in \mathbb{R}^2_+\}. \tag{13}$$

Moreover, thanks to the homogeneity assumption and the assumption that households are heterogeneous only in terms of their reaction to a change in prices, prices can be normalized so that the price space can be identified to $\mathbb{U}_+ := \mathbb{U} \cap \mathbb{C}_+$, where $\mathbb{U} := \{z \in \mathbb{C} : |z| = 1\}$. To define our population, we introduce a second function $\tilde{w}$ defined over $\mathbb{U} \times \mathbb{R}_{++}$ by

$$\tilde{w}(z_p, x) = \overline{w}(p_1, p_2, x). \tag{15}$$

The population is a collection of functions $\{w_\theta\}$ where for $\theta \in \mathbb{U}$, w_θ is defined by:

$$w_\theta(p, x) = T_\theta[\overline{w}](p, x) := \overline{w}\big(\theta z_p, x\big), \tag{14}$$

where $\theta \in \mathbb{U}$ is some unitary complex number (hence essentially acts as a rotation). We shall now specify the set of parameters θ. From (12) we deduce that for all $n \in \mathbb{Z}$, one has

$$\tilde{w}(e^{in\frac{\Pi}{2}} z_p, x) = \tilde{w}(z_p, x). \tag{15}$$

We can therefore consider the equivalence relation

$$z \sim z' \Leftrightarrow \exists n \in \mathbb{Z} \quad / \quad z' = i^n z. \tag{16}$$

The quotiented space $\mathbb{U}_+/\sim$ is denoted Π. Let us define the set of feasible budget share functions $\mathcal{W}$ by (the uniform closure of) the collection of functions $\{w_\theta\}_{\theta \in \Pi}$. Since the price space $\mathbb{U}_+$ is compact, one can apply Ascoli's Theorem and prove that the family of such transformed budget share functions is relatively compact for the uniform topology. Taking its uniform closure yields compactness. Thus, according to Theorem 1, we can conclude that $\mathcal{W}$ admits a probability measure with respect to which the market budget share function is constant. In addition, thanks to the compactness of the set of rotation parameters, Π, one can prove along the same lines as in the previous example that Assumptions 2 and 3 hold.

As in in the previous subsection, this second Example leaves us with an interpretational issue. (12) requires indeed a specific household behavior on the boundary of the price space, namely that budget shares be identical whether the price of the first commodity or the price of the second commodity is equal to zero. On the one hand, this can be viewed as unrealistic. On the other, whatever being the household behavior over K the budget share function can always be extended by continuity outside K in order to fulfill (12). Hence, as long as one is only interested in the behavior of market demand on the compact price set K, assumptions made at the household level in this example are rather weak. The major requirement is that households are not victims of money illusion — an assumption that is commonly made in demand theory.

Remarks.

• According to the angle of attack adopted in this paper, three ingredients drive the insensitivity of aggregate budget share: (1) one needs a "large" population (this is Assumption 1(iii)), (2) whose characteristics are in a compact set (Assumptions 1(ii)) and (3) are uniformly distributed (this is the G-invariance of λ). For this uniformity requirement to describe a perfectly heterogeneous population of households, we need the "specific type of heterogeneity" requirement (Assumption 2, Proposition 1). Finally, for it to be unambiguously defined, we need an additional assumption (Assumption 3(i)).

• In this paper, the 'uniform distribution' describes a population of households that are heterogeneous in terms of their reaction to changes in prices and income. Note, however, that under the additional requirement that every individual budget share function is homogeneous of degree zero in (p, x) and satisfies the budget identity, one easily sees that Theorem 1 remains valid for the set of transformations T_Δ on the space $\mathcal{W}$ defined by perturbations of the price space solely, i.e. for the class of affine transformations,

$$\forall w \in \mathcal{W}, \forall \Delta \in \mathbb{R}^L_{++}, \forall (p, x) \in \mathbb{R}^L_{++} \times \mathbb{R}_{++} \quad T_\Delta[w](p, x) = w(\Delta \quad p, x). \tag{17}$$

• Suppose that income x at price p is defined by:

$$x := p \cdot \omega \tag{18}$$

where $\omega \in \mathbb{R}^L_+$ is the initial endowment in commodities of any household in our subpopulation. It is a routine matter to prove that, if the aggregate budget share function is insensitive to changes in prices and income, as it follows from our Theorem 1, then market demand F satisfies the gross substitutability property. Indeed, consider two price systems p and q such that $q_l > p_l$ and $q_k = p_k$ for $k \neq l$. Denote by $F_k(p, p \cdot \omega)$ the market demand for commodity k at the price system p. The insensitivity property implies that

$$\frac{p_k F_k(p, p \cdot \omega)}{p \cdot \omega} = \frac{q_k F_k(q, p' \cdot \omega)}{q \cdot \omega}, \tag{19}$$

where by assumption $p \cdot \omega < q \cdot \omega$. Hence, for any pair $(p, q) \in (\mathbb{R}^L_{++})^2$ if $q_l > p_l$ and $q_k = p_k$ for $k \neq l$ one has

$$F_k(q, q \cdot \omega) > F_k(p, p \cdot \omega). \tag{20}$$

Thus, there is a unique equilibrium price, which is moreover globally stable in any standard tâtonnement process. Similarly, it is easy to prove that, if the aggregate budget share function is approximately insensitive to changes in prices and income, as it follows from our Corollary 1, then, for any given compact price set K and ε small enough, market demand F satisfies the gross substitutability property on K. As already said, under some standard, addititional assumption which guarantees that no equilibrium price exists outside the compact set of prices K, this again ensures uniqueness and global stability of the price equilibrium. Observe moreover, that nothing prevents from interpreting the collection $\{1, ..., L\}$ of "commodities" as composed of consumption goods and securities, possibly within an incomplete markets setting.

• In [24], the celebrated example of [2] is formalized by a "uniform" distribution over the space of individual characteristics which induces the insensitivity of the aggregate budget share function. The mathematical structure of this example is essentially the following: Consider the set of budget share functions $\mathcal{W} := \Sigma^{\mathbb{R}_{++}^{(L+1)}}$ as an uncountable product over the unit-simplex $\Sigma := \{x \in \mathbb{R}_+^L : \sum_i x_i = 1\}$. Equip this space with the product topology. Thanks to Tychonov's theorem, $\mathcal{W}$ is compact. Kolmogorov's extension theorem insures furthermore that the infinite product of the Lebesgue measure $\lambda^{\mathbb{R}_{++}^{(L+1)}}$ is well-defined. John [24] proves that a population whose budget share functions are distributed according to the "uniform" probability $\lambda^{\mathbb{R}_{++}^{(L+1)}}$ has a market demand of the symmetric Cobb-Douglas type. The strength of this example is that, like in the general theory developed in this paper, no continuity assumption on the budget share functions is required. Since $\mathbb{R}_{++}^{(L+1)}$ is non-countable, however, the product topology is not metrizable, so that Theorem 1 does not apply to this setting. Moreover, it is difficult to think of any analogue of our Proposition 1 within John's framework.

4. Proofs of the results

Some preliminary remarks: By assumption each transformation $T \in \mathcal{T}$ is distance-preserving on the space $(\mathcal{W}, d)$, one-to-one and onto (see [26] for details in the specific case of affine transformations), where d denotes the distance induced by the sup-norm. Consider therefore the (Abelian) group G spanned by the transformations T. A generic element $g \in G$ is defined by:

$$g = T_1 \circ ... \circ T_N \quad T_i \in \mathcal{T}, \forall i. \tag{21}$$

By assumption 1 (ii), $\mathcal{W}$ is stable by any transformation T, thus, it must also be stable by the operation of the group G. Moreover, G operates isometrically on $\mathcal{W}$. Otherwise stated:

$$\forall g \in G, w, v \in \mathcal{W},\ gw \in W \text{ and } d(gw, gv) = d(w, v), \qquad (22).$$

4.1. Proof of Theorem 1

Since $\mathcal{W}$ is pre-compact with respect to d, for any $\varepsilon > 0$, there exists at least one finite subset $R(\varepsilon)$ of $\mathcal{W}$, such that, for any $w \in \mathcal{W}$, $\inf\{d(w, r) : r \in R(\varepsilon)\}$ □ ε. Let call $R(\varepsilon)$ a ε-network, and denote by $N(\varepsilon)$ the minimal cardinality of such ε-networks.

Claim. Let $\varepsilon > 0$, and R and R' two ε-networks of $\mathcal{W}$, of minimal cardinality $N(\varepsilon)$. There exists a bijection $\psi : R \to R'$, such that:

$$d(w, \psi(w)) \leq 2\varepsilon \quad \forall w \in \mathcal{W}. \tag{23}$$

To prove this claim, take $w \in R$, and consider the following set $A_w \subset R'$ of elements of R' which are "closely related" to w:

$$A_w = \big\{ v \in R' \ : \ B(w, \varepsilon) \cap B(v, \varepsilon) \neq \emptyset \big\}. \tag{24}$$

Take, now, any subset $I \subset R$, and consider the set R'' obtained by replacing every element from I by the family of its "close" points:

$$R'' := \big(R \setminus I\big) \cup \big(\cup_{w \in I} A_w\big). \tag{25}$$

It is not difficult to see that R'' is still an ε-network of $\mathcal{W}$. Indeed, for any $x \in \mathcal{W}$, there exists some $w \in R$. If $w \notin I$, we are done. Otherwise, there must also exist some $v \in R'$ such that $d(v, x) \leq \varepsilon$. Hence $v \in A_w$, which implies that $v \in R''$.

Before going further, let us first recall the following well-known "wedding lemma" (see, for example, [17]):[15]

Lemma 1 *Let Y be a nonempty set, n some integer ≥ 1 and $A_1, ..., A_n$ be finite subsets of Y such that:*

$$\forall I \subset \{1, ..., n\}, \quad \#\big(\cup_{i \in I} A_i\big) \geq \#I. \tag{26}$$

Then, there exists a one-to-one mapping from I to $\prod_i A_i$.

In order to apply our wedding lemma, we need to verify:

$$\#R \leq \#R'' \leq \#(R \setminus I) + \#\big(\cup_{w \in I} A_w\big) \tag{27}$$

$$= \#R - \#I + \#\big(\cup_{w \in I} A_w\big). \tag{28}$$

This implies that $\#\big(\cup_{w \in I} A_w\big) \geq \#I$. Hence, there exists some one-to-one mapping $\psi : R \to \cup_{w \in R} A_w \subset R'$ such that $\psi(w) \in A_w$, $\forall w \in R$. Since $\#R = \#R'$, ψ is also onto. The inequality announced in the claim follows from the triangle inequality of the distance d.

In order to prove the theorem, take any sequence $(\varepsilon_n)_n$ of positive real numbers $\varepsilon_n \leq \varepsilon$ converging to 0 and, for every n, an ε_n-network R_n, of minimal cardinality $N(\varepsilon_n) = N_n$. Let us denote by:

$$\lambda_n := \frac{1}{n} \sum_{w \in R_n} \delta_w \tag{29}$$

the uniform probability measure over R_n. For any element $g \in G$, the finite network $R'_n := gR_n$ is still an ε-network of $\mathcal{W}$. Indeed, if $w \in \mathcal{W}$ and $x \in R_n$ such that $d(g^{-1}w, x) \leq \varepsilon_n$, one has:

$$d(w, gx) = d(g^{-1}w, x) \leq \varepsilon_n \leq \varepsilon. \tag{30}$$

Take any bijection ψ as in the preceding claim, any function $F \in \mathcal{C}^0(\mathcal{W})$, and denote:

$$\alpha_n := \sup\big\{|F(w) - F(v)|,\ w, v \in \mathcal{W}\ /\ d(w, v) \leq 2\varepsilon_n\big\}. \tag{31}$$

We have:

$$\int_{\mathcal{W}} F(gw)\lambda_n(dw) - \int_{\mathcal{W}} F(w)\lambda_n(dw) = \frac{1}{N_n}\Big[\sum_{w\in R_n} F(gw) - \sum_{w\in R_n} F(w)\Big] \tag{32}$$

$$= \frac{1}{N_n}\Big[\sum_{w\in R'_n} F(w) - \sum_{w\in R_n} F(w)\Big] = \frac{1}{N_n}\Big[\sum_{w\in R_n} (F(\psi(w) - F(w))\Big]. \tag{33}$$

It follows that:

$$|\int_{\mathcal{W}} F(gw)\lambda_n(dw) - \int_{\mathcal{W}} F(w)\lambda_n(dw)| = \frac{1}{N_n}\sum_{w\in R_n} |F(\psi(w) - F(w)| \leq \alpha_n. \tag{34}$$

Banach-Alaoglu's theorem implies that the sequence $(\lambda_n)_n$ of probability measures admits a subsequence that converges for the weak-$*$ topology to some probability measure, say, λ. On the other hand, since $\mathcal{W}$ is $\sigma(L_\infty, L_1)$-compact, F is uniformly continuous, so that $\alpha_n \to 0$ as n grows to infinity. Moreover, the mapping $w \mapsto F \circ g(w)$ is $\sigma(L_\infty, L_1)$-continuous. Hence, (34) yields, by passing to the limit:

$$\int_{\mathcal{W}} F(gw)\lambda(dw) = \int_{\mathcal{W}} F(w)\lambda(dw), \quad \forall g \in G \qquad (35).$$

In order to conclude the proof of the Theorem, consider the application $F_{p,x} : w \mapsto w(p, x)$. $F_{p,x} : \big(\mathcal{W}, d\big) \to \mathbb{R}$ is continuous. Thus, the preceding equality yields:

$$\int_{\mathcal{W}} T[w](p, x)d\lambda = \int_{\mathcal{W}} wd\lambda, \quad \forall p, x, T \in \mathcal{T} \tag{36}$$

□

It should be noted that the measure λ is, in general, <u>not</u> the Haar measure of any (locally compact) group. What theorem 2 does is essentially to provide sufficient conditions ensuring that λ can be viewed as <u>the</u> Haar measure on the group G, and to take advantage from the uniqueness of this last measure.

4.2. Proof of Corollary 1

Since $\mathcal{W}$ is compact, the space of continuous functions on $\mathcal{W}$ is separable, i.e., admits a countable and dense subset $(f_n)_n$. Hence, the weak$-*$ topology on $\Delta(\mathcal{W})$ can be metrized by, e.g., the distance induced by the countable collection of semi-norms $p_f(\mu) := \int_{\mathcal{W}} |f(w)|\mu(dw)$:

$$d(\mu, \nu) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{p_{f_i}(\nu - \mu)}{1 + p_{f_i}(\nu - \mu)}. \tag{37}$$

With the notations introduced in the proof of Theorem 1:

$$| \int_{\mathcal{W}} F(gw)\lambda_n(dw) - \int_{\mathcal{W}} F(w)\lambda_n(dw)| \square | \int_{\mathcal{W}} F(gw)\lambda_n(dw) - \int_{\mathcal{W}} f_i(gw)\lambda_n(dw)| +$$

$$| \int_{\mathcal{W}} f_i(gw)\lambda_n(dw) - \int_{\mathcal{W}} f_i(w)\lambda_n(dw)| + | \int_{\mathcal{W}} F(w)\lambda_n(dw) - \int_{\mathcal{W}} f_i(w)\lambda_n(dw)|. \tag{38}$$

For f_i $\frac{\varepsilon}{3}$-close to F, if $\alpha_n = \frac{\varepsilon}{3}$, this yields:

$$| \int_{\mathcal{W}} F(gw)d\lambda_n - \int_{\mathcal{W}} F(w)d\lambda_n| \square \varepsilon. \tag{39}$$

Hence, it suffices to take $\nu = \lambda_n$ for n large enough.

□

4.3. Proof of Proposition 1

It suffices to show that, for any $w \in \mathcal{W}$ and any $\varepsilon > 0$, there exists a collection $(g_1, ..., g_n) \in G^n$ such that

$$\mathcal{W} = \cup_{i=1}^{n} B(g_i w, \varepsilon). \tag{40}$$

This easily follows from the pre-compactness of $\mathcal{W}$ and assumption 2. In turn, (40) implies that each open ball $B(g_i w, \varepsilon)$ must be non-negligible with respect to λ. Indeed,

$$1 = \lambda(\mathcal{W}) \square \sum_i \lambda\big(B(g_i w, \varepsilon)\big) = \sum_i \lambda\big(g_i B(w, \varepsilon)\big)$$

$$= \sum_i \lambda\big(B(w, \varepsilon)\big) = n\lambda\big(B(w, \varepsilon)\big). \tag{41}$$

The first equality comes from the fact that G operates isometrically; the second from the G-invariance of λ.

□

4.4. Proof of Theorem 2

Thanks to assumption 3 (i), we can identify each element $g \in G$ with its (continuous) canonically associated mapping on $\mathcal{W}$, $\varphi_g : \mathcal{W} \to \mathcal{W}$:

$$\forall w \in \mathcal{W}, \quad \varphi_g(w) = gw. \tag{42}$$

On the other hand, let endow G with the following metric:

$$\delta(g,h) := \sup_{w\in\mathcal{W}} d(g(w),h(w)) \quad g,h\in G. \tag{43}$$

It easily follows that the family of mappings $\varphi_g : w \mapsto gw,\ g\in G$ is equicontinuous. Indeed, for any $\varepsilon > 0$, one has:

$$\forall w,v\in\mathcal{W}, \forall g\in G,\ d(w,v) \Box\, \varepsilon \Rightarrow d(g(w),g(v)) \Box\, \varepsilon. \tag{44}$$

Thanks to Ascoli's theorem, (G,δ) is therefore relatively compact. But assumption 3 (ii) says precisely that (G,δ) is closed. Hence, G is now a compact topological group. Consider the right-hand translation:

$$R_g(h) = hg \quad g,h\in G. \tag{45}$$

One has:

$$\delta(R_g(h) - R_g(h')) = \sup_w d(hg(w),h'g(w)) = \sup_w d(h(w),h'(w)) \tag{46}$$

because of assumption 2 and of the distance-preserving property of any g in G. Thus, $R_g(\cdot)$ is an isometry. We therefore can apply Theorem 1 on the group G itself, viewed as operating on itself <u>via</u> $R_g(\cdot)$. Thus, that there exists a probability μ on (G,d) verifying, for any continuous mapping $F : (G,d) \to (G,d)$:

$$\int_G F(hg)\mu(dh) = \int_G F(h)\mu(dh) \quad g\in G. \tag{47}$$

Obviously, μ is the Haar measure associated with (G,d). Let fix λ, a 'uniform distribution', $w\in\mathcal{W}$, $g\in G$ and $F\in\mathcal{C}^0\big(\mathcal{W},\mathcal{W}\big)$. One has:

$$\int_{\mathcal{W}} F(w)\lambda(dw) = \int_{\mathcal{W}} F(gw)\lambda(dw). \tag{48}$$

Let us integrate both terms of the last equality with respect to $\mu(dg)$. Since F is continuous, hence bounded, Fubini's theorem yields:

$$\int_{\mathcal{W}} F(w)\lambda(dw) = \int_{\mathcal{W}} \lambda(dw)\Big[\int_G F(gw)\mu(dg)\Big]. \tag{49}$$

By assumption 2, for each w, there exists a h such that $w = hv$. Thus,

$$\int_G F(gw)\mu(dg) = \int_G F(ghv)\mu(dg) = \int_G F(gv)\mu(dg). \tag{50}$$

It follows that:

$$\int_{\mathcal{W}} F(w)\lambda(dw) = \int_{\mathcal{W}} \lambda(dw)\Big[\int_G F(gw)\mu(dg)\Big] = \int_G F(gw)\mu(dg). \tag{51}$$

Hence the result follows from the uniqueness of the Haar measure (see [7, chap. 7(1), Theorem 1, p.13]).

□

4.5. Proof of Prop. 2.

Since the arguments are the same in both cases, we focus on case (i). Suppose that $w_j(\overline{p},\cdot)$ is not constant, and that $s := \lim_{x\to\infty}(\overline{p},x)$ exists. By Assumption 1(iii), $T_\Delta[w]\in\mathcal{W}$ for all $\Delta=(1,...,1,\alpha)$, $\alpha>0$. Define $w^\alpha(p,x):=w(p,\alpha x)=T_\Delta[w](p,x)$. Then, $(w^n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{W}$ such that, for any fixed $x>0$, the sequence $\big(w_j^n(\overline{p},x)\big)_n=\big(w_j(\overline{p},nx)\big)_n$ converges to s, i.e., the sequence $\big(w_j^n(\overline{p},\cdot)\big)_n$ of functions converges pointwise to the constant function $w_j^\infty(\overline{p},\cdot)\equiv s$.

Since $w_j(\overline{p},\cdot)$ is not constant, there must exist some $x_0>0$ such that $w_j(\overline{p},x_0)\neq s$. This implies for $n\in\mathbb{N}$:

$$\begin{aligned}\|w_j^n(\overline{p},\cdot)-w_j^\infty(\overline{p},\cdot)\|_\infty &= \sup_{x>0}|w_j^n(\overline{p},x)-s|\\ &\geq |w_j(\overline{p},n\frac{x_0}{n})-s|\\ &= |w_j(\overline{p},x_0)-s|>0.\end{aligned}$$

Hence, any subsequence of $\big(w_j^n(\overline{p},\cdot)\big)_n$ does not converge uniformly to the pointwise limit $w^\infty(\overline{p},\cdot)$. As a consequence, $(w_j^n)_n$ and $(w^n)_n$ have no converging subsequence in $\big(\mathcal{W},\|\cdot\|_\infty\big)$. This implies that $\big(\mathcal{W},\|\cdot\|_\infty\big)$ is not compact, contradicting Assumption 1(ii).

□

4.6. Analysis of the case $\tilde{w}(t)=1+\sin t$

Take $L=1$, and $\mathcal{W}$ as a subset of all functions $w:\mathbb{R}^2_{++}\to[0,2]$ defined as follows: let $\overline{w}(p,x):=1+\sin\ln(x)$ be the generator, and each 2π-periodic budget share function $\tilde{w}(t):=\overline{w}(p,e^t)=1+\sin t$. Define $w_\alpha(p,x):=\overline{w}(p,e^\alpha x)=1+\sin(\alpha+\ln x)$ and $\mathcal{W}:=\big\{w_\alpha\ :\ \alpha\in\mathbb{R}\big\}$.

In order for Assumption 3(i) to be satisfied, we restrict ourselves to affine transformations on income only, of the form T_Δ, with $\Delta\in\mathbb{R}/e^{2\pi}\mathbb{Z}$. Assumption 2 is easily verified, as well as Assumption 1 (i) and (iii). In order to prove Assumption 1(ii), i.e., that $\big(\mathcal{W},\|\cdot\|_\infty\big)$ is compact, one can observe that $\mathbb{R}/e^{2\pi}\mathbb{Z}$ is compact, and apply Ascoli's theorem. Alternatively, one can proceed as follows: Let $(\alpha_n)_n$ be a sequence in $\mathbb{R}$, and consider the sequence $(w_{\alpha_n})_n$ in $\mathcal{W}$. This latter sequence is identical to the sequence $(w_{\beta_n})_n$ where $\beta_n\in[0,2\pi]$ and $\beta\equiv\alpha\ (\mathrm{mod}2\pi)$ for all n. Assume (passing to a subsequence if necessary), without lost of generality, that $(\beta_n)_n$ converges to some $\beta\in[0,2\pi]$. We now prove that $(w_{\beta_n})_n$ converges to w_β with respect to the sup-norm topology.

Take $\varepsilon>0$ and choose some integer n such that $|\sin\beta_n-\sin\beta|,|\cos\beta_n-\cos\beta|\ \square\ \frac{\varepsilon}{2}$ for all $n\geq N$. One has:

$$\|w_{\beta_n}-w_\beta\|_\infty = \sup_x|\sin(\beta_n+\ln x)-\sin(\beta+\ln x)|.$$

Observing that $\sin(\beta_n + \ln x) - \sin(\beta + \ln x) = (\sin\beta_n - \sin\beta)\cos\ln x + (cos\beta_n - \cos\beta)\sin\ln x$, yields for all $n \geq N$:

$$
\begin{aligned}
|\sin(\beta_n + \ln x) - \sin(\beta + \ln x)| &\leq |\cos\ln x||\sin\beta_n - \sin\beta| + |\sin\ln x||\cos\beta_n - \cos\beta| \\
&\leq \varepsilon
\end{aligned}
$$

Consequently, $\|w_{\beta_n} - w_\beta\|_\infty \leq \varepsilon$ for n large enough. Hence, the compactness of $\mathcal{W}$ for the strong topology.

4. References

1. Aumann, R.: Markets with a Continuum of Traders. Econometrica **32**, 39-50 (1964).

2. Becker, G.S.: Irrational Behavior and Economic Theory. Journal of Political Economy **70**, 1-13 (1962).

3. Billette de Villemeur, E.: Heterogeneity and Stability : Variations on Scarf's Processes. Florence: European University Institute, Working Paper ECO No. 98/38 (1998)

4. ————: Aggregation of Demand and Distribution of Characteristics: a Difficulty in Modelling Behavioural Heterogeneity. Université de Cergy-Pontoise: Document de travail 99/38 (1999)

5. ————: Behavioural Complementariy (not Heterogeneity) causes the Law of Demand. Gremaq-Université de Toulouse: Working-Paper 20.14.545 (2001)

6. Bottazzi, J.-M., Hens, T.: On the Excess Demand Functions with Incomplete Markets. Journal of Economic Theory **68**(1), 149-163 (1996).

7. Bottazzi, J.-M., Lö- er, A., Hens, T.: Market Demand in the CAPM, Journal of Economic Theory **79** 192-206 (1998)

8. Bourbaki, N.: "Eléments de mathématiques", Livre IV : Intégration. Paris: Hermann 1963

9. Calvet L., Grandmont, J.-M., Lemaire, I.: Heterogeneous Probabilities in Complete Asset Markets. Université Catholique de Louvain-la-Neuve : CORE DP 9819 (1998)

10. Chiappori, P.-A.: Distribution of income and the "Law of Demand". Econometrica **53**(1), 109-127 (1985)

11. Chiappori, P.-A., Ekeland I.: Aggregation and Market Demand: An Exterior Differential Calculus Viewpoint. Econometrica **67**, 1435-1457 (1999a)

12. ————: Disaggregation of Excess Demand Functions in Incomplete Markets. Journal of Mathematical Economics **31**, 111-129 (1999b)

13. Chiappori, P.-A., Ekeland, I., Kubler, F., Polemarchakis, H.: The Identification of Preferences from Equilibrium Prices under Uncertainty. Université Catholique de Louvain-la-Neuve: CORE DP 2000/25 (2000)

14. Debreu, G.: Excess Demand Functions. Journal of Mathematical Econonomics **1**, 15-21 (1974)

15. Dierker, E., Dierker, H., Trockel, W.: Price Dispersed Preferences and $\mathcal{C}^1$ mean demand. Journal of Mathematical Economics **13**, 11-42 (1984)

16. Grandmont, J.-M.: Transformations of the Commodity Space, Behavioral Heterogeneity, and the Aggregation Problem'. Journal of Economic Theory **57**, 1-35 (1992)

17. Halmos, P., Vaugham, J.: The Marriage Problem', A.J.M. **72**, 214-215 (1950)

18. Hicks, J.R.: Value and Capital. Oxford: Clarendon Press, fourth edition 1953.

19. Hildenbrand, K.: On J.-M. Grandmont's Modelling of Behavioral Heterogeneity. Bonn Universität: Discussion Paper No. A-580 (1998)

20. Hildenbrand, W.: Core and Equilibria of a Large Economy. Princeton: Princeton University Press 1974
21. Hildenbrand, W.: On the Law of Demand. Econometrica **51**, 997-1019 (1983)
22. Hildenbrand, W.: Market Demand: Theory and Empirical Evidence. Princeton: Princeton University Press 1994
23. Hildenbrand, W., Kneip, A.: On Behavioral Heterogeneity. SFB 303 Universität Bonn: Discussion Paper N. A589 (1999)
24. John, R.: Equal Distribution of Consumers' Tastes and Perfect Behavioral Heterogeneity. SFB 303 Universität Bonn (mimeo) (2001)
25. Kirman, A.: The Intrinsic Limits of Modern Economic Theory: The Emperor has no Clothes. The Economic Journal **Conference 99**, 126-139 (1989)
26. Kneip, A.: Behavioral Heterogeneity and Structural Properties of Aggregate Demand. Journal of Mathematical Economics **31**, 49-79 (1999)
27. Maret, I.: Modeling Behavioral Heterogeneity in Demand Theory. Université Louis Pasteur, Strasbourg: Working Paper BETA No. 2001-04 (2001), forthcoming in Lecture Notes in Economics and Mathematical Systems. Berlin Heidelberg: Springer 2002
28. Marhuenda F.: Distribution of Income and Aggregation of Demand. Econometrica **63**(3), 647-666 (1995)
29. Mas-Colell, A.: On the Price Equilibrium Set of a Pure Exchange Economy. Journal of Mathematical Economics **4**(2), 117-126 (1977)
30. Mas-Colell, A., Neuefeind, W.: Some Generic Properties of Aggregate Excess Demand and Applications. Econometrica **45**, 591-599 (1977)
31. Mertens, J.-F.: Stable Equilibria — A Reformulation, Part I: Definition and Basic Properties. Math. of Oper. Research **14**, 575-624 (1989)
32. Momi, T.: Excess Demand functions with Incomplete Markets — a Global Result, Graduate school of economics. University of Tokyo (mimeo) (2001)
33. Quah, J.: The Law of Demand when Income is Price Dependent. Econometrica **65**(6), 1421-1442 (1997)
34. ————: The Monotonicity of Individual and Market Demand. Econometrica **68**(4), 911-930 (2000)
35. ————: Demand is heterogeneous in Grandmont's model. Nuffield College Working Paper (2001)
36. Shafer, W., Sonnenschein, H.: Market Demand and Excess Demand Functions, chap. 14 in Handbook of Mathematical Economics vol. II, edited by K. Arrow & M. Intriligator. Asmterdam: North-Holland 1982.
37. Trockel, W.: Market Demand, Lecture Notes in Economics and Mathematical Systems. New-York: Springer-Verlag 1984

FOOTNOTES

[2]. Interestingly enough, an analogous conclusion has been drawn, for completely different reasons, in strategic game theory. Indeed, even the most demanding refinements of strategic stability lead to some intrinsically unavoidable indeterminacy of equilibria (see [31])). On the other hand, the *aggregation* problem faced here is definitely not to be confused with the *identification* issue, treated, e.g., in [13]. In the former case, one tries to deduce from micro-economic assumptions some sound restrictions on the macro-economic level; in the latter, one deduces the micro-economic characteristics of an economy from macro-economic observations. [36] is an excellent survey on all these issues.

[3]. Notice that [10] and more recently [28] and [34] obtained the Law of Demand for more general income distributions. However, this was done at the cost of additional requirements on individual behaviors (or on the aggregate substitution effect matrix). Hildenbrand subsequently showed in [22] that an assumption over the distribution of individual demand vectors ensures the positive semidefiniteness in the aggregate of the income effect matrix. The Law of Demand follows then from the Slutsky decomposition of the Jacobian matrix of market demand. In this approach, individual rationality was still required to give an account of the negative semidefiniteness in the aggregate of the substitution effect matrix.

[4]. Some of the ideas related to heterogeneity have been also applied in [9] to financial asset economies with heterogeneous beliefs, showing the versatility of this approach.

[5]. These terminologies follow respectively [5] and [27]. Note that a similar alternative explanation of the Law of Demand was already proposed by [18, p.64] for the excess demand function : Hicks underlines, indeed, that the Law of Demand emerges in the aggregate if either the income effect is negligible at the micro-economic level or income effects *cancel out when aggregating* over buyers and sellers.

[6]. Note that this approximate insensitivity is sufficient to get the Law of Demand.

[7]. See also [37] and the references therein, especially [30].

[8]. On the other hand, our angle of attack is quite different: neither do we need to rely on the existence proof of a Haar measure on some locally compact topological group in order to exhibit a 'uniform' distribution on agents' characteristics, nor do we restrict ourselves to individual preferences that are representable by smooth utility functions or to homogeneous budget constraints.

[9]. **Notations:** For any pair of vectors $x, y \in \mathbb{R}^L$, $x \cdot y$ denotes the Euclidean scalar product, and $x \quad y = (x_1 y_1, ..., x_L y_L)$ the tensor product. If $p \in \mathbb{R}^L_{++}$, p^{-1} denotes the vector $(\frac{1}{p_1}, ..., \frac{1}{p_L})$. For any bijective mapping $T : X \to X$ and any integer n, T^n stands for $T \circ ... \circ T$, the n^{th} composition of T with itself. Any

Euclidean space is equipped with its Euclidean norm. $B(x, \varepsilon)$ is the open ball of center x and of radius ε. δ_x is the Dirac measure with support $\{x\}$; $\#X$ is the cardinality of the set X. For any topological space $X, \mathcal{C}^0(X)$ [resp. $L_\infty(X)$] is the space of continuous functions [resp. equivalence classes of bounded functions] $f : X \to X$.

[10]. This follows from the fact that $\forall p, q \in \mathbb{R}^L_{++}$,

$$(p-q)\cdot\Big(p^{-1}\quad W(p,x)-q^{-1}\quad W(q,x)\Big) \;\square\; (p-q)(p^{-1}-q^{-1})\quad W(q)+(p-q)(p^{-1}-q^{-1})\quad \varepsilon.$$

[11]. Under individual rationality, the last property would typically result from the local non-satiation of households' preferences, and would imply that $\sum_{l=1}^{L} w_l^i(p, x) = 1, \forall i, p, x$. This is required, e.g., in [16] and [33].

[12]. Observe that [16] and [26] do not need to assume the WARP at the individual level, while [33] does require it, because his assumptions on behavioral heterogeneity are weaker.

[13]. It is worth emphasizing that this property, while being sufficient, is not *necessary* to prove that an economy is heterogeneous. The arguments provided in [35], for instance, are sufficient. Thus, the following property is a much stronger by-product.

[14] We thank an anonymous Referee for having pointed out this to us.

[15]. The interpretation should be clear: A_i is the set of boyfriends of Ms.i; if a certain collection of ladies I put their boyfriends in common, the number of men they get is at least as high as $\#I$. The conclusion of the lemma becomes then obvious: the n ladies will be able to marry without practicing polyandry.