Strategic Knowledge Sharing in Bayesian Games: A General Model *

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Abstract

This paper provides a general model for the study of strategic knowledge sharing in any finite Bayesian game. Unlike earlier papers on strategic information revelation, higher-order uncertainties can be incorporated into the model. We propose an equilibrium concept, called *knowledge equilibrium*, which takes into account voluntary, public, and decentralized communication possibilities of exogenously certifiable statements. After the communication stage, beliefs are explicitly deduced from consistent possibility correspondences, without making reference to sequences of perturbed games. Several properties of knowledge equilibria and revised knowledge are examined. In particular, it is shown that a knowledge equilibrium is always a sequential equilibrium of the associated extensive form game with communication.

KEYWORDS: Strategic information revelation; Interactive knowledge; Bayesian games; Knowledge revision; Consistent beliefs.

JEL CLASSIFICATION: C72; D82.

1 Introduction

Interactive decision situations, or simply games, are usually based on an endogenous knowledge structure. Indeed, agents' uncertainty can be modified and reduced through the information reflection of aggregated variables (like a price system), by individual experimentation, or by the observation of other agents' actions. In some circumstances, knowledge can also be directly exchanged via verbal or written revelations. In this case, by communicating voluntarily with each other, agents can actively modify the information structure of the game they are playing, even if the strategic context does not involve, a priori, any possibility of information exchange.

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This paper is concerned with strategic and direct knowledge sharing in incomplete information games. More precisely, we add, to any given Bayesian game (or game with differential information), a first stage of non-cooperative communication which is not modeled in the basic interactive decision situation. Several assumptions will be made concerning the features of communication. Among them, we require that only truthful—but possibly very vague—revelations are allowed. That is, players are free to make uninformative, partially informative, or complete disclosures concerning their own information, but they cannot disclose knowledge they do not possess. This is possible if, e.g., information can be certified, proved, or verified, or if there exists injuries against lying agents. The truth telling constraint is also justified if players are able to transmit exogenous messages they initially received from the "external world". Since these exogenous messages allowed them to construct their initial knowledge (i.e., they allowed them to exclude some states of the world), they might also convey information to other players when they are transmitted.[1] A second important assumption concerns the mechanism of communication. Communication is assumed direct in the sense that there is no centralized mechanism to ensure knowledge sharing. In particular, players cannot commit to exchange information before they actually receive it. Finally, we assume that information revelation does not directly affect players' payoffs, but only the information structure. This excludes, e.g., communication through the observation of others' payoff-relevant actions in a sequential game, or indirect communication through a price system.

Considering direct and strategic communication is important for at least three reasons. First, it may radically affect the outcomes predicted with solution concepts like the Bayesian–Nash equilibrium, where the knowledge structure is fixed throughout the analysis. Hence, pre-play and voluntary knowledge sharing is of great interest for applied game-theoretical researches since, in many economic, legal, political or financial models, an exogenous information is assumed, but seems often inappropriate in real-world problems. Second, such an analysis provides some characterizations of endogenous information structures which are likely to arise in practice. For example, it helps to characterize kinds of incomplete information games in which distributed knowledge can become common knowledge through voluntary disclosures. Finally, it enables to study players' strategic behavior and knowledge updating when the information structure can be manipulated.

Game-theoretical papers dealing with voluntary knowledge sharing are suggestive, but still new and far from being definitive. The pioneering contributions made on the topic of strategic information revelation with provable (or certifiable, verifiable) statements are models of *persuasion* from a seller to a buyer where the seller can reveal or conceal the quality of his product at no cost. This literature has been initiated by Grossman (1981) and Milgrom (1981) who showed that the seller is not able to mislead the potential buyer about the quality of his product, even in a monopolistic market without reputation possibilities. Other papers generalized this result by considering that full certifiable disclosure is not possible, or that many interested groups are involved, in sometimes more general contexts (see, e.g., Milgrom and Roberts, 1986; Okuno-Fujiwara et al., 1990; Shin, 1994a,b; Lipman and Seppi, 1995). In a game-theoretical point of view, Okuno-Fujiwara et al. (1990) extended substantially these models because they considered *many* privately informed decision makers, whereas in all other papers the decision maker is assumed completely uninformed. Hence, they provided a model of strategic information revelation for "real" incomplete information games (with at least two asymmetrically informed decision makers), in which all players may have some information to disclose before choosing an action. In particular, they gave some commonly

[1] Other justifications and various examples may be found in the literature in the field of persuasion and communication games; see (among others) Grossman (1981), Milgrom (1981), Milgrom and Roberts (1986), Okuno-Fujiwara, Postlewaite, and Suzumura (1990), Shin (1994a,b), Lipman and Seppi (1995), Seidmann and Winter (1997), and Glazer and Rubinstein (2001).

analyzed duopoly examples with incomplete information (e.g., about firms' production costs or demand functions) where their assumptions were, or were not, sufficient for perfect revelation.

It is worth mentioning that this literature differs from the literature on *cheap talk games*, i.e., sender-receiver games where non-binding, non-certifiable and costless communication takes place before the receiver chooses his action.[2] The equilibrium set is always enlarged by cheap talk because all messages can be sent whatever the information senders have, i.e., messages have no intrinsic meaning and prove nothing. In particular, every game of costless and non-certifiable communication has "babbling" equilibria, where messages are ignored. In Section 5 we discuss how the model developed in this paper could be modified in order to allow cheap talk communication.

As far as we know, existing studies on strategic information revelation consider information structures with uncorrelated types, which implies that beliefs (probabilities assignments) about others' knowledge are common to all agents. In particular, there is surprisingly no attempt to study strategic information revelation in environments with higher-order uncertainty (i.e., uncertainty about others' knowledge).[3] It is well known, however, that interactive knowledge and higher-order uncertainty play a crucial role in interactive decision situations.[4] As matter stands, the explicit evolution of information structures and of interactive knowledge was only studied with *exogenous* and *mandatory* communication, in the literature about the emergence of common knowledge and consensus (see, e.g., Aumann, 1976; Geanakoplos and Polemarchakis, 1982; Parikh and Krasucki, 1990; Krasucki, 1996). In short, it was shown that, by communicating, agents may agree and reach a consensus about their claims and beliefs. The analysis was first done with perfectly reliable communication about posterior probabilities. It was shown that the communication of posterior probabilities continues until they are equal and common knowledge. The study of more general information exchange also leads, with some assumptions on the communication protocol and on the type of sent messages, to such a consensus. However, in these models, strategic consideration is completely absent because agents are not involved in any game.[5]

This paper is an attempt to "mix" the strategic approach to information disclosure and the approach on exogenous communication protocols in environments with higher-order uncertainty. More precisely, we develop a general game-theoretical model in which, before playing a Bayesian game with a partitional information structure, players can publicly and costlessly exchange exogenously certifiable information in a first stage game. Information revelation is done at an interim stage (i.e., after each one got his initial private information) and voluntarily. In this framework, we construct an equilibrium concept, called *knowledge equilibrium*, in which revision rules and updated information structures are obtained from *knowledge consistency conditions*. We examine several characterizations of players' communication possibilities, revision rules and updated knowledge, and we study the properties of the resulting payoff-relevant strategies and equilibria. Since we use the information partition approach to model the differential information rather than the type set approach, we are able to eliminate the independent type assumption made in previous work on strategic information revelation.

In consequence, our model of strategic information revelation extends most of previous

[2]See, e.g., Crawford and Sobel (1982), and Farrell and Rabin (1996). A larger class of games, information structures, and communication possibilities allowing many communication stages are considered, e.g., by Forges (1990) and Ben-Porath (2001). The perspective of this literature differs however from ours.

[3]Shin (1994a,b) considered a particular information structure in which a decision maker does not know the interested party's knowledge about fundamental states. However, the depth of knowledge of the information structure does not exceed one.

[4]See, e.g., the surveys of Geanakoplos (1994) and Morris and Shin (2000) to name just a few.

[5]An influential strand in the computer science literature has also examined knowledge and communication with hierarchical knowledge reasoning (see, e.g. Fagin, Halpern, Moses, and Vardi, 1995). However, to the best of our knowledge, none of these contributions try to integrate agents' *incentives* to share knowledge.

papers, since any information structure, Bayesian game, and certifiability possibility can be considered. The only restrictions is that we do not allow repeated and networked communication, and the state space must be finite.[6] As shown, substantial difficulties arise from these generalizations, particularly in characterizing beliefs when unexpected messages are received. To deal with these difficulties by keeping the analysis and possible applications tractable, we define knowledge consistency conditions for revised information by relying on explicit "epistemic" inferences. These inferences are carried out without reference to the sequences of trembles used to define a sequential equilibrium. Interestingly, our conditions *imply* Kreps and Wilson's (1982) consistency conditions. Nevertheless, with such a "standard" approach without bounded rationality, it is shown that, outside equilibrium, second stage information structures over the set of states of the world do not necessarily satisfy truth (non delusion) and introspection axioms of knowledge, except for some particular configurations. Hence, outside the equilibrium path, a player's information cannot necessarily be represented by a partition of the state space because unexpected messages may yield many possible and reasonable interpretations.

In short, updated knowledge in the second stage game, after all messages have been received, are constructed using the following inference rule. Each player, given the vector of messages received from the others, verifies if there exists an equilibrium (expected) vector of messages which is compatible with the actual messages in one state he considers as possible. In that case, there is either no deviation in the communication stage or the deviation is not observable by the player. Hence, he applies Bayes' rule by inverting all players' communication strategies. Otherwise, he knows that at least one player has deviated. If he can identify the player who has deviated, then he continues to apply Bayes' rule on others' communication strategies, and excludes the states in which the identified deviant is the most likely to deviate. If a non-degenerated set of players might be the potential deviants, then the same procedure is performed by considering the deviant player as the potential deviant which is the most likely to deviate. The "most likely" relations are common to all players, which ensures that this procedure generates a sequential equilibrium. As shown, however, given that players are endowed with different initial information, they can make entirely different interpretations from a vector of messages. For example, they may not observe the same deviations, they may not identify the same deviant, and they may not exclude the same states of the world in which a deviant is the most likely to deviate. In our knowledge, such a construction is in itself a substantial originality and richness of our model comparing to previous papers on strategic information revelation.

In Section 2 we present the general framework of this paper. First, we describe the initial information structure and the initial Bayesian game based on it. Second, we construct the communication stage (first stage game) and we characterize communication possibilities and strategies. Then, we present the second stage game in which the initial Bayesian game is played according to players' first stage communication choices. Finally, we define the sequential equilibrium of the complete two-stage communication game.

In Section 3 we introduce an alternative equilibrium, called knowledge equilibrium. Contrary to the sequential equilibrium approach, we consider outside equilibrium information structures in terms of *possibility correspondences* such that outside equilibrium beliefs are simply obtained by applying Bayes' rule on them. To restrict outside equilibrium beliefs, we elaborate some natural restrictions for the possibility correspondences obtained outside the equilibrium path. The cognitive requirements behind the conditions we impose are explicitly defined in terms of players' inferences about possible worlds. In addition, we explicitly characterize observable and identifiable deviations, and we demonstrate some properties of second

[6] Multi-stage communication has been considered by Lipman and Seppi (1995) with one decision maker and symmetrically informed interested parties.

stage information structures.

In Section 4 we compare knowledge and sequential equilibria. First, we show that a knowledge equilibrium is always a sequential equilibrium of the two-stage communication game. That is, the conditions on beliefs generated by our possibility correspondences satisfy Kreps and Wilson's (1982) consistency condition in the associated extensive form game. A very simple example proves, however, that a sequential equilibrium may not be a knowledge equilibrium. In this example, the difference stems from the fact that there is only a finite set of possible outside equilibrium beliefs which can be generated from possibility correspondences and prior probabilities. Other examples of revised information are presented to illustrate the additional requirements we impose on outside equilibrium beliefs. These additional requirements result from the fact that players only consider unilateral deviations as possible, a restriction which is not imposed by Kreps and Wilson's (1982) consistency condition. However, some conditions ensuring the equivalence of the two approaches are developed.

Section 5 concludes by a discussion and gives some lines for further researches and developments. We have collected the main technical proofs, lemmas, and constructions in the Appendices. Applications to some classes of games and information structures are investigated in a companion paper (Koessler, 2002).

2 General Framework

In this section we describe a general class of initial information structures and Bayesian games, and we construct the pre-play communication stage in which agents strategically modify the initial Bayesian game through their influence on its information structure. The spirit of the equilibrium of the complete game will be to require that information disclosures of the first stage game of communication are optimal and that every profile of strategies of the "continuation Bayesian games" generated by communication forms a Bayesian equilibrium.[7]

2.1 Initial Information Structure and Bayesian Game

We consider a measure space $(\Omega, \mathcal{F}, p)$, where Ω is a finite state space (the set of states of the world), $\mathcal{F}$ is a σ-algebra of events on Ω, and p is a full-support probability distribution on Ω, i.e., $p(\omega) > 0$ for all $\omega \in \Omega$. The power set 2^Ω is the set of subsets (or events) of Ω. A state ω characterizes the fundamentals of the game (e.g., players' preferences), as well as players' uncertainty about the fundamentals and about others' knowledge. The probability distribution p determines players' common priors about the states of the world.

Let $N = \{1, \dots, n\}$ be the finite set of players ($n \geq 2$), and let $h_i : \Omega \to 2^\Omega \backslash \{\emptyset\}$ be player i's *initial information function*. We denote by $h = (h_i)_{i \in N}$ the *initial information structure* on Ω. It gives exactly how knowledge is distributed between players. It is assumed that the initial information structure is *partitional* and *correct*, that is, $\{h_i(\omega) : \omega \in \Omega\}$ forms a partition H_i of Ω and $\omega \in h_i(\omega)$ for all $\omega \in \Omega$ and $i \in N$.[8] Therefore, the initial information structure can also be described by the n-tuple of partitions $H = (H_i)_{i \in N}$. The *initial information set* (partition cell) $h_i(\omega)$ of player i at ω is the element of H_i containing ω. When player i is at an information set $h_i(\omega)$, he cannot distinguish the states in it. In other words, he knows that a corresponding state is realized but he cannot say which one.

[7] Of course, these continuation Bayesian games will not be proper subgames since they do not stem from a singleton information set.

[8] This last requirement is equivalent to the truth axiom in epistemic models: what an agent knows for sure is always true. Actually, the knowledge structure behind this modeling is a strong epistemic model, which mainly corresponds to the epistemic logic S5.

Strategic concerns are introduced by considering an *initial Bayesian game* in which each player i has a finite set of *effective* or *payoff-relevant actions* A_i and a von Neumann Morgenstern *utility function* $u_i : A \times \Omega \to \mathbb{R}$, where $A = \prod_{i \in N} A_i$.[9] This Bayesian game is characterized by a tuple $G \equiv \langle N, \Omega, p, h, A, (u_i)_{i \in N} \rangle$. We will add a first strategic communication stage which can modify the initial information structure h of this game. By $G(h') \equiv \langle N, \Omega, p, h', A, (u_i)_{i \in N} \rangle$ we denote the Bayesian game which is the same as G, except that the information structure is h' instead of h.

Having defined initial information structures and Bayesian games, we turn to the definitions of strategies and equilibria of such Bayesian games. We denote by $\Delta(A_i)$ the set of probability distributions over A_i. A (mixed) strategy of player i in the Bayesian game G is given by a H_i measurable function $\phi_i : \Omega \to \Delta(A_i)$. A profile of strategies of G is given by $\phi = (\phi_i)_{i \in N} \in \Phi = \prod_{i \in N} \Phi_i$, where Φ_i is player i's set of strategies in G. We shall sometimes write $\Phi_i(h)$ and $\Phi(h)$ to specify the information structure we consider. With some abuse of notations, utility functions u_i are naturally extended to mixed strategies of G by $u_i(\phi, \omega) = \sum_{a \in A} \phi(a \mid \omega)\, u_i(a, \omega)$. Then, player i's expected utility when he is at his information set $h_i(\omega)$ and the strategy profile $\phi \in \Phi$ is used is given by

$$\begin{aligned} U_i(\phi \mid h_i(\omega)) \equiv E_p\big(u_i(\phi, \cdot) \mid h_i(\omega)\big) &= \sum_{\omega' \in \Omega} p\left(\omega' \mid h_i(\omega)\right) u_i(\phi, \omega') \\ &= \sum_{\omega' \in \Omega} p\left(\omega' \mid h_i(\omega)\right) \sum_{a \in A} \phi(a \mid \omega')\, u_i(a, \omega'). \end{aligned} \tag{1}$$

Accordingly, $\phi \in \Phi(h)$ is a *Bayesian–Nash equilibrium* of $G(h)$ if for all $i \in N$ and $\omega \in \Omega$ we have[10]

$$U_i(\phi \mid h_i(\omega)) \geq U_i(a_i, \phi_{-i} \mid h_i(\omega)), \quad \forall\, a_i \in A_i. \tag{2}$$

We denote the set of equilibria of $G(h)$ by $\Phi^*(h) \subseteq \Phi(h)$. Since finite Bayesian games are investigated, we have $\Phi^*(h) \neq \emptyset$ for all initial information structures h on Ω.

In order to modify the information structure h, players will be able to share their knowledge by communicating information about Ω in a first stage game. Then, the profile of actions $a \in A$ can be modified because a new Bayesian game will eventually be played (new in the sense that the information structure has changed via communication). Communication will modify the expected utility of a player through the direct modification of his information and through the modification of the action profile. We will impose that such a profile of actions is chosen optimally for each information structure that could be generated by communication. Given these "subgame" decisions, players will compare their utility for each information structure and will try to generate their most favorable information structure by communicating noncooperatively with each other.

2.2 Communication Stage

Before the Bayesian game G is played, but *after* each player received his private information, we allow players to *publicly* and *simultaneously* send *one* explicit message containing some of

[9] Finiteness of actions' sets is not necessary in our model. We only need the existence of a Bayesian–Nash equilibrium for each information structure on Ω. This is automatically verified with finite sets of actions. If the sets $(A_i)_{i \in N}$ are infinite, some usual conditions on the utility functions and the actions space are needed for the existence of a second stage Bayesian equilibrium. Finiteness of the state space is however required.

[10] Standard game-theoretical conventions are used throughout the paper. In particular, for any variable, we denote its profile over all agents except that of player i by the corresponding letter with subscript $-i$. With some abuse of notation, a_i will sometimes denote the strategy assigning probability one to the action a_i.

their private information. As will be seen, allowing only one message to be sent is without loss of generality (see Remark 1).[11] Information is certified since only truthful reports are allowed. Formally, each player i, when he is at his information set $h_i(\omega)$, chooses to reveal an event $x_i \subseteq \Omega$ to all the other players. The condition that sent information x_i of player i at ω is true is formally equivalent to $h_i(\omega) \subseteq x_i$.[12] Said differently, player i can reveal x_i at ω only if he knows x_i at ω. In such a setting, each player can exactly reveal what he knows, just a part, or nothing, but he cannot reveal information he does not possess. In other words, agents must tell the truth, but not necessarily the whole truth. This assumption is called the *certifiability assumption.*

For the purpose of characterizing communication possibilities, let $\mathcal{Y}_i$ be the σ-algebra generated by H_i, minus the empty set. That is, $\mathcal{Y}_i$ is the family of all unions of events in H_i. For any $i \in N$ and $\omega \in \Omega$, the set $Y_i(\omega) \equiv \{y_i \in \mathcal{Y}_i : \omega \in y_i\}$ contains relevant knowledge player i has at ω.[13] More precisely, $Y_i(\omega)$ is the set of player i's self-evident events containing ω.[14] Let $\mathcal{X}_i \subseteq \mathcal{Y}_i$ be a *set of messages* (in terms of events) such that $\mathcal{X}_i \cup \{\emptyset\}$ is a set of events closed under intersection. What player i can reveal at $\omega \in \Omega$ is given by the subset $X_i(\omega) = \{x_i \in \mathcal{X}_i : \omega \in x_i\}$ of $Y_i(\omega)$. Let $\mathcal{Y} = \prod_{i \in N} \mathcal{Y}_i$, $\mathcal{X} = \prod_{i \in N} \mathcal{X}_i$, $Y = (Y_i)_{i \in N}$, and denote by $X = (X_i)_{i \in N}$ the general *level of certifiability.* This certifiability level characterizes the preciseness of information that can be transmitted by each player at each state of the world. It is reasonable to impose that $\Omega \in X_i(\omega)$ for all $\omega \in \Omega$ and $i \in N$ (i.e., $\Omega \in \mathcal{X}_i$ for all $i \in N$), which means that players always have the possibility to reveal nothing. This assumption can be dropped easily in order to analyze, e.g., the effect of mandatory communication which limit players' discretion.[15] The following proposition results directly from the modeling of certifiability and from the correctness of the initial information structure.

Proposition 1 *If player i can reveal x_i at ω (i.e., $x_i \in X_i(\omega)$), then player i knows x_i at ω (i.e., $h_i(\omega) \subseteq x_i$) and x_i is true at ω (i.e., $\omega \in x_i$).*

Proof. Obvious. □

Remark 1 Assuming that $\mathcal{Y}_i \cup \{\emptyset\}$ is a σ-algebra and that $\mathcal{X}_i \cup \{\emptyset\}$ is closed under intersection is without loss of generality. If $\mathcal{Y}_i \cup \{\emptyset\}$ is not a σ-algebra but also contains all events player i can possibly know (i.e., events of the set $\{E \subseteq \Omega : \exists\, \omega \in \Omega \text{ s.t. } h_i(\omega) \subseteq E\}$), then revealing such an event is tantamount to revealing an event of $\mathcal{Y}_i$ since others can infer player i's information sets in which this revelation is possible. The reason is that players' partitions are known by each other (see also Example 1 on page 9). On the other hand, if $\mathcal{X}_i \cup \{\emptyset\}$ is not closed under intersection, this means that there are two events E and F in $\mathcal{X}_i$ such that $E \cap F \neq \emptyset$ and $E \cap F \notin \mathcal{X}_i$. But then, it suffices that player i sends both E and F to certify the event $E \cap F$. Nonetheless, $\mathcal{X}_i$ might not be closed under union (or complementation). Indeed, there are many instances in which it is possible to prove a fact, but it is not possible to prove that a fact is not true. This arises, for example, if players are only able to certify fundamental (payoff-relevant) events,[16] but are not always informed about them.

[11] However, sequential communication may markedly complicate the model.

[12] Since the initial information structure is correct ($\omega \in h_i(\omega)$ for all $i \in N$ and $\omega \in \Omega$), an event initially perceived as true by an agent is effectively true.

[13] An illustration of these objects is given in Example 1 on page 9.

[14] A self-evident event (or truism) for player i is an event which is known by player i whenever it occurs. That is, $h_i(\omega) \subseteq E$ for all $\omega \in E$ or, equivalently, $\{\omega \in \Omega : h_i(\omega) \subseteq E\} = E$. Such an event cannot happen unless player i knows it.

[15] An analysis of limited discretion in information disclosure has been reported in a persuasion game by Fishman and Hagerty (1990).

[16] Fundamental or payoff-relevant events are events of the σ-algebra generated by the set of utility functions $(u_i)_{i \in N}$.

The certifiability level is called *perfect* if $X = Y$ (i.e., $\mathcal{X} = \mathcal{Y}$). When certifiability is perfect, players can reveal any piece of knowledge they have. If $X_i(\omega) = \{\Omega\}$ for all $i \in N$ and $\omega \in \Omega$, the certifiability level is null, which implies that communication is irrelevant. Indeed, if each player is only able to send the same message whatever his information set, then the information structure is unchanged and the game with communication is equivalent to the initial Bayesian game G. On the contrary, when players send different messages, they anticipate the effect on the information structure and on players' behaviors in the second stage game.

The following proposition shows that if $x_i \in \mathcal{X}_i$ is revealed by player i, then the set of states $X_i^{-1}(x_i) \equiv \{\omega' \in \Omega : x_i \in X_i(\omega')\}$ in which this player can send x_i is exactly the set x_i. Thus, when x_i is revealed by player i at ω, other players learn that the real state is in x_i. Therefore, the set x_i can be interpreted as the *pure informational content* of the message x_i. Consequently, a message $x_i' \subseteq x_i$ is, according to its pure informational content, at least as informative as x_i.[17]

Proposition 2 *For any certifiability level X, any player $i \in N$, and any message $x_i \in \mathcal{X}_i$, the pure informational content of the message x_i is the event x_i itself. That is, $X_i^{-1}(x_i) \equiv \{\omega \in \Omega : x_i \in X_i(\omega)\} = x_i$.*

Proof. On the one hand, if $\omega \in X_i^{-1}(x_i)$ then $x_i \in X_i(\omega)$. But, by definition, $\omega \in x_i'$ for all $x_i' \in X_i(\omega)$. Thus, $\omega \in x_i$. On the other hand, if $\omega \notin X_i^{-1}(x_i)$ then $x_i \notin X_i(\omega)$. This means that $\omega \notin x_i$ or $x_i \notin \mathcal{X}_i$. Since $x_i \in \mathcal{X}_i$ (by assumption), we necessarily have $\omega \notin x_i$. $\square$

Remark 2 A different but equivalent formalization of certifiability is to consider a set of available messages which are not in terms of events, but which also differ at each player's information set (see, e.g., Lipman and Seppi, 1995; Seidmann and Winter, 1997). From our point of view the modeling of messages in terms of events is more natural and convenient because messages have a direct and explicit semantical content. Moreover, what is certifiable is very easy to characterize. Indeed, from the previous proposition, an event E is *certifiable by player i* iff $E \in \mathcal{X}_i$, and a message $E \in \mathcal{X}_i$ simply *certifies* the event E. Finally, the certifiability level corresponds to a rich language in the sense of Seidmann and Winter (1997) if $\mathcal{X} = \mathcal{Y}$, i.e., if the certifiability level is perfect.

Besides adapting the certifiability level to the context analyzed (e.g., by allowing only some agents to communicate, or by allowing only payoff-relevant events to be certified), allowing partial certifiability is interesting owing to its impact on communication behaviors and on knowledge sharing possibilities. Indeed, for distributed knowledge to be perfectly shared not all information has to be certifiable. To see this, consider two players with the partitions $H_1 = \{\{\omega_1\}, \{\omega_2, \omega_3\}\}$ and $H_2 = \{\{\omega_1, \omega_2\}, \{\omega_3\}\}$, and assume that at ω_2 player 1 reveals $\{\omega_2, \omega_3\}$ and player 2 reveals $\{\omega_1, \omega_2\}$. In this case, both players necessarily know that the event $\{\omega_2\} = \{\omega_1, \omega_2\} \cap \{\omega_2, \omega_3\}$ is true, even if it cannot be certified and is not initially known by any player.[18] What is more, a certifiability level may be sufficient to share all knowledge when communication is mandatory, but such knowledge sharing might be obtained strategically only with more certifiability possibilities. Such a phenomenon can be obtained, e.g., in persuasion games or Cournot games with communication and incomplete information about fundamentals and about others' knowledge about the fundamentals. Indeed, in such games, payoff-relevant certifiability is not always sufficient to get a perfectly revealing equilibrium, although full

[17] According to an *equilibrium* informational content, this is not necessarily true. Indeed, when communication is strategic, the informativeness of a message can go beyond its pure informational content.

[18] Of course, an event which is learnt is necessarily "distributed knowledge", i.e., it must be a superset of an event belonging to the σ-algebra generated by the *Join*, $\bigvee_{i \in N} H_i$, of players' initial partitions.

certifiability leads to complete knowledge sharing (see Okuno-Fujiwara et al., 1990; Shin, 1994a).

In the following example we illustrate different certifiability possibilities.

Example 1 Let $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6\}$ be the set of states of the world, and let $H_1 = \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}, \{\omega_5, \omega_6\}\}$ be player 1's partition. We get

$$Y_1(\omega_1) = \{\{\omega_1, \omega_2\}, \{\omega_1, \omega_2, \omega_3, \omega_4\}, \{\omega_1, \omega_2, \omega_5, \omega_6\}, \Omega\}.$$

Events of $Y_1(\omega_1)$ are player 1's self-evident events containing ω_1. They represent all relevant information player 1 has at ω_1 because revealing another (true) event for him at ω_1 is equivalent to revealing an event of $Y_1(\omega_1)$. For example, revealing respectively $\{\omega_1, \omega_2, \omega_3\}$ and $\{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\}$ is equivalent to revealing respectively $\{\omega_1, \omega_2\}$ and $\{\omega_1, \omega_2, \omega_3, \omega_4\}$. Indeed, the certifiability assumption implies that if player 1 reveals x_1 at ω_1, then $h_1(\omega_1) \subseteq x_1$. Thus, if $\{\omega_1, \omega_2, \omega_3\}$ is revealed, then receivers know that $h_1(\omega_1) = \{\omega_1, \omega_2\}$. If $\{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\}$ is revealed, then they know that $h_1(\omega_1) \subseteq \{\omega_1, \omega_2, \omega_3, \omega_4\}$. If we posit that $X_1(\omega_1) = \{\{\omega_1, \omega_2, \omega_3, \omega_4\}, \Omega\}$, then player 1 can only reveal that his information set belongs to one of the events in $X_1(\omega_1)$. He cannot certify that he knows $\{\omega_1, \omega_2\}$ or that he knows $\{\omega_1, \omega_2, \omega_5, \omega_6\}$, but he can certify that he knows $\{\omega_1, \omega_2, \omega_3, \omega_4\}$ or he can certify nothing (i.e., Ω).

Definition 1 A pure *communication strategy* for player i is a H_i measurable function $c_i : \Omega \to \mathcal{X}_i$ such that $c_i(\omega) \in X_i(\omega)$ for all $\omega \in \Omega$. A mixed communication strategy for player i is a H_i measurable function $\pi_i : \Omega \to \Delta(\mathcal{X}_i)$, such that the support of $\pi_i(\omega)$ is included in $X_i(\omega)$ for all $\omega \in \Omega$.

The conditions $c_i(\omega) \in X_i(\omega)$ and $\text{supp}(\pi_i(\omega)) \subseteq X_i(\omega)$ mean that in any state $\omega \in \Omega$, player i can only reveal directly an event he knows and which he can certify at ω. Given a certifiability level X, the set of pure communication strategies profiles is denoted by $C(X) = \prod_{i \in N} C_i(X_i)$, where $C_i(X_i)$ is the set of pure communication strategies of player i. To simplify the notations we will simply denote by C_i the set of communication strategies and by $C = \prod_{i \in N} C_i$ the set of communication strategy profiles. Similarly, let Π_i be the set of mixed communication strategies of player i and let $\Pi = \prod_{i \in N} \Pi_i$ be the set of mixed communication strategy profiles.

2.3 Continuation Bayesian Games

The communication game in which the initial Bayesian game $G \equiv \langle N, \Omega, p, h, A, (u_i)_{i \in N} \rangle$ is preceded by the first communication stage described in the previous subsection is denoted by (G, X), where $X : \Omega \to (2^{\mathcal{X}_i})_{i \in N}$ is the certifiability level. For each state of the world $\omega \in \Omega$, $X(\omega) = (X_1(\omega), \ldots, X_n(\omega))$ specifies the set of events certifiable by each player at ω. In the communication game (G, X), there are two types of behavioral strategies. Communication strategies defined before specify which messages players will send at each of their initial information set. Effective (payoff-relevant) strategies specify the actions chosen in the second stage continuation Bayesian games, after a vector of messages $x = (x_1, \ldots, x_n) \in \mathcal{X} = (\mathcal{X}_1, \ldots, \mathcal{X}_n)$ has been sent. Formally:

Definition 2 A pure *effective* or *payoff-relevant strategy* for player i is a function $s_i : \mathcal{X} \times \Omega \to A_i$ such that $s_i(x, \cdot)$ is H_i measurable for all x. A mixed effective or payoff-relevant strategy for player i is a function $\sigma_i : \mathcal{X} \times \Omega \to \Delta(A_i)$ such that $\sigma_i(x, \cdot)$ is H_i measurable for all x.

Hence, $\sigma_i(a_i \mid x, \omega)$ is the probability that player i chooses action $a_i \in A_i$ at $\omega \in \Omega$, when the vector of events $x = (x_1, \ldots, x_n) \in X(\omega)$ has been revealed during the communication stage. We denote by $\Sigma = \prod_{i \in N} \Sigma_i$ the set of mixed effective strategy profiles.

Similarly, the set of pure effective strategy profiles is denoted by $S = \prod_{i \in N} S_i$.[19] Utility functions are also extended (with some abuse of notations) to effective strategies by $u_i(\sigma, x, \omega) = \sum_{a \in A} \sigma(a \mid x, \omega) u_i(a, \omega)$. To characterize second stage expected utilities, and then rational effective strategies, we have to specify the *second stage information structure*, i.e., the information structure of the continuation Bayesian games generated by every possible vector of events $x \in \mathcal{X}$ revealed in the communication stage. As already mentioned, information structures of continuation Bayesian games may be non-partitional and incorrect if a deviation from an expected communication strategy profile $c \in C$ occurs in the communication stage. The intuition is that players perfectly know, at equilibrium, the communication behaviors of the others, but they cannot always make correct and rational inferences about the real states of the world when a deviation occurs. Thus, they may not be able to construct a "true" partition from the messages of a deviant player (from outside equilibrium messages). Moreover, agents' information outside the equilibrium path may be incorrect if they make bad interpretations from a partially revealing message, or if an agent sends a message used in the equilibrium path, but in a state of the world where he was not expected to use it.

Before presenting second stage information structures formally, let us illustrate the intuition of our previous claims concerning outside equilibrium information. Consider two individuals, Alice and Bob, with the partitions $H_A = \{\{\omega_1\}, \{\omega_2\}\}$ and $H_B = \{\{\omega_1, \omega_2\}\}$ of $\Omega = \{\omega_1, \omega_2\}$. First, assume that at every state $\omega \in \Omega$ Alice certifies that the event $\{\omega\}$ has occurred. Consequently, Bob learns the real state and his partition becomes the same as Alice's one inasmuch as only truthful messages can be sent (i.e., information is certified). If Alice deviates in one state (for example, at ω_1) by revealing the event $\{\omega_1, \omega_2\}$ instead of $\{\omega_1\}$, then Bob can reasonably infer that either $\{\omega_1, \omega_2\}$, or $\{\omega_1\}$, or $\{\omega_2\}$ are realized events. With the first inference his interpretation is partial; with the second it is complete and correct; with the last it is incorrect.

Now, assume that Alice certifies the event $\{\omega_1\}$ at ω_1 and the event $\{\omega_1, \omega_2\}$ at ω_2. If this behavior is believed by Bob, he will infer the event $\{\omega_1\}$ at ω_1 and $\{\omega_2\}$ at ω_2 because the message $\{\omega_1, \omega_2\}$ is only sent at ω_2. Here, the formal argument is not only linked to certifiability but also to Bayesian updating along the equilibrium path.[20] In this case, Bob's partition also becomes the same as Alice's one. However, if Alice deviates by sending $\{\omega_1, \omega_2\}$ at ω_1, and this is not anticipated by Bob (as it will be the case outside the equilibrium path), he will infer the event $\{\omega_2\}$ at ω_1. In other words, he will consider the state ω_2 as the only possible state at ω_1.

It is clear from this example that Bob's information can be non-partitional, and even incorrect, without considering bounded cognitive rationality.[21] However, we will see that, along the equilibrium path, the information structure remains partitional and correct, by construction. More details on the characteristics of endogenous information structures will be exposed in Section 3.

To represent players' knowledge after the communication stage, with possible errors in information processing, we will use *possibility* or *accessibility correspondences* (instead of information functions), which do not necessarily provide truthful and partitional information to players. Such possibility correspondences are presented in the next section. Alternatively, the two-stage game being completely characterized, the sequential equilibrium of the communication game (G, X) can be defined. Such an equilibrium is characterized in details in

[19] It is important to differentiate strategy profiles in $\Phi(h)$ (i.e., strategy profiles of the Bayesian game $G(h)$ without pre-play communication) from strategy profiles in Σ or S which depend on the messages sent during the communication stage.

[20] In the literature on strategic information revelation this argument is known as the "unravelling argument".

[21] Non-partitional information structures are analyzed in details in, e.g., Geanakoplos (1989), Dekel, Lipman, and Rustichini (1998), and Modica and Rustichini (1999).

Appendix A.

3 Knowledge Equilibrium

In this section we define an alternative equilibrium to the sequential equilibrium which is substantially more tractable, explicit, and selective. More precisely, we define an equilibrium concept, called knowledge equilibrium, for the two-stage game (G, X) in which outside equilibrium beliefs are deduced from consistent possibility correspondences (i.e., consistent updated and revised knowledge). In Subsection 3.1, we begin to characterize the information structure conditionally to the messages sent in the communication stage. Some illustrations of second stage information structures are presented in Subsection 3.2. Then, in Subsection 3.3, we determine traditional knowledge axioms satisfied by these information structures in different possible configurations. Finally, in Subsection 3.4, we define the knowledge equilibrium according to our knowledge consistency condition.

3.1 Revision Rules and Knowledge Consistency

In this subsection we successively present several reasonable conditions on players' inferences from the first stage of public communication. We also give some definitions and properties concerning these inferences. Then, we link them to a general knowledge consistency condition used to define the knowledge equilibrium.

3.1.1 Equilibrium Inferences

After the communication stage, if players communicate according to the strategy profile $c \in C$, the vector of messages $c(\omega) \in X(\omega)$ is publicly observed at ω, and so it becomes common knowledge. In addition, if players are sufficiently introspective, messages which are publicly announced should often convey information beyond what they certify. That is, a reported event E—which has the pure informational content "the real state of the world belongs to E" by Proposition 2—can still provide significant evidence about an event $F \subsetneq E$. Such an additional information relies on the fact that players are aware of others' incentive to manipulate information. For example, if a particular message is only sent in certain states of the world, then its meaning is that one of these states must be realized. Hence, the *fact* that the message is sent can itself signal some of the sender's information.

Rational inferences are obtained with the "minimal" requirement that players use Bayes' rule along the equilibrium path. With such a requirement, inferences do not come purely from the content of a sentence in isolation from its context. Since our framework is mainly set-theoretical, we write this requirement by defining states of the world excluded by players when they receive a vector of messages. Formally, given an initial information structure $h = (h_i)_{i \in N}$ on Ω and a profile of communication strategies $c \in C$ *which is used and rationally anticipated*, players' information functions $h_i^c : \Omega \to 2^\Omega \backslash \{\emptyset\}$ after the communication stage are defined by the following equilibrium inference:

$$h_i^c(\omega) \equiv h_i(\omega) \cap c^{-1}(c(\omega)) \quad \forall\, i \in N,\ \omega \in \Omega, \tag{3}$$

where $c^{-1}(c(\omega)) \equiv \{\omega' \in \Omega : c(\omega') = c(\omega)\}$. The intuition behind this simple epistemic learning process is that if a player sends different messages at different information sets, then the others will differentiate these information sets. It corresponds to a kind of rational expectation learning. In a state ω, players do not only learn the informational content of communicated messages $c(\omega) = (c_1(\omega), \ldots, c_n(\omega))$, but they also take into account what would have been revealed in other states than ω. Therefore, *along the equilibrium path*, revealing $c(\omega)$

is equivalent to revealing $c^{-1}(c(\omega))$. Of course, the event $c^{-1}(c(\omega))$ conveys at least as much information as the vector of events $c(\omega)$ since the assumptions that the initial information is correct and that revealed information is certified imply that $\omega \in h_i(\omega) \subseteq c_i(\omega)$ for all $\omega \in \Omega$ and $i \in N$, and thus $c_i^{-1}(c_i(\omega)) = \{\omega' \in \Omega : c_i(\omega') = c_i(\omega)\} \subseteq \{\omega' \in \Omega : h_i(\omega') \subseteq c_i(\omega)\} \subseteq c_i(\omega)$.

The partition of player i generated by h_i^c is denoted by H_i^c and is called the *equilibrium partition* of player i given c. Of course, H_i^c is at least as fine as player i's initial partition H_i. The information structure given c is denoted by $h^c \equiv (h_i^c)_{i \in N}$ (or, equivalently, $H^c \equiv (H_i^c)_{i \in N}$). This information structure can also be constructed in the following way. Let $W(c_i)$ be the partition generated by the function c_i, and denote by $W(c) = \bigvee_{i \in N} W(c_i)$ the partition generated by c. That is, the element of $W(c)$ containing ω is $c^{-1}(c(\omega))$. The partition $W(c)$ summarizes the information publicly and strategically revealed along the equilibrium path (i.e., when the profile of communication strategies c was used and anticipated). Given an equilibrium communication profile c, players partitions after the communication stage can be rewritten $H_i^c = H_i \vee W(c)$ for all $i \in N$.

Therefore, by construction, the information structure will be partitional along the equilibrium path. This clearly shows that players are able to learn, even if nothing is revealed in the current state of the world. A player might infer others' information from their silence because he is aware of their strategic incentives in disclosing or not information in a self-interested manner.

Remark 3 The same learning process was introduced by Parikh and Krasucki (1990), and was used extensively in the common knowledge literature. In particular, the partition $W(c)$ is called the working partition. In our setting, this working partition is endogenous because we consider *voluntary* communication. That is, the functions c_i, $i \in N$, are determined by strategic behaviors, whereas they are specified, in the previous literature, by exogenous functions (e.g., by posterior probabilities, or by union consistent, convex or injective functions) which are the same for all agents.

Given a communication strategy profile $c \in C$, we have seen that the information structure becomes h^c after the communication stage if c is actually used by all players. Hence, after the communication stage, the Bayesian game $G(h^c)$ will be played. If players do not always send the message Ω at each of their information sets, then the Bayesian game $G(h^c)$ is clearly different from the initial Bayesian game $G(h)$ and, in general, other actions will be played since the set of Bayesian equilibria $\Phi^*(h)$ of $G(h)$ usually differs from the set of Bayesian equilibria $\Phi^*(h^c)$ of $G(h^c)$. To determine rational communication strategies, the comparison of players' payoffs associated with these equilibria is however not sufficient because we must characterize players' behavior outside the equilibrium path.[22] To do this, we have to characterize players' second stage information when a deviation from the communication strategy profile c occurs. Considering such deviations is necessary to support an equilibrium. Hence, we are led to consider what players would do in any counterfactual event that somebody deviates from its equilibrium communication strategy.

3.1.2 Necessary Conditions on Outside Equilibrium Inferences

The above-mentioned learning process (3) will apply *at equilibrium*, when the profile of communication strategies c is chosen and anticipated by every player. However, these inferences, which go beyond the pure informational content of sent messages, are fragile when deviations

[22] Comparative static results are sufficient to examine *ex ante* incentives to share information, when players can commit to reveal their information before they receive it as it is the case in the literature on information exchange in oligopoly.

can occur. We will explicitly derive outside equilibrium inferences from *outside equilibrium possibility correspondences* (instead of beliefs directly). This permits a deeper study of the endogenous structure of knowledge. Moreover, cognitive requirements can be described without reference to the sequences of trembles used to define the sequential equilibrium. As will be proved, our conditions *imply* Kreps and Wilson's (1982) conditions, and in some cases they are equivalent to theirs.

More precisely, given the sequencing of the complete game (G, X), if a player deviates from $c(\omega)$ at ω, then each player i's information in the second stage game is determined by his initial information $h_i(\omega)$ and by the vector of messages $x = (x_i)_{i \in N} \in X(\omega)$ sent at ω during the communication stage. Player i's *possibility correspondence* is a function $\mathcal{P}_i : \mathcal{X} \times \Omega \to 2^{\Omega} \backslash \{\emptyset\}$. For each $\omega \in \Omega$ and $x \in X(\omega)$, $\mathcal{P}_i(x, \omega)$ is called the *possibility set* of player i, and is interpreted to mean the collection of states player i thinks are possible at ω when the vector of messages $x \in \mathcal{X}$ has been sent in the communication stage. The second stage information structure is then denoted by $\mathcal{P} = (\mathcal{P}_i)_{i \in N}$. The characterization of such information is necessary to analyze what players will know (and thus, will do) after the communication stage (along and outside the equilibrium path), and thus to test the stability of a given communication behavior. The bulk of the work involved in defining completely our equilibrium concept consists in characterizing "acceptable", or consistent, possibility sets $\mathcal{P}_i(x, \omega)$. In this paragraph we provide several conditions that such possibility sets have to satisfy. In the next paragraph we give a general condition which ensures all conditions reported here.

A first obvious requirement is that each player $i \in N$ excludes the states of the world he considered as impossible at the beginning of the game (perfect recall) and the states of the world which are proved to be unrealized. In particular, if a player $k \in N$ reveals x_k, then others exclude the realization of the states $\omega \notin x_k$. Such a condition is a structural condition.[23]

RR1 (Certifiability Constraint).

$$\mathcal{P}_i(x, \omega) \subseteq h_i(\omega) \cap \bigcap_{k \in N} x_k.$$

From the preceding discussion, the rational expectation learning rule (3) will apply along the equilibrium path, i.e., when no deviation is observed. An observable deviation from c can be defined as follows:

Definition 3 A vector of messages $x \in X(\omega)$ is an *observable deviation* from $c \in C$ by player i at ω if for all $\omega' \in h_i(\omega)$ we have $c(\omega') \neq x$ or, equivalently, if $h_i(\omega) \cap c^{-1}(x) = \emptyset$.

Said differently, a deviation is observable by player i if the vector of messages he receives is not compatible with an equilibrium vector of messages in any state player i considers as possible. A non-observable deviation gives rise to the following revision rule condition:

RR2 (Bayesian Updating). If $x \in X(\omega)$ is not an observable deviation from c by player i at ω (i.e., there exists $\omega' \in h_i(\omega)$ such that $c(\omega') = x$), then

$$\mathcal{P}_i(x, \omega) = h_i(\omega) \cap c^{-1}(x).$$

This condition is a "standard" condition of Bayesian updating along the equilibrium path which implies that (but is not equivalent to) $\mathcal{P}_i(x, \omega) = h_i^c(\omega)$ if $x = c(\omega)$. In particular, if this condition is always satisfied, then $\{\mathcal{P}_i(c(\omega), \omega) : \omega \in \Omega\} = H_i^c$.

[23] For the moment, conditions we impose on possibility sets refer to a specified player $i \in N$, a specified state of the world $\omega \in \Omega$, and a specified vector of messages $x \in X(\omega)$.

Remark 4 It is worth noticing that a deviation x can be observable at ω even if x is in the range of c. This feature contrasts with previous work on strategic information revelation in which either decision makers are uninformed or players' signals are independent. In these cases, a deviation is observable if and *only if* it does not belong to the range of c. In our framework, this condition is sufficient but not necessary for a deviation to be observable.

As a third condition we reasonably need that each player i makes the same inferences when he receives the same messages in two states belonging to his same initial information set. That is, $\mathcal{P}_i(x,\cdot):\Omega \to 2^{\Omega}\backslash\{\emptyset\}$ is measurable with respect to H_i for all x.

RR3 (Admissible Revision). If $\omega' \in h_i(\omega)$, then

$$\mathcal{P}_i(x,\omega) = \mathcal{P}_i(x,\omega').$$

As a fourth condition we reasonably need that players should not signal information that they do not possess. In other words, every player i's interpretation from others' messages should be compatible with others' information, i.e., player i's inference from each player k must be a union of some of player k's information sets.[24]

RR4 (Admissible Interpretation). There exists $y = (y_1, \ldots, y_n) \in \prod_{k\in N} \mathcal{Y}_k$ such that

$$\mathcal{P}_i(x,\omega) = h_i(\omega) \cap \bigcap_{k\in N} y_k.$$

The fifth condition we will impose on revised knowledge is not always required by Kreps and Wilson's (1982) consistency condition (this claim is proved in Examples 6 and 7 in Section 4). Our additional condition stipulates that players, when updating their knowledge, are aware that only unilateral deviations from c are possible. In the terminology of Kreps and Wilson this is equivalent to the fact that unilateral deviations are infinitely more likely than any multilateral deviations. Therefore, we will formulate the last revision rule, as well as the characterization of consistent information structures, only for unilateral deviations. We denote by $X(c,\omega)$ the set of *unilateral deviations from c at ω*. That is, $X(c,\omega) \equiv \{x \in X(\omega) : \exists\, i \in N,\ x = (x_i, c_{-i}(\omega))\}$. Note that $c(\omega) \in X(c,\omega)$, i.e., $c(\omega)$ is also a unilateral deviation from c at ω, with some abuse of language.

A consequence of the restriction to unilateral deviations is that if a deviation from c is observable and identifiable by a player, then he will only interpret the outside equilibrium vector of messages as a deviation by the identifiable player. That is, he will continue to apply Bayes's rule on others' communication strategies. More precisely, we will say that a deviation $x \in X(c,\omega)$ from c is *j-identifiable* by player i at $\omega \in \Omega$ if player j's message is the only one which is incompatible with others' messages. Formally:

Definition 4 A deviation $x \in X(c,\omega)$ from c is *j-identifiable* by player i at $\omega \in \Omega$ if there is one and only one player $j \in N$ such that $h_i(\omega) \cap c_{-j}^{-1}(x_{-j}) \cap x_j \neq \emptyset$. A deviation $x \in X(c,\omega)$ from c is *identifiable* by player i at $\omega \in \Omega$ if there exists $j \in N$ such that x is j-identifiable by player i at $\omega \in \Omega$.

A deviation which is identifiable by player i at ω is also observable by player i at ω. Indeed, if the deviation x is identifiable by player i at ω, then there is some $j \in N$ such that $h_i(\omega) \cap c_{-l}^{-1}(x_{-l}) \cap x_l = \emptyset$ for all $l \neq j$, and thus $h_i(\omega) \cap c^{-1}(x) = \emptyset$. What is more, it is easy to see that if x_j does not belong to the range c_j, then the deviation $(x_j, c_{-j}(\omega))$ is observable and j-identifiable by any player $i \in N$ at ω. Notice also that if $x_i \neq c_i(\omega)$, then $(x_i, c_{-i}(\omega))$ is

[24] This condition reflects the fact that, if "trembles" are considered as in Selten (1975) or Kreps and Wilson (1982), players' probability of trembling is measurable with respect to their own information.

always observable and i-identifiable by player $i \in N$ at ω. This simply means that player i can always observe and identify his own deviations. Finally, as shown in Lemma 2 in Appendix B, when a unilateral deviation x is j-identifiable at ω by some player, then player j is effectively the deviant player at ω, i.e., $x_j \neq c_j(\omega)$. This might not be true for multilateral deviations.

Of course, an observable deviation is not always identifiable. In this case, several players might be the deviants. Figure 1 presents a very simple example of a deviation which is observable but not identifiable by a third player with no information (i.e., $H_3 = \{\{\omega_1, \omega_2\}\}$).

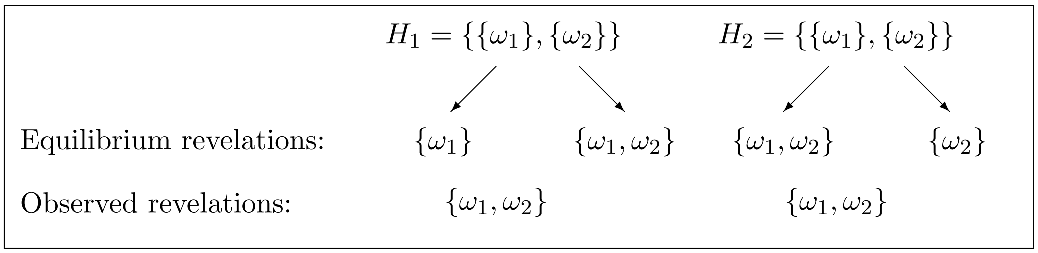

Figure 1: An Observable and Non-Identifiable Deviation.

We denote by $N_i(c, x, \omega)$ the set of (unilateral) *potential deviants* from the communication strategy profile c at ω to $x \in X(c, \omega)$ for player i. Formally, this set is defined by

$$N_i(c, x, \omega) \equiv \{j \in N : h_i(\omega) \cap c_{-j}^{-1}(x_{-j}) \cap x_j \neq \emptyset\}.$$

For example, in the deviation considered in Figure 1, $N_3(c, x, \omega) = N_3(c, (\Omega, \Omega), \omega) = \{1, 2\}$ for all $\omega \in \Omega = \{\omega_1, \omega_2\}$. Notice that as long as $x \in X(c, \omega)$, i.e., x is a unilateral deviation from c at ω, it is easy to see that $N_i(c, x, \omega) \neq \emptyset$.[25] Moreover, from the definition of an identifiable deviation, it is clear that if x is a j-identifiable deviation by player i at $\omega \in \Omega$, then $N_i(c, x, \omega) = \{j\}$.

Our condition RR5 can now be formulated in the following way. This condition is illustrated in details in Example 3 in Subsection 3.2.

RR5 (Unilateral Deviations). If $x \in X(c, \omega)$ is observable by player $i \in N$ at ω, then there exists $j \in N_i(c, x, \omega)$ and $y_j \in \mathcal{Y}_j$ such that

$$\mathcal{P}_i(x, \omega) = h_i(\omega) \cap c_{-j}^{-1}(x_{-j}) \cap x_j \cap y_j.$$

Of course, the revision rule RR5 implies the following revision rule:

RR5' (Deviant Identification). If $x \in X(c, \omega)$ is j-identifiable by player $i \in N$ at ω, then there exists $y_j \in \mathcal{Y}_j$ such that

$$\mathcal{P}_i(x, \omega) = h_i(\omega) \cap c_{-j}^{-1}(x_{-j}) \cap x_j \cap y_j.$$

The following proposition gives some relations between our different revision rules.

Proposition 3 *If $x \in X(c, \omega)$ is not an observable deviation from c by player $i \in N$ at $\omega \in \Omega$, then condition RR2 implies conditions RR1, RR3 and RR4 at i, ω, x. If $x \in X(c, \omega)$ is an observable deviation from c by player $i \in N$ at $\omega \in \Omega$, then condition RR5 implies conditions RR1 and RR4 at i, ω, x.*

[25] If one wants to consider multilateral deviations, then one can define $N_i(c, x, \omega)$ as the set of *subsets* of potential deviants. In this way, the condition $N_i(c, x, \omega) \neq \emptyset$ is restored for any deviation $x \in \mathcal{X}$ (unilateral and multilateral).

Proof. Let $i \in N$, $\omega \in \Omega$, and $x \in X(c,\omega)$. Assume that x is not an observable deviation by player i at ω and that RR2 is satisfied, i.e., $\mathcal{P}_i(x,\omega) = h_i(\omega) \cap c^{-1}(x)$. Since $c_k^{-1}(x_k) \subseteq x_k$, RR1 is immediately satisfied. Moreover, we have $c_k^{-1}(x_k) \in \mathcal{Y}_k$ because c_k is measurable with respect to H_k. Thus, RR4 is also satisfied. Finally, notice that if $\omega' \in h_i(\omega)$, then $h_i(\omega) = h_i(\omega')$, and thus $h_i(\omega) \cap c^{-1}(x) = h_i(\omega') \cap c^{-1}(x)$. Consequently, RR2 gives $\mathcal{P}_i(x,\omega) = \mathcal{P}_i(x,\omega')$, i.e., RR3 is satisfied. When x is an observable deviation by player i at ω, then condition RR5 implies conditions RR1 and RR4 because $c_k^{-1}(x_k) \subseteq x_k$ and $c_k^{-1}(x_k) \in \mathcal{Y}_k$. This completes the proof. $\square$

It is convenient to maintain all revision rules separately because some of them are sometimes sufficient to characterize a unique possibility correspondence for each player. In this case, these possibility correspondences will satisfy all of our conditions, as well as our knowledge consistency condition. This general consistency condition is presented in the next paragraph.

3.1.3 Consistent Second Stage Information Structures

Conditions RR1–RR5 induce stronger requirements on beliefs than those of the weakest version of the perfect Bayesian equilibrium which is used in most economic applications of dynamic games of incomplete information. Indeed, this weakest version of the perfect Bayesian equilibrium places no restrictions at all on the beliefs off the equilibrium path (along the equilibrium path, Bayes' rule is applied).[26] To see that stronger requirements on beliefs are imposed here, note that when player i observes the vector of messages $x \in X(\omega)$ from all players at ω, his second stage (revised) belief at ω about the state ω' is given by $p(\omega' \mid \mathcal{P}_i(x,\omega))$ (see Appendix C for more details). That is, second stage beliefs are obtained by conditioning prior beliefs on possibility correspondences. Before the communication stage, player i's belief about an event E at ω is given by the posterior probability $p(E \mid h_i(\omega))$ whereas, after the communication stage, his belief is given by $p(E \mid \mathcal{P}_i(x,\omega))$ when the vector of messages $x \in X(\omega)$ has been sent. From condition RR2 these beliefs satisfy Bayesian updating along the equilibrium path since

$$p(\omega' \mid h_i^c(\omega)) = \begin{cases} \dfrac{p(\omega')}{\sum_{\omega'' \in h_i(\omega) \cap c^{-1}(c(\omega))} p(\omega'')} & \text{if } \omega' \in h_i(\omega) \cap c^{-1}(c(\omega)) \\ 0 & \text{otherwise.} \end{cases}$$

Nevertheless, conditions RR1–RR5 are still not sufficiently restrictive, in general, to ensure that associated beliefs are consistent in the sense of Kreps and Wilson. Indeed, in Kreps and Wilson's sequential equilibrium, there is an agreement on the ranking of the relative probability of each player's zero probability information sets, this agreement being generated by arbitrary small perturbations of the game, with the implicit assumption that the equilibrium history of the play is common knowledge. In a terminology closer to our setting this means that, after the communication stage and for every player $j \in N$, there is a commonly agreed set of player j's information sets that are compatible with an unexpected and identifiable deviation by player j. When the deviation is observable and can be assigned to various players (or set of players), then Kreps and Wilson's belief consistency condition implies that there is even a commonly agreed set of players' information sets that are compatible with the deviation.

The requirements described just above are very critical ones. Indeed, it is not always reasonable to assume a collective agreement when, even individually and independently, players cannot always make perfectly rational inferences (remember that possibility correspondences might entail players knowing false events; see Subsection 3.2 for some illustrations). Nonetheless, we still keep this restriction because it is largely accepted in applied game theory and

[26] See Fudenberg and Tirole (1991) for more details.

in the refinement literature.[27] In consequence, our work can also be viewed as providing some explanations and illustrations of what the sequential equilibrium restrictions entail when strategic communication is possible. It also permits us to analyze what kind of information structures can endogenously emerge. In particular, it gives some light on the question of what types of non-partitional information structures are likely to arise (see Subsection 3.3 for more details).[28]

In an attempt to get the requirement on outside equilibrium information structures discussed before, we should impose an additional revision rule condition in order to get Kreps and Wilson's (1982) consistency condition. However, given the complexity of the interaction between several players' partitions and messages, we do not succeed in this task. Instead, we construct a relatively simple and cognitively explicit method for characterizing outside equilibrium information structures. This construction ensures conditions RR1–RR5, as well as the conditions of the sequential equilibrium, whatever the initial Bayesian game G and the certifiability level X. As mentioned before, our consistency condition is even stronger than Kreps and Wilson's (1982) condition. In particular, we assume that deviations are always interpreted as unilateral deviations. This is done for the sake of tractability more than for a hope of refinement or selection. Section 4 compares the two approaches.

For every player $j \in N$, given a commonly expected communication profile $c \in C$, let $\succeq_j$ be a complete, reflexive, and transitive ordering over the set H_j of player j's information sets. Hence, the pair $(H_j, \succeq_j)$ is a partially ordered set. The relation $h_j(\omega) \sim_j h_j(\omega')$ means that player j is equally likely to deviate (from his communication strategy c_j) at his information sets $h_j(\omega)$ and $h_j(\omega')$. When $h_j(\omega) \succ_j h_j(\omega')$, player j is infinitely more likely to deviate at $h_j(\omega)$ than at $h_j(\omega')$. Therefore, $(H_j, \succeq_j)$ represents the common interpretations of player j's deviation.[29] Players' inferences are obtained from this ordering in the following manner. Denote by $\mathcal{I}_j$ the partition of Ω generated by the equivalence relation $\sim_j$ over H_j. Let $I_i^j : C \times \mathcal{X} \times \Omega \to \mathcal{I}_j$ be the *interpretation function* of player i from j's deviation. For each communication strategy profile $c \in C$, each state $\omega \in \Omega$, and each vector of messages $x \in X(c, \omega)$, the set $I_i^j(c, x, \omega)$ gives the states player i considers as possible when interpreting j's deviation if the vector of messages x has been revealed but does not conform with the communication strategy profile c. More precisely, it is the set of possible states of the world for player i when he excludes the states in which player j is "infinitely less likely" to deviate.

For all $j \in N$ and $E \subseteq \Omega$, define

$$\text{Maxi}\{E \mid H_j, \succeq_j\} \equiv \left\{\omega \in E : h_j(\omega) \succeq_j h_j(\omega'),\ \forall\, \omega' \in E\right\}.$$

That is, $\text{Maxi}\{E \mid H_j, \succeq_j\}$ is the $\succeq_j$-maximum component (set of states of the world) of $\mathcal{I}_j$ in E. Given a deviation $x \in X(c, \omega)$, the interpretation function of player i from player j is defined by

$$I_i^j(c, x, \omega) \equiv \text{Maxi}\{h_i(\omega) \cap c_{-j}^{-1}(x_{-j}) \cap x_j \mid H_j, \succeq_j\}. \tag{4}$$

Of course, this interpretation will apply only if player i thinks that the deviant is player j.

[27] Our equilibrium concept can however be redefined by imposing only conditions RR1–RR5 (or even weaker conditions) on second stage information structures without difficulties.

[28] Shin (1994b) also proposes an explanation for non-partitional information structures in a context of strategic communication. However, in his example, information structures are non-partitional because he does not consider an information structure on the whole state space, but only on the payoff-relevant (fundamental) state space. Here, information might be non-partitional on the whole space of exogenous states of the world.

[29] It is worth noticing that the ordering on player j's information sets does not depend on the type of deviation. One can extend our framework by conditioning every $\succeq_j$ on player j's message $x_j \in \mathcal{X}_j$, but it is unnecessary for most applications. Our results do not depend on this restriction. The fact that a sequential equilibrium is not necessarily a knowledge equilibrium in Example 5 in the next section does not rely on this restriction.

This is for example the case when the deviation is j-identifiable by player i. When $N_i(c, x, \omega)$ is not a singleton, we will select a particular player in the set of potential deviants. This selection process will be the same for all players. This does not imply that all players agree on the same deviant, since some players can identify a deviant, without the others being able to identify him (this claim is illustrated in the next subsection, in Example 4).

Consider now a bijection $\rho : N \to N$. This bijection generates a permutation of the set of players N and induces a strict ordering on N, interpreted in the following way: if $\rho(i) > \rho(j)$, then player i is infinitely more likely to deviate than player j.[30] Hence, the player which is the most likely to deviate at ω for player i when the vector of messages $x \in \mathcal{X}$ has been revealed and corresponds to an observable deviation for player i is

$$\overline{N}_i(c, x, \omega \mid \rho) \in \arg\max_{k \in N_i(c,x,\omega)} \rho(k).$$

Of course, $\arg\max_{k \in N_i(c,x,\omega)} \rho(k)$ is always a singleton, and if x is j-identifiable by player i, then $\overline{N}_i(c, x, \omega \mid \rho) = j$, whatever the permutation ρ. We can now define the knowledge consistency condition. This condition is only defined for unilateral deviations. As mentioned before, since our equilibrium concept only allows unilateral deviations, it is without loss of generality to restrict our conditions on information structures which can be generated by unilateral deviations.[31]

Definition 5 (Knowledge Consistency) Consider a certifiability level X and a communication strategy profile c. A second stage information structure $\mathcal{P} = (\mathcal{P}_i)_{i \in N}$ is *consistent* with (c, X) if there exists a system of complete, reflexive, and transitive orderings $(H_k, \succeq_k)_{k \in N}$, and a bijection $\rho : N \to N$ such that for all $\omega \in \Omega$, $i \in N$, and $x \in X(c, \omega)$ we have

$$\mathcal{P}_i(x, \omega) = \begin{cases} h_i(\omega) \cap c^{-1}(x) & \text{if } h_i(\omega) \cap c^{-1}(x) \neq \emptyset \\ I_i^{\overline{N}_i(c,x,\omega|\rho)}(c, x, \omega) & \text{otherwise.} \end{cases} \tag{5}$$

Remark 5 Given that outside equilibrium information is explicitly characterized, it should be possible to strengthen our consistency condition by introducing criteria that exclude unreasonable states of the world given the message of each player *and* the payoffs' structure of the initial Bayesian game G. Such additional restrictions can be done by conditioning the orderings $(\succeq_k)_{k \in N}$ and the bijection ρ to the particular type of game considered.

As shown in the following proposition, a consistent information structure always satisfies revision rules RR1–RR5.

Proposition 4 *If the second stage information structure $\mathcal{P} = (\mathcal{P}_i)_{i \in N}$ is consistent with (c, X), then conditions RR1–RR5 are satisfied for all $i \in N$, $\omega \in \Omega$, and $x \in X(c, \omega)$.*

Proof. On the one hand, if x is not an observable deviation for player i at ω, then knowledge consistency is equivalent to RR2. Hence, conditions RR1, RR3, and RR4 are also satisfied by Proposition 3. (Condition RR5 is irrelevant for non-observable deviations.) On the other hand, if x is an observable deviation for player i at ω, then, from the definition of $I_i^j(c, x, \omega)$ given by Equation (4) we have, by construction, $I_i^j(c, x, \omega) = I_i^j(c, x, \omega')$ if $h_i(\omega) = h_i(\omega')$. Thus, $\mathcal{P}_i(x, \omega) = \mathcal{P}_i(x, \omega')$, i.e., admissible learning (condition RR3) is

[30] As for the ordering on information sets, this ordering could be generalized by conditioning it on the deviation $x \in \mathcal{X}$.

[31] The same is true for the sequential equilibrium. That is, to characterize a sequential equilibrium, the characterization of players' outside equilibrium beliefs at information sets which cannot be reached by unilateral deviations is irrelevant.

satisfied. Moreover, from the definition of the application Maxi, there exists $y_j \in \mathcal{Y}_j$ such that $I_i^j(c, x, \omega) = h_i(\omega) \cap c_{-j}^{-1}(x_{-j}) \cap x_j \cap y_j$, which shows that condition RR5 is satisfied. Finally, since RR5 is satisfied, Proposition 3 gives RR1 and RR4 (condition RR2 is irrelevant for observable deviations). □

In the next subsection we illustrate revision rules RR1–RR5 and the knowledge consistency condition. We also show that the revisions rules are not always sufficient to ensure the knowledge consistency condition.

3.2 Illustrations

Let $C(c) \equiv \{\tilde{c} \in C : \forall\ \omega \in \Omega,\ \exists\ i \in N,\ \tilde{c}_{-i}(\omega) = c_{-i}(\omega)\}$ be the set of unilateral deviations from the communication strategy profile $c \in C$. That is, if $\tilde{c} \in C(c)$, then $\tilde{c}(\omega) \in X(c, \omega)$. To scrutinize second stage information structures when a communication strategy profile $\tilde{c} \in C(c)$ is used, we define $\mathcal{P}_i^{\tilde{c}}(\omega) \equiv \mathcal{P}_i(\tilde{c}(\omega), \omega)$ for all $\omega \in \Omega$. That is, $\mathcal{P}_i^{\tilde{c}}(\omega)$ is the set of states player i conceives as possible in the second stage game at ω, when the profile of communication strategies $\tilde{c}$ has been used in the first stage game. With some abuse of language, we also call the function $\mathcal{P}_i^{\tilde{c}} : \Omega \to 2^{\Omega}\backslash\{\emptyset\}$ player i's possibility correspondence. Note that, by definition, if $\mathcal{P}$ is consistent with (c, X), then $\mathcal{P}_i^{c}(\omega) = h_i^c(\omega)$ for all $\omega \in \Omega$. We call the possibility correspondence $\mathcal{P}_i^{\tilde{c}}$ consistent with (c, X) if it can be obtained from a consistent second stage information structure $\mathcal{P} = (\mathcal{P}_i)_{i \in N}$.

Condition RR2 (Bayesian updating) implies that if a player j deviates at one of his information set by sending a message he should have sent in an other state (according to his communication strategy c_j), other players can make errors in information updating, i.e., we can have non-partitional and erroneous information structures. To see this, simply assume (as in the example with Alice and Bob on page 10) that $H_1 = \{\{\omega_1\}, \{\omega_2\}\}$, $H_2 = \{\{\omega_1, \omega_2\}\}$, $c_1(\omega_1) = \{\omega_1\}$, and $c_1(\omega_2) = \Omega$. We get $H_2^c = \{h_2^c(\omega_1), h_2^c(\omega_2)\} = \{\{\omega_1\}, \{\omega_2\}\}$. By construction, this information structure is partitional. Consider an other communication strategy profile $\tilde{c} \in C(c)$ satisfying $\tilde{c}_1(\omega_1) = \tilde{c}_1(\omega_2) = \Omega$. That is, player 1 deviates at ω_1 by revealing Ω instead of $c_1(\omega_1) = \{\omega_1\}$. In that case, if c remains the expected communication profile, the consistent possibility correspondence of player 2 verifies $\mathcal{P}_2^{\tilde{c}}(\omega_1) = \mathcal{P}_2^{\tilde{c}}(\omega_2)$ $(= \mathcal{P}_2(\Omega, \omega_1) = \mathcal{P}_2(\Omega, \omega_2)) = \{\omega_2\}$ by RR2. The information structure associated with the (outside equilibrium) possibility correspondence $\mathcal{P}_2^{\tilde{c}}$, when player 1 reveals Ω at ω_1 and ω_2, is clearly non-partitional and erroneous: $\mathcal{P}_2^{\tilde{c}}(\omega_1) \cup \mathcal{P}_2^{\tilde{c}}(\omega_2) \neq \Omega$, and at ω_1 player 2 knows $\{\omega_2\}$ (because $\mathcal{P}_2^{\tilde{c}}(\omega_1) = \{\omega_2\}$) which is a wrong knowledge (because $\omega_1 \notin \mathcal{P}_2^{\tilde{c}}(\omega_1)$).

Another situation in which players can have wrong knowledge occurs when they make interpretations from observable deviations which are finer than the interpretations restricted to the pure informational content of a message. For instance, if $c_1(\omega) = \{\omega\}$ and $\tilde{c}_1(\omega) = \Omega$ for all $\omega \in \Omega$ in the last example, then we can have $\mathcal{P}_2^{\tilde{c}}(\omega_1) = \mathcal{P}_2^{\tilde{c}}(\omega_2) = \{\omega_2\}$ if player 2 infers the event $\{\omega_2\}$ when he observes the message Ω from player 1 (this is obtained if $\{\omega_2\} \succ_1 \{\omega_1\}$). In this case, the reason for wrong knowledge differs from the last one where the deviation from c was not observable. As will be shown in Theorem 1, the only case in which we are sure that an information structure is correct (i.e., satisfies the truth axiom of knowledge) is when no deviation occurs at all (i.e., $\tilde{c} = c$) or when the deviation is identifiable and when players do not make interpretations which go beyond the pure informational content of the deviant's message (i.e., $\mathcal{I}_j = \{\Omega\}$ for all $j \in N$). From this theorem we also know that a partitional information structure is only guaranteed when no deviation occurs at all.

The fact that players update their outside equilibrium beliefs by conditioning on non-partitional and possibly erroneous information structures means that outside equilibrium, a substantial degree of information processing error might happen. Some players might be unaware of the subtle informational content of some messages by perceiving only their face value, by making false interpretations, or by applying erroneously Bayes' rule on unobservable deviations.

In the following example we give a detailed illustration of the knowledge consistency condition and we analyze the properties of the information structure along and outside the equilibrium path.

Example 2 Let $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6\}$, and let

$$\begin{aligned} H_1 &= \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}, \{\omega_5, \omega_6\}\}, \\ H_2 &= \{\{\omega_1\}, \{\omega_2, \omega_3, \omega_4, \omega_5\}, \{\omega_6\}\}, \end{aligned}$$

be the initial information partitions of players 1 and 2. Consider the following communication strategies:

$$\begin{array}{rcll} c_1(\omega) &=& \{\omega_1, \omega_2\}, & \text{if } \omega \in \{\omega_1, \omega_2\}, \\ c_1(\omega) &=& \{\omega_3, \omega_4\}, & \text{if } \omega \in \{\omega_3, \omega_4\}, \\ c_1(\omega) &=& \Omega, & \text{if } \omega \in \{\omega_5, \omega_6\}, \\ c_2(\omega) &=& \Omega, & \text{for all } \omega \in \Omega. \end{array}$$

Player 1 reveals all his information at ω_1, ω_2, ω_3 and ω_4, he reveals nothing at ω_5 and ω_6, and player 2 never reveals anything. We obtain $H_1^c = H_1$ and

$$H_2^c = \{\{\omega_1\}, \{\omega_2\}, \{\omega_3, \omega_4\}, \{\omega_5\}, \{\omega_6\}\},$$

since

$$W(c) = \{c^{-1}(c(\omega)) : \omega \in \Omega\} = \{c_1^{-1}(c_1(\omega)) : \omega \in \Omega\} = \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}, \{\omega_5, \omega_6\}\}.$$

Let ω_3 be the real state of the world and assume that player 1 deviates from his communication strategy c_1 to the communication strategy $\tilde{c}_1$ characterized by

$$\tilde{c}_1(\omega) = \begin{cases} \{\omega_1, \omega_2, \omega_3, \omega_4\} & \text{if } \omega \in \{\omega_3, \omega_4\} \\ c_1(\omega) & \text{otherwise.} \end{cases} \tag{6}$$

This deviation is observable and 1-identifiable by both players at ω_3. Player 1's knowledge is unchanged but the (outside equilibrium) possibility correspondence of player 2 must verify $\mathcal{P}_2^{\tilde{c}}(\omega_3)$ $(= \mathcal{P}_2(\tilde{c}_1(\omega_3), \omega_3)) \subseteq h_2(\omega_3) \cap \tilde{c}_1(\omega_3) = \{\omega_2, \omega_3, \omega_4\}$ by the certifiability constraint (condition RR1). Moreover, we have either $\mathcal{P}_2^{\tilde{c}}(\omega_3) = \{\omega_2\}$, or $\mathcal{P}_2^{\tilde{c}}(\omega_3) = \{\omega_3, \omega_4\}$, or $\mathcal{P}_2^{\tilde{c}}(\omega_3) = \{\omega_2, \omega_3, \omega_4\}$ by admissible interpretation (condition RR4).[32]

Assume that the prior probabilities are $p(\omega) = 1/6$ for all $\omega \in \Omega$. At ω_3, without player 1's deviation, player 2 necessarily believes both states ω_3 and ω_4 with probability 1/2 (because $h_2^c(\omega_3) = \{\omega_3, \omega_4\}$, and thus $p(\omega_3 \mid \{\omega_3, \omega_4\}) = p(\omega_4 \mid \{\omega_3, \omega_4\}) = 1/2$). After the deviation, player 2 might consider the states ω_2, ω_3 and ω_4 as possible (when $\mathcal{P}_2^{\tilde{c}}(\omega_3) = \{\omega_2, \omega_3, \omega_4\}$) and should believe each of those states with probability 1/3 (because $p(\omega \mid \{\omega_2, \omega_3, \omega_4\}) = 1/3$ for $\omega \in \{\omega_2, \omega_3, \omega_4\}$). Then, if the deviation to $\tilde{c}_1$ occurs, the information of player 2 is non-partitional: his possibility set at ω_2 is $\mathcal{P}_2^{\tilde{c}}(\omega_2) = h_2^c(\omega_2) = \{\omega_2\}$ by Bayesian updating (condition RR2) whereas his possibility set at ω_3 is $\mathcal{P}_2^{\tilde{c}}(\omega_3) = \{\omega_2, \omega_3, \omega_4\}$. Nevertheless, in

[32] The first inference $\mathcal{P}_2^{\tilde{c}}(\omega_3) = \{\omega_2\}$ is obtained when $\{\omega_1, \omega_2\} \succ_1 \{\omega_3, \omega_4\}$; the second inference $\mathcal{P}_2^{\tilde{c}}(\omega_3) = \{\omega_3, \omega_4\}$ is obtained when $\{\omega_1, \omega_2\} \prec_1 \{\omega_3, \omega_4\}$; finally, the inference $\mathcal{P}_2^{\tilde{c}}(\omega_3) = \{\omega_2, \omega_3, \omega_4\}$ is obtained when $\{\omega_1, \omega_2\} \sim_1 \{\omega_3, \omega_4\}$.

this case, player 2 does not exclude the true state at ω_3. He does exclude the true state, however, if $\mathcal{P}_2^{\tilde{c}}(\omega_3) = \{\omega_2\}$ because $\omega_3 \notin \mathcal{P}_2^{\tilde{c}}(\omega_3)$.

Now, let ω_1 be the real state and assume that player 2 deviates from the communication strategy c_2 to

$$\tilde{c}_2(\omega) = \begin{cases} \{\omega_1\} & \text{if } \omega \in \{\omega_1\} \\ c_2(\omega) & \text{otherwise.} \end{cases}$$

This deviation is observable and 2-identifiable by both players. Moreover, the outside equilibrium belief of player 1 is uniquely defined by the certifiability constraint, and we necessarily get $\mathcal{P}_1^{\tilde{c}}(\omega_1) = \{\omega_1\}$. With such a deviation player 1 necessarily knows the event $\{\omega_1\}$ at ω_1 although he did not know it with the initial communication strategies. Notice that the outside equilibrium information structure is also non-partitional because $\mathcal{P}_1^{\tilde{c}}(\omega_2) = h_1^c(\omega_2) = \{\omega_1, \omega_2\}$. However, player 1 has never a wrong knowledge, i.e., never knows an unrealized event.

To illustrate that conditions RR1–RR5 may not be sufficient for knowledge consistency, consider a third player with the partition

$$H_3 = \{\{\omega_1, \omega_2, \omega_3\}, \{\omega_4\}, \{\omega_5, \omega_6\}\},$$

and (for simplicity) without communication possibilities (i.e., $\mathcal{X}_3 = \{\Omega\}$). Assume that player 1 deviates according to (6). The deviation is also 1-identifiable by player 3. We have seen that there are three consistent revisions for player 2 at ω_3. These revisions imply the following conditions on the ordering $\succeq_1$:

(i) $\mathcal{P}_2^{\tilde{c}}(\omega_3) = \{\omega_2\} \Rightarrow \{\omega_1, \omega_2\} \succ_1 \{\omega_3, \omega_4\}$;

(ii) $\mathcal{P}_2^{\tilde{c}}(\omega_3) = \{\omega_3, \omega_4\} \Rightarrow \{\omega_3, \omega_4\} \succ_1 \{\omega_1, \omega_2\}$;

(iii) $\mathcal{P}_2^{\tilde{c}}(\omega_3) = \{\omega_2, \omega_3, \omega_4\} \Rightarrow \{\omega_1, \omega_2\} \sim_1 \{\omega_3, \omega_4\}$.

In case (i) we get, from the definition of the interpretation function (4),

$$\begin{aligned} \mathcal{P}_3^{\tilde{c}}(\omega_3) &= \mathcal{P}_3(\tilde{c}(\omega_3), \omega_3) = I_3^1(c, \tilde{c}(\omega_3), \omega_3) \\ &= \big\{\omega \in h_3(\omega_3) \cap \{\omega_1, \omega_2, \omega_3, \omega_4\} : h_1(\omega) \succeq_1 h_1(\omega'),\ \forall\, \omega' \in \{\omega_1, \omega_2, \omega_3\}\big\} \\ &= \{\omega_1, \omega_2\}. \end{aligned}$$

Similarly, in case (ii) we have $\mathcal{P}_3^{\tilde{c}}(\omega_3) = \{\omega_3\}$, and in case (iii) we have $\mathcal{P}_3^{\tilde{c}} = \{\omega_1, \omega_2, \omega_3\}$. These relations between players' inferences are clearly not imposed by RR1–RR5, but will be necessary to get a sequential equilibrium (see Section 4).

The following example illustrates condition RR5 by showing some restrictions it imposes on players' revised knowledge.

Example 3 Let

$$\begin{aligned} H_1 &= \{\{\omega_1\}, \{\omega_2, \omega_3\}, \{\omega_4\}\} \\ H_2 &= \{\{\omega_1, \omega_2\}, \{\omega_3\}, \{\omega_4\}\} \\ H_3 &= \{\{\omega_1, \omega_2, \omega_3, \omega_4\}\}, \end{aligned}$$

and consider the following anticipated profile of communication strategies:

$$
\begin{aligned}
&c_1(\omega_1) = \{\omega_1\}\\
&c_1(\omega_2) = c_1(\omega_3) = c_1(\omega_4) = \{\omega_2, \omega_3, \omega_4\}\\
&c_2(\omega_1) = c_2(\omega_2) = c_2(\omega_3) = \{\omega_1, \omega_2, \omega_3\}\\
&c_2(\omega_4) = \{\omega_4\}.
\end{aligned}
$$

Let $\tilde{c} \in C(c)$ be the unilateral deviation from c satisfying

$$
\begin{aligned}
&\tilde{c}_1(\omega) = \Omega && \text{for all } \omega \in \Omega\\
&\tilde{c}_2(\omega) = c_2(\omega) && \text{for all } \omega \in \Omega.
\end{aligned}
$$

This deviation is always observable and 1-identifiable. Assume that the real state is $\omega \neq \omega_4$, i.e, the vector of observed messages is $\tilde{c}(\omega) = (\Omega, \{\omega_1, \omega_2, \omega_3\})$. It is easy to verify that player 3's possibility correspondence characterized by $\mathcal{P}_3^{\tilde{c}}(\omega) = \{\omega_1, \omega_2\}$ or $\mathcal{P}_3^{\tilde{c}}(\omega) = \{\omega_3\}$ satisfies conditions RR1–RR4, but not condition RR5. The only possibility correspondence satisfying conditions RR1–RR5 must be characterized by $\mathcal{P}_3^{\tilde{c}}(\omega) = \{\omega_1\}$ or $\mathcal{P}_3^{\tilde{c}}(\omega) = \{\omega_1, \omega_2, \omega_3\}$ or $\mathcal{P}_3^{\tilde{c}}(\omega) = \{\omega_2, \omega_3\}$.[33]

The following example shows that a deviation can be identified by a player, without being identifiable by an other. Revised knowledge is characterized in such a setting.

Example 4 Let

$$
\begin{aligned}
H_1 &= \{\{\omega_1\}, \{\omega_2, \omega_3\}, \{\omega_4\}\}\\
H_2 &= \{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_4\}\}\\
H_3 &= \{\{\omega_1, \omega_2, \omega_3, \omega_4\}\}\\
H_4 &= \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}\},
\end{aligned}
$$

and consider the following communication strategies of players 1 and 2 (communication of players 3 and 4 is ignored):

$$
c_1(\omega) = \begin{cases} \Omega & \text{if } \omega = \omega_4 \\ h_1(\omega) & \text{otherwise} \end{cases} \qquad c_2(\omega) = \begin{cases} \Omega & \text{if } \omega = \omega_1 \\ h_2(\omega) & \text{otherwise.} \end{cases}
$$

Consider the following deviation from c: $\tilde{c}_1(\omega_1) = \Omega$, $\tilde{c}_1(\omega) = c_1(\omega)$ for all $\omega \neq \omega_1$, $\tilde{c}_2(\omega_4) = \Omega$, and $\tilde{c}_2(\omega) = c_2(\omega)$ for all $\omega \neq \omega_4$. Let $\omega = \omega_1$ be the real state. In this case, the deviation is not identifiable by player 3 since $N_3(c, \tilde{c}(\omega_1), \omega_1) = \{1, 2\}$. On the contrary, it is 1-identifiable by player 4 at ω_1 since $N_4(c, \tilde{c}(\omega_1), \omega_1) = \{1\}$. Consider the bijection ρ satisfying $\rho(i) = i$ for all $i \in N$. Hence, $\overline{N}_3(c, \tilde{c}(\omega_1), \omega_1 \mid \rho) = 2$. Consequently, player 3 will interpret the deviation as a deviation by player 2, whereas player 4 will interpret the deviation as a deviation by player 1. Whatever the orderings $(\succeq_j)_j$ over players' partitions, we obtain $\mathcal{P}_3^{\tilde{c}}(\omega_1) = \{\omega_4\}$ and $\mathcal{P}_4^{\tilde{c}}(\omega_1) = \{\omega_1\}$. It is interesting to note that, in this example, player 4 does not make errors in information processing, contrary to player 3, the reason being that player 4 can correctly identify the deviant, whereas player 3 cannot identify it correctly because his initial information (represented by H_3) is not sufficient to make such an identification.

[33] It can be shown, *in this example*, that Kreps and Wilson's (1982) consistency condition imposes a similar condition, mainly that beliefs about ω_2 and ω_3 are the same for the third player. Example 7 illustrates a similar configuration in which our condition differs from Kreps and Wilson's (1982) one.

3.3 Rationality of Consistent Knowledge

Usually, economists assume that agents are rational both in their act and in their representation. This second assumption requires that agents' knowledge is derived from consistent and perfectly rational (but possibly incomplete) inferences. Rational inferences are often associated with strong requirements on agents' knowledge. Agents' knowledge is generally described by *knowledge operators*. Remember that a knowledge operator for player $i \in N$ is a function $K_i^{\tilde{c}} : 2^\Omega \to 2^\Omega$. Given a second stage information structure $\mathcal{P} = (\mathcal{P}_i)_{i \in N}$ and a profile of communication strategies $\tilde{c} \in C$, each player i's knowledge operator is characterized by

$$K_i^{\tilde{c}} E \equiv \{\omega \in \Omega : \mathcal{P}_i^{\tilde{c}}(\omega) \subseteq E\}, \quad \forall\, E \in 2^\Omega,$$

where, as before, $\mathcal{P}_i^{\tilde{c}}(\omega) \equiv \mathcal{P}_i(\tilde{c}(\omega), \omega)$.

In this subsection we analyze the properties of second stage information structures on Ω by investigating if they satisfy the five axioms assumed in the standard model of knowledge. As was already mentioned and illustrated, it turns out that few of these axioms are satisfied in the continuation Bayesian games, except in the continuation Bayesian game of the equilibrium path. These axioms are given below.

(A0) $K_i^{\tilde{c}} \Omega = \Omega$: an agent always knows that the universal event Ω is realized.

(A1) $K_i^{\tilde{c}}(E \cap F) = K_i^{\tilde{c}} E \cap K_i^{\tilde{c}} F$ (distribution axiom): an agent knows E and F iff he knows E and he knows F.

(A2) $K_i^{\tilde{c}} E \subseteq E$ (non delusion or truth axiom): all what an agent knows is true.

(A3) $K_i^{\tilde{c}} E \subseteq K_i^{\tilde{c}} K_i^{\tilde{c}} E$ (positive introspection axiom): if an agent knows E, then he knows that he knows E.

(A4) $\overline{K_i^{\tilde{c}}} E \subseteq K_i^{\tilde{c}} \overline{K_i^{\tilde{c}}} E$ (negative introspection axiom): if an agent does not know E, then he knows that he does not know it.

The next theorem shows that, outside equilibrium, non delusion and introspection axioms are not necessarily satisfied. Nevertheless, if there are only two states of the world, the positive introspection axiom is always satisfied. Moreover, when the deviation can be identified and players restrict their interpretation to the pure informational content of the deviant's message (and continue to apply Bayes' rule on others' communication strategies), the truth axiom is satisfied. Finally, and not surprisingly, all axioms are satisfied along the equilibrium path, i.e., when no player deviates from his communication strategy. Axioms (A0) and (A1) will always be satisfied by construction of the knowledge operator.

Theorem 1 *Let $\mathcal{P} = (\mathcal{P}_i)_{i \in N}$ be a second stage information structure consistent with (c, X), let $\tilde{c} \in C(c)$ be a unilateral deviation from c, and let $(K_i^{\tilde{c}})_{i \in N}$ be the associated knowledge operators. Then, for all $i \in N$:*

(i) $K_i^{\tilde{c}}$ satisfies (A0) and (A1);

(ii) If $\tilde{c} = c$, then $K_i^{\tilde{c}}$ satisfies (A0)–(A4);

(iii) If $\tilde{c} \neq c$, then $K_i^{\tilde{c}}$ does not necessarily satisfy (A2), (A3) and (A4);

(iv) If $|\Omega| = 2$, then $K_i^{\tilde{c}}$ satisfies (A3);

(v) If $\tilde{c}(\omega)$ is an identifiable deviation by player i at ω and $\mathcal{I}_j = \{\Omega\}$ for all $j \in N$, then $\omega \in \mathcal{P}_i^{\tilde{c}}(\omega)$. Hence, if $\tilde{c}(\omega)$ is identifiable by player i in all states $\omega \in \Omega$ and if $\mathcal{I}_j = \{\Omega\}$ for all $j \in N$, then $K_i^{\tilde{c}}$ satisfies (A2).

Proof.

(i) This property results directly from the definition of the knowledge operator.

(ii) By construction, $\{\mathcal{P}_i^c(\omega) : \omega \in \Omega\} = \{h_i^c(\omega) : \omega \in \Omega\} = H_i^c$ is a partition of Ω and $\omega \in \mathcal{P}_i^c(\omega)$ for all $\omega \in \Omega$. The fact that such a partitional information structure satisfies axioms (A0)–(A4) is a well known result (actually, a partitional and correct information structure is equivalent to (A0)–(A4); see, for example, Geanakoplos, 1994).

(iii) The fact that the truth axiom is not necessarily satisfied when $\tilde{c} \neq c$, even when $|\Omega| = 2$, was shown with the example of Alice and Bob. Indeed, in this example, if $c_1(\omega_1) = \{\omega_1\}$, $c_1(\omega_2) = \Omega$, and $\tilde{c}_1(\omega) = \Omega$ for all $\omega \in \Omega$, then $\mathcal{P}_2^{\tilde{c}}(\omega_1) = \{\omega_2\}$. Hence, $K_2^{\tilde{c}}\{\omega_2\} = \{\omega_1\} \nsubseteq \{\omega_2\}$. To see that the negative introspection axiom may not be satisfied let $c_1(\omega) = \{\omega\}$ for all $\omega \in \Omega$, $\tilde{c}_1(\omega_1) = \{\omega_1\}$ and $\tilde{c}_1(\omega_2) = \Omega$. Then we can have $\mathcal{P}_2^{\tilde{c}}(\omega_1) = \{\omega_1\}$ and $\mathcal{P}_2^{\tilde{c}}(\omega_2) = \Omega$. Hence, $\overline{K_2^{\tilde{c}}\{\omega_1\}} = \{\omega_2\} \nsubseteq K_2^{\tilde{c}}\overline{K_2^{\tilde{c}}\{\omega_1\}} = \emptyset$. To show that the positive introspection axiom may not be satisfied, consider three states of the world[34] $\Omega = \{\omega_1, \omega_2, \omega_3\}$ and let $H_1 = \{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}\}$, $H_2 = \{\Omega\}$, $c_1(\omega) = \Omega$ for all $\omega \in \Omega$, $\tilde{c}_1(\omega_1) = \{\omega_1, \omega_2\}$ and $\tilde{c}_1(\omega) = \Omega = c(\omega)$ if $\omega \neq \omega_1$. In this case, we can have $\mathcal{P}_2^{\tilde{c}}(\omega_1) = \{\omega_1, \omega_2\}$ and $\mathcal{P}_2^{\tilde{c}}(\omega) = \Omega$ for $\omega \neq \omega_1$. We get $K_2^{\tilde{c}}\{\omega_1, \omega_2\} = \{\omega_1\} \nsubseteq K_2^{\tilde{c}}K_2^{\tilde{c}}\{\omega_1, \omega_2\} = \emptyset$.

(iv) Let $\Omega = \{\omega_1, \omega_2\}$. Note first that for any $E \subseteq \Omega$ we either have $K_i^{\tilde{c}}E = \Omega$, or $K_i^{\tilde{c}}E = \emptyset$, or $K_i^{\tilde{c}}E = \{\omega\}$ for some $\omega \in \Omega$. In the first case we necessarily have $K_i^{\tilde{c}}K_i^{\tilde{c}}E = \Omega$ and thus $K_i^{\tilde{c}}E \subseteq K_i^{\tilde{c}}K_i^{\tilde{c}}E$. In the second case we necessarily have $K_i^{\tilde{c}}K_i^{\tilde{c}}E = \emptyset$, and thus positive introspection is also satisfied. Thus, assume that $K_i^{\tilde{c}}E = \{\omega\}$ for some $\omega \in \Omega$. Hence, either $K_i^{\tilde{c}}E = \{\omega_1\}$ or $K_i^{\tilde{c}}E = \{\omega_2\}$. Without loss of generality let $K_i^{\tilde{c}}E = \{\omega_1\}$. If $E = \{\omega_1\}$, then $K_i^{\tilde{c}}K_i^{\tilde{c}}E = \{\omega_1\}$, i.e., $K_i^{\tilde{c}}K_i^{\tilde{c}}E \subseteq K_i^{\tilde{c}}E$. If $E = \{\omega_2\}$ then $\mathcal{P}_i^{\tilde{c}}(\omega_1) = \{\omega_2\}$ from the definition of the knowledge operator. In this case, positive introspection is not satisfied iff

$$\begin{aligned}
& K_i^{\tilde{c}}E \nsubseteq K_i^{\tilde{c}}K_i^{\tilde{c}}E \\
\Leftrightarrow\quad & \{\omega_1\} \nsubseteq K_i^{\tilde{c}}\{\omega_1\} \\
\Leftrightarrow\quad & K_i^{\tilde{c}}\{\omega_1\} = \{\omega_2\} \text{ or } K_i^{\tilde{c}}\{\omega_1\} = \emptyset \\
\Leftrightarrow\quad & \mathcal{P}_i^{\tilde{c}}(\omega_2) = \{\omega_1\} \text{ or } \mathcal{P}_i^{\tilde{c}}(\omega_2) = \Omega.
\end{aligned}$$

Since player i does not know the true state both at ω_1 and ω_2, we necessarily have $\bigcap_{i \in N} \tilde{c}_i(\omega) = \Omega$ for all $\omega \in \Omega$ by the certifiability constraint (condition RR1, which is necessarily satisfied from Proposition 4). But then, from admissible revision (condition RR3) we have $\mathcal{P}_i^{\tilde{c}}(\omega_1) = \mathcal{P}_i^{\tilde{c}}(\omega_2)$, leading to a contradiction.

(v) Let j be the identified deviant at ω. Then, from the knowledge consistency condition we get (since an identifiable deviation by i is also observable by i):

$$\mathcal{P}_i^{\tilde{c}}(\omega) = \text{Maxi}\{h_i(\omega) \cap c_{-j}^{-1}(\tilde{c}_{-j}(\omega)) \cap \tilde{c}_j(\omega) \mid H_j, \succeq_j\}.$$

Since by assumption $\mathcal{I}_j = \{\Omega\}$, this gives

$$\mathcal{P}_i^{\tilde{c}}(\omega) = h_i(\omega) \cap c_{-j}^{-1}(\tilde{c}_{-j}(\omega)) \cap \tilde{c}_j(\omega).$$

Moreover, from Lemma 2 in Appendix B we have $\tilde{c}_k(\omega) = c_k(\omega)$ for all $k \neq j$. Thus, $\omega \in c_{-j}^{-1}(\tilde{c}_{-j}(\omega))$. Therefore, since $\omega \in h_i(\omega)$ and $\omega \in \tilde{c}_j(\omega)$, we obtain $\omega \in \mathcal{P}_i^{\tilde{c}}(\omega)$. If this is true for all $\omega \in \Omega$, then the condition $\mathcal{P}_i^{\tilde{c}}(\omega) \subseteq E$ implies $\omega \in E$, i.e., (A2) is satisfied. This completes the proof. $\square$

[34] Three states are necessary to get a failure of the positive introspection axiom given property (iv).

3.4 Equilibrium of the Complete Game

Having defined consistency of second stage information structures, we can now define a related equilibrium for the complete game with the communication stage and the second stage Bayesian games based on information consistent with previous messages.

Given a second stage information structure $\mathcal{P} = (\mathcal{P}_i)_{i\in N}$ and a profile of effective strategies $\sigma \in \Sigma$, player i's expected utility in the continuation Bayesian game following a vector of messages $x \in \mathcal{X}$ is given by

$$\begin{aligned} U_i(\sigma, x, \mathcal{P}_i, \omega) &\equiv E_p\big(u_i(\sigma, x, \cdot) \mid \mathcal{P}_i(x, \omega)\big) = \sum_{\omega' \in \Omega} p\left(\omega' \mid \mathcal{P}_i(x,\omega)\right) u_i(\sigma, x, \omega') \\ &= \sum_{\omega' \in \Omega} p\left(\omega' \mid \mathcal{P}_i(x,\omega)\right) \sum_{a \in A} \sigma(a \mid x, \omega')\, u_i(a, \omega'). \end{aligned} \tag{7}$$

A profile of effective strategies $\sigma \in \Sigma$ is at equilibrium in the continuation Bayesian game generated by the vector of messages $x \in \mathcal{X}$ if we have

$$U_i(\sigma, x, \mathcal{P}_i, \omega) \geq U_i(a_i, \sigma_{-i}, x, \mathcal{P}_i, \omega), \quad \forall\, i \in N,\ \omega \in \bigcap_{k\in N} x_k,\ a_i \in A_i. \tag{8}$$

The set of effective strategy profiles satisfying Equation (8) is denoted by $\Sigma^*(\mathcal{P}, x) \subseteq \Sigma$. Accordingly, the set of effective strategy profiles satisfying Equation (8) for all $x \in \mathcal{X}$ is denoted by $\Sigma^*(\mathcal{P}) = \bigcap_{x\in\mathcal{X}} \Sigma^*(\mathcal{P}, x) \subseteq \Sigma$. Hence, an effective strategy profile $\sigma \in \Sigma^*(\mathcal{P})$ satisfies sequential rationality at the second stage game when the second stage information structure is $\mathcal{P}$. That is, the profile of effective strategies always forms some kind of Bayesian equilibrium of the second stage game given the information structure generated by the profile of communication strategies (*outside* and along the equilibrium path). This is the first condition for a knowledge equilibrium. According to the second condition, each player $i \in N$ has never any incentive to change his communication strategy given the second stage strategies and others' communication behaviors. In a state $\omega \in \Omega$, if player i communicates $x_i \in X_i(\omega)$ instead of $c_i(\omega) \in X_i(\omega)$, his expected utility at the beginning of the first stage game, given the second stage information structure $\mathcal{P}$ and the profile of effective and optimal strategies $\sigma \in \Sigma^*(\mathcal{P})$, does not strictly increase. The third condition is the condition of knowledge consistency (Definition 5).

Definition 6 A *knowledge equilibrium* of the game (G, X) is a profile of effective strategies $\sigma \in \Sigma$, a profile of communication strategies $c \in C$, and a second stage information structure $\mathcal{P} = (\mathcal{P}_i)_{i\in N}$ satisfying the following conditions:

1. *Second Stage Rationality*: $\sigma \in \Sigma^*(\mathcal{P})$;

2. *Rational Communication*: For all $i \in N$, $\omega \in \Omega$, and $x_i \in X_i(\omega)$,

$$EU_i\big(\sigma, c, \mathcal{P}_i \mid h_i(\omega)\big) \equiv \sum_{\omega' \in \Omega} p\big(\omega' \mid h_i(\omega)\big)\, U_i\big(\sigma, c(\omega'), \mathcal{P}_i, \omega'\big) \geq$$
$$EU_i\big(\sigma, x_i, c_{-i}, \mathcal{P}_i \mid h_i(\omega)\big) \equiv \sum_{\omega' \in \Omega} p\big(\omega' \mid h_i(\omega)\big)\, U_i\big(\sigma, x_i, c_{-i}(\omega'), \mathcal{P}_i, \omega'\big);$$

3. *Consistent Knowledge*: $\mathcal{P}$ is consistent with (c, X).

Note that if $(\sigma, c, \mathcal{P})$ is a knowledge equilibrium, then there exists a Bayesian equilibrium $\phi \in \Phi^*(h^c)$ of the Bayesian game $G(h^c)$ such that $\sigma(c(\omega), \omega) = \phi(\omega)$ for all $\omega \in \Omega$. That is, along the equilibrium path, the second stage strategies correspond to a Bayesian equilibrium of the game $G(h^c)$. If $c(\omega) = \Omega$ for all $\omega \in \Omega$ and $i \in N$, then the information structure is not modified (i.e., $h^c = h$) and thus payoff-relevant actions chosen along the equilibrium path will be the same as those induced by a Bayesian equilibrium of the initial Bayesian game G without communication.

4 Knowledge and Sequential Equilibria

In this section we analyze some links between knowledge equilibria and sequential equilibria. Theorem 2 shows our main result: a knowledge equilibrium of a Bayesian game G given a certifiability level X is always a Kreps and Wilson's (1982) sequential equilibrium of the communication game (G, X). Thus, existence of particular knowledge equilibria (as perfectly revealing or non-revealing ones) in some classes of games will imply existence of sequential equilibria with the same properties. Furthermore, we keep all properties of sequential equilibria without referring to sequences of perturbed and strictly positive strategy profiles. Additionally, we show in Proposition 5 that if informative disclosures are restrained to complete disclosures (i.e., for all $\omega \in \Omega$ and $i \in N$ either $c_i(\omega) = \Omega$ or $c_i(\omega) = h_i(\omega)$), then the set of non-revealing knowledge equilibria and the set of non-revealing sequential equilibria coincide. Several examples illustrate the difference between the two approaches. This difference stems from the restrictions imposed by the knowledge equilibrium on outside equilibrium beliefs which are not required by Kreps and Wilson's (1982) consistency condition.

Theorem 2 *If a profile of communication and effective strategies forms a knowledge equilibrium of the communication game (G, X), then it also forms a sequential equilibrium.*

Proof. See Appendix C. □

Remark 6 As already mentioned, if players' beliefs are compatible with the certifiability constraint and are obtained by Bayes' rule whenever possible, then these beliefs are *not* necessarily consistent. Hence, they are *not* sufficient to get a sequential or knowledge equilibrium. Such weak and basic conditions on beliefs are used in almost all existing papers on strategic information revelation. They are not sufficient here because we consider substantially more general and complex information structures. In particular, the corresponding Harsanyi's (1967–1968) information structure of the initial partitional information structure $h = (h_i)_{i \in N}$ involves implicitly correlated types. (See Fudenberg and Tirole, 1991 for more details on the links between various consistency conditions and Kreps and Wilson's consistency condition.)

Remark 7 Since a knowledge equilibrium only admits pure communication strategies, such an equilibrium might not exist (contrary to the sequential equilibrium defined in Appendix A). However, a knowledge equilibrium with pure communication strategies remains an equilibrium when players are allowed to use mixed communication strategies. Thus, considering pure communication strategies is w.l.o.g. for the existence of, e.g., perfectly revealing or non-revealing equilibria. Moreover, we expect that most models and applications considered in the literature on strategic information revelation apply equivalently by using the knowledge equilibrium, with certainly substantially more tractability.

The following very simple example shows that a sequential equilibrium—even in pure communication and effective strategies—is not necessarily a knowledge equilibrium. The reason is that, contrary to the sequential equilibrium, a knowledge equilibrium can be supported only by a finite set of outside equilibrium beliefs, and these beliefs depend on prior probabilities. This feature may be seen either as an advantage or as a restriction of our consistency condition. In the sequential equilibrium, prior beliefs only matter along the equilibrium path, but not at information sets reached with zero probability. In the knowledge equilibrium, prior beliefs always characterize a player's second stage belief (after this player has excluded some states, of course).

Example 5 Let $N = \{1, 2\}$ be a set of players, $\Omega = \{\omega_1, \omega_2\}$ a payoff-relevant state space, $p(\omega_1) = p(\omega_2) = 1/2$ a prior probability distribution on Ω, and $H_1 = \{\{\omega_1\}, \{\omega_2\}\}$ and $H_2 = \{\{\omega_1, \omega_2\}\}$ an initial information structure. Therefore, only player 1 is informed about the payoff-relevant state and it is common knowledge that he is informed. His communication strategy is denoted by $c : \Omega \to \mathcal{X}_1$. Consider the initial Bayesian game of Figure 2 (where payoff-relevant actions A, B, C, and D are only available to player 2).

ω_1	A	B	C	D
	$(0, 6)$	$(1, 5)$	$(-2, 0)$	$(1, -6)$

ω_2	A	B	C	D
	$(1, -6)$	$(1, 1)$	$(-2, 2)$	$(0, 3)$

Figure 2: Bayesian Game of Example 5.

In the unique Bayesian equilibrium of this game, player 2 plays B, i.e., $\phi_2(B \mid \omega) = 1$ for all $\omega \in \Omega$. Adding the communication stage, and assuming perfect certifiability (i.e., $\mathcal{X}_1 = \mathcal{Y}_1 = \{\{\omega_1\}, \{\omega_2\}, \{\omega_1, \omega_2\}\}$) we can represent the complete game in extensive form as in Figure 3.

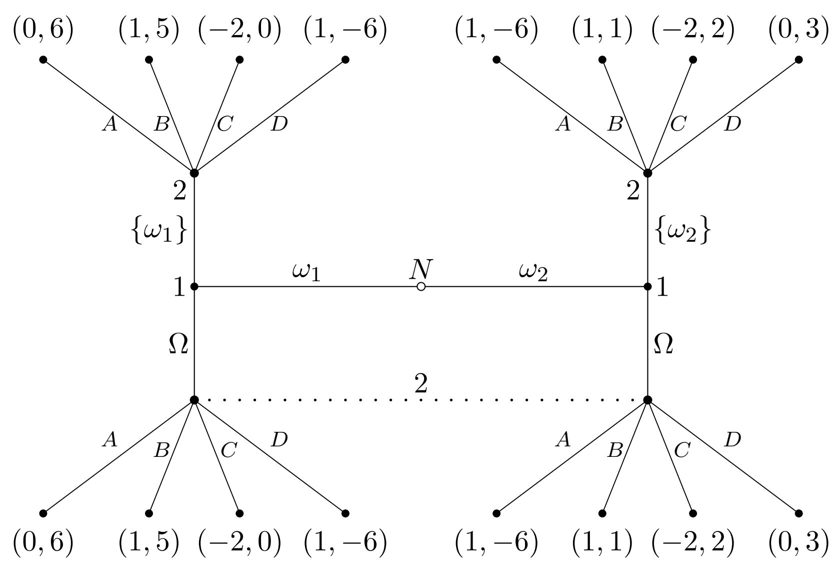

Figure 3: Two-Stage Extensive Form Game of Example 5.

We can easily verify that there exists a non-revealing knowledge equilibrium and sequential equilibrium, i.e., an equilibrium where player 1 reveals $c(\omega) = \Omega$ in any state

$\omega \in \Omega$. In this case, outside equilibrium beliefs and possibility correspondences are unique (by the certifiability constraint) and player 2 plays B if he receives the message $x = \Omega$, A if he receives the message $x = \{\omega_1\}$, and D if he receives the message $x = \{\omega_2\}$ (i.e., $\sigma_2(B \mid \Omega, \omega) = \sigma_2(A \mid \{\omega_1\}, \omega) = \sigma_2(D \mid \{\omega_2\}, \omega) = 1$ for all $\omega \in \Omega$).

There is also a perfectly revealing sequential equilibrium where player 1 reveals $c(\omega) = \{\omega\}$ in any state $\omega \in \Omega$. To support this equilibrium player 2 has to play C when he receives an outside equilibrium message, i.e., when he receives the message $x = \Omega$ from player 1. Player 2 plays this action if and only if his belief about ω_1 when he receives the message Ω belongs to the interval $[\frac{1}{7}, \frac{1}{6}]$. Such an outside equilibrium belief cannot be achieved with our approach in terms of possibility correspondences. Indeed, we have either $\mathcal{P}_2(\Omega, \omega) = \{\omega\}$, or $\mathcal{P}_2(\Omega, \omega) = \Omega$, or $\mathcal{P}_2(\Omega, \omega) = \Omega \backslash \{\omega\}$. In any case, player 2's belief about ω is either 1, 1/2 ($= p(\omega)$), or 0. Therefore, he never plays action C, and thus player 1 has always an incentive to deviate from full revelation in at least one state.[35] As mentioned before, this can be seen either as a weakness or as a advantage of our consistency condition because it restricts significantly the set of outside equilibrium beliefs. Other restrictions not required by the sequential equilibrium, but required by our knowledge consistency condition are presented in the following examples. Contrary to the present example, these restrictions are due to the fact that we do not allow players to interpret an observable deviation as a multilateral deviation. As already mentioned, multilateral deviations could be allowed in our model without substantial difficulties (see Footnote 25).

Example 6 Let $N = \{1, 2, 3, 4\}$, $\Omega = \{\omega_1, \omega_2\}$, $H_1 = H_2 = H_3 = \{\{\omega_1\}, \{\omega_2\}\}$, $H_4 = \{\Omega\}$, $c_1(\omega_1) = c_2(\omega_1) = \{\omega_1\}$, $c_3(\omega_2) = \{\omega_2\}$, $c_1(\omega_2) = c_2(\omega_2) = c_3(\omega_1) = \Omega$, and consider the deviation to the vector of messages $x = (\Omega, \Omega, \Omega)$ at ω_2. This deviation is 3-identifiable at ω_2 by all players, and in particular by player 4. Hence, $\mathcal{P}_4(x, \omega_2) = \{\omega_2\}$, i.e., player 4's belief about ω_1 is $p(\omega_1 \mid \mathcal{P}_4(x, \omega_2)) = 0$. Nevertheless, Kreps and Wilson's (1982) consistency condition allows this belief to be equal to one, i.e., $\mu_4(\omega_1 \mid x, \omega) = 1$ for all $\omega \in \Omega$. To see this, consider the "trembling" communication strategies[36] $(\pi_i^t)_{i \in N}$ satisfying $\lim_{t \to \infty} \pi_i^t(c(\omega) \mid \omega) = 1$, $\pi_1^t(\Omega \mid \omega_1) = \pi_2^t(\Omega \mid \omega_1) = \varepsilon_t$, and $\pi_3^t(\Omega \mid \omega_2) = (\varepsilon_t)^3$, where $\lim_{t \to \infty} \varepsilon_t = 0$. We get, for all $\omega \in \Omega$,

$$\mu_4^t(\omega_1 \mid (\Omega, \Omega, \Omega), \omega) =$$
$$\frac{p(\omega_1)\pi_1^t(\Omega \mid \omega_1)\pi_2^t(\Omega \mid \omega_1)\pi_3^t(\Omega \mid \omega_1)}{p(\omega_1)\pi_1^t(\Omega \mid \omega_1)\pi_2^t(\Omega \mid \omega_1)\pi_3^t(\Omega \mid \omega_1) + p(\omega_2)\pi_1^t(\Omega \mid \omega_2)\pi_2^t(\Omega \mid \omega_2)\pi_3^t(\Omega \mid \omega_2)},$$

and thus

$$\lim_{t \to \infty} \mu_4^t(\omega_1 \mid (\Omega, \Omega, \Omega), \omega) = \lim_{t \to \infty} \frac{p(\omega_1)(\varepsilon_t)^2}{p(\omega_1)(\varepsilon_t)^2 + p(\omega_2)(\varepsilon_t)^3} = \lim_{t \to \infty} \frac{p(\omega_1)(\varepsilon_t)^2}{p(\omega_1)(\varepsilon_t)^2} = 1.$$

Example 7 Let $N = \{1, 2, 3\}$, $\Omega = \{\omega_1, \omega_2, \omega_3\}$,

$$\begin{aligned} H_1 &= \{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}\} \\ H_2 &= \{\{\omega_1, \omega_2\}, \{\omega_3\}\} \\ H_3 &= \{\omega_1, \omega_2, \omega_3\}, \end{aligned}$$

[35] One can verify that in this game there is a perfectly revealing knowledge equilibrium iff $p(\omega_1) \in [\frac{1}{7}, \frac{1}{6}]$.

[36] See Appendix A for more details.

$$c_1(\omega) = \{\omega\} \text{ for all } \omega \in \Omega, \qquad c_2(\omega) = \begin{cases} \Omega & \text{if } \omega \in \{\omega_1, \omega_2\} \\ \{\omega\} & \text{if } \omega = \omega_3, \end{cases}$$

and consider player 1's deviation to $x_1 = \Omega$. Since Ω does not belong to the range of c_1, this deviation is always 1-identifiable. Condition RR5 implies $\mathcal{P}_3((x_1, c_2(\omega)), \omega) \subseteq c_2^{-1}(c_2(\omega)) = c_2^{-1}(\Omega) = \{\omega_1, \omega_2\}$ for $\omega \in \{\omega_1, \omega_2\}$. Thus, player 3's belief about ω_3 is null at ω_1 and at ω_2 since $p(\omega_3 \mid \mathcal{P}_3((x_1, c_2(\omega_1)), \omega_1)) = p(\omega_3 \mid \mathcal{P}_3((x_1, c_2(\omega_2)), \omega_2)) = p(\omega_3 \mid \{\omega_1, \omega_2\}) = 0$. However, the belief $\mu_3(\omega_3 \mid (x_1, c_2(\omega_1)), \omega) = \mu_3(\omega_3 \mid (x_1, c_2(\omega_2)), \omega) = 1$ for all $\omega \in \Omega$ is compatible with Kreps and Wilson's (1982) consistency condition. To see this, consider the "trembling" communication strategies satisfying $\pi_1^t(\Omega \mid \omega_3) = \pi_2^t(\Omega \mid \omega_3) = \varepsilon_t$ and $\pi_1^t(\Omega \mid \omega_1) = \pi_1^t(\Omega \mid \omega_2) = (\varepsilon_t)^3$. We get, for all $\omega \in \Omega$,

$$\mu_3^t(\omega_3 \mid (\Omega, \Omega), \omega) = \frac{p(\omega_3)\pi_1^t(\Omega \mid \omega_3)\pi_2^t(\Omega \mid \omega_3)}{p(\omega_1)\pi_1^t(\Omega \mid \omega_1)\pi_2^t(\Omega \mid \omega_1) + p(\omega_2)\pi_1^t(\Omega \mid \omega_2)\pi_2^t(\Omega \mid \omega_2) + p(\omega_3)\pi_1^t(\Omega \mid \omega_3)\pi_2^t(\Omega \mid \omega_3)},$$

and thus

$$\lim_{t\to\infty} \mu_3^t(\omega_3 \mid (\Omega, \Omega), \omega) = \lim_{t\to\infty} \frac{p(\omega_3)(\varepsilon_t)^2}{p(\omega_1)(\varepsilon_t)^3 + p(\omega_2)(\varepsilon_t)^3 + p(\omega_3)(\varepsilon_t)^2} = 1.$$

In Example 5 on page 27 we have seen that we can have a perfectly revealing sequential equilibrium without having a perfectly revealing knowledge equilibrium, even with a very simple information structure. However, in this example, the non-revealing sequential equilibrium was a knowledge equilibrium. In the next proposition we show, under a particular certifiability level called *radical certifiability level*, that if a sequential equilibrium is non-revealing, then it is a knowledge equilibrium. In other words, if all knowledge equilibria involve some knowledge sharing (i.e., are not non-revealing) this is also true for all sequential equilibria. With a radical certifiability level players can only reveal exactly what they know or nothing. More precisely:

Definition 7 A certifiability level X is a *radical certifiability level* if for all $i \in N$ and $\omega \in \Omega$ we have either $X_i(\omega) = \{\Omega\}$ or $X_i(\omega) = \{\Omega, h_i(\omega)\}$.

In particular, if $|\Omega| = 2$ (as in Example 5) the certifiability level is necessarily radical.

Proposition 5 *Under the radical certifiability level, if a sequential equilibrium is non-revealing, then it is a knowledge equilibrium.*

Proof. The only differences between sequential equilibria of a communication game and knowledge equilibria are due to the use of pure communication strategies and particular degenerate second stage beliefs in the knowledge equilibrium (see Appendices A and C). Of course, the restriction to pure communication strategies has no impact when considering deviations from a non-revealing equilibrium, which is in pure strategies. Hence, we have to show that consistent beliefs (in the sense of Kreps and Wilson, 1982) are the same as beliefs generated by consistent possibility correspondences when a unilateral deviation occurs. Let $c \in C$ be a non-revealing equilibrium, i.e., $c_i(\omega) = \Omega$ for all $\omega \in \Omega$ and $i \in N$. A unilateral deviation $x \in X(c, \omega)$ by any player $i \in N$ at ω is necessarily identifiable by all players since it is characterized by a public disclosure $x = (\Omega \ldots, h_i(\omega), \ldots, \Omega)$. Therefore, from revision rule RR5' (deviant identification) we have $\mathcal{P}_j(x,\omega) = h_j(\omega) \cap h_i(\omega)$ for all $j \in N$. We have to show that for any consistent belief μ (in the sense of Kreps and Wilson), $\mu_j(\omega' \mid x,\omega) = p(\omega' \mid \mathcal{P}_j(x,\omega))$ for all $\omega' \in \Omega$ and $j \in N$. Using the fact that $c_j(\omega) = \Omega$ for all $\omega \in \Omega$ and $j \in N$, that player i's (trembling) communication strategy is measurable with respect to H_i, and the fact that player i can reveal $x_i = h_i(\omega)$ only when $h_i(\omega)$ is its actual information set, we get the following equalities:

$$
\begin{aligned}
&\lim_{t\to\infty} \pi_j^t(\Omega \mid \omega'') = 1 \text{ for all } j \neq i \text{ and } \omega'' \in \Omega, \\
&\lim_{t\to\infty} \pi_i^t(x_i \mid \omega'') = \lim_{t\to\infty} \pi_i^t(x_i \mid \omega''') = 0 \text{ for all } \omega'',\, \omega''' \in h_i(\omega), \\
\text{and} \quad & \pi_i^t(x_i \mid \omega'') = 0 \text{ for all } \omega'' \notin h_i(\omega).
\end{aligned}
$$

Finally, the result is obtained by applying Bayes' rule on these (unspecified) trembling communication strategies (see Equation (10) on page 34 in Appendix C). □

Thus, under the radical certifiability level we know by Theorem 2 and the last proposition that there exists a non-revealing sequential equilibrium if and only if there exists a non-revealing knowledge equilibrium. This result is of particular interest since the non-existence of a non-revealing knowledge equilibrium might suggest that Bayesian equilibria of the initial Bayesian game $G = G(h)$ without communication are not robust against strategic information revelation.

5 Conclusion and Discussion

It is a commonplace that voluntary and direct knowledge sharing may have relevant effects on information structures, and thus on individual and collective behaviors. In this paper, we have proposed a model that can effectively deal with such effects in a very large class of problems. In particular, our model can be applied, contrary to previous models in the literature on strategic information revelation, to environments with higher-order uncertainty and several privately informed decision makers. In addition, our model allows explicit characterizations of endogenous knowledge structures, along and outside the equilibrium path. As shown, all properties of a sequential equilibrium are satisfied by the knowledge equilibrium, but it is not necessary to consider perturbed games. Instead, outside equilibrium beliefs are determined by using natural and consistent interpretations of public revelation choices. Under these conditions we have seen, however, that traditional knowledge properties may not be satisfied, but remain perfectly manageable. In the following lines we discuss some of our results and assumptions, especially the assumption that information can be certified. We also justify and link our knowledge consistency condition by relying on other studies in the literature. Finally, we expose various possible extensions.

The Certifiability Constraint. A seemingly strong assumption regarding the feature of communication has been made throughout the paper since we always assumed that players can only reveal information they possess. Such an assumption may or may not be satisfied depending on the interactive context we want to analyze. We made this assumption so as to be in accordance with the literature on strategic information revelation. Nevertheless, we do not think that dropping the truth telling assumption changes substantially our model and our constructions. The only significant change is that certified information loses all its sense, i.e., the pure informational content of a message is irrelevant. Technically, this implies that the intersections with all disclosed events $x_j \in \mathcal{X}_j$ by any player j in any state ω must be dropped. In particular, the interpretation function should be redefined by $I_i^j(c,x,\omega) \equiv \mathrm{Maxi}\{h_i(\omega) \cap c_{-j}^{-1}(x_{-j}) \mid H_j, \succeq_j\}$, i.e., by dropping x_j in Equation (4) on page 17. Additionally, we should require that the set of available messages does not change across states of the world, i.e., $X(\omega) = X(\omega')$ for all ω, $\omega' \in \Omega$. In that case, it is easy to see that a non-revealing equilibrium always exists. Indeed, if $\mathcal{I}_i = \{\Omega\}$ and $c_i(\omega) = \Omega$ for all $i \in N$, then $\mathcal{P}_i(x,\omega) = I_i^j(c,x,\omega) = \{\Omega\}$ for all i, $j \in N$ and for any deviation $x \in X(c,\omega)$, a well known result in the literature on cheap talk games.

Knowledge Consistency. Several comments can be made on the links between our knowledge consistency condition and Kreps and Wilson's (1982) belief consistency condition. As seen in various examples and in the construction of consistent possibility correspondences, the differences concern the requirement we made on the identification of unilateral deviations and on the restrictions to a finite set of possible outside equilibrium beliefs. Indeed, we assumed that, when possible, outside equilibrium disclosures are assigned to a unique deviant, and deviations of this deviant are associated homogeneously to a set of his information sets compatible with others' disclosures and certified information. Our knowledge consistency conditions could also be justified from revision rules used in cognitive science (see, e.g., Gärdenfors, 1988). Actually, along the equilibrium path, the second stage information structure of the associated continuation Bayesian game is *uniquely* obtained from a knowledge *expansion*, since the vector of messages does not conflict with any agent's initial knowledge and equilibrium expectation. On the other hand, when a deviation is observed by an agent, second stage information is obtained from different possible knowledge *revisions* since the agent receives information that is inconsistent with the expected equilibrium play. This revising process is not uniquely defined because there is no well defined logical rule governing a unique rational belief revision, albeit some conditions should be required (as those induced by our revision rules RR1–RR4). When a multitude of beliefs can be logically selected, the literature on rational revisions of epistemic states consider a central rationality criterion on revisions in that they should be the minimal changes that accommodate and include the epistemic input (here, an epistemic input is a vector of messages received from the others). This postulate is motivated by a conservativity principle according to which, when changing beliefs in response to new evidence (in our case, an observable deviation), agents should continue to believe as many of the old beliefs as possible. In some sense, assigning a deviation to only one player satisfies this informational economy criterion since equilibrium interpretations are kept for all other players (our condition RR5). Finally, the orderings $\succeq_j$ we constructed on each player j's partition, as well as the ordering generated by the bijection ρ on the set of agents, could be assimilated to what is called *epistemic entrenchments* in that potential deviants and their information sets are ordered without connection to players' initial information and the prior probabilities (Gärdenfors, 1988, Chapter 4).

Strategic Delay in Information Disclosures. In many instances each player, before choosing which information he will disclose, may have an incentive to wait for others' public revelations. Such a possibility was not explored in the paper since we considered simultaneous and one-stage communication. If waiting is a strategic possibility, then a knowledge equilibrium will remain an equilibrium in our model if each player, after having waited one period, already want to disclose the same information. This no regret property can be obtained by dropping the sum over the set of states of the world in the definition of rational communication of the knowledge equilibrium (Condition 2 in Definition 6). In that case, even after a player waits one period and receives others' messages, he will continue to rationally disclose his information according to the same communication strategy. If an equilibrium exists under this condition, then it is stronger in the sense that it is robust to strategic incentives to wait others' information disclosures.

Extensions. By specifying the utility function for some adequate interactive decision situations we expect that the model presented here can easily be applied to specific or stylized economic problems, as well as to some classes of games. Some possible applications and several examples are presented in a companion paper (Koessler, 2002). Alternatively, we could eventually envisage to extend our theoretical analysis to repeated or networked communications. It could also be interesting to analyze when agents will rationally refuse to participate in some public revelation meetings, and thus will intentionally neglect some new possible information. Phenomena of bounded rationality due to a myopic learning or to a problem of information treatment (not only outside the equilibrium path) may also be envisaged. Another probably enriching and related research project would be to analyze strategic knowledge sharing without an equilibrium perspective. This should allow a characterization of endogenous knowledge structures in some classes of interactive decision situations where, e.g., common knowledge of rationality is assumed, but others' information-contingent strategies are initially unknown.

Appendices

A. Sequential Equilibrium of (G, X)

In this section we define Kreps and Wilson's (1982) sequential equilibrium of the extensive form communication game (G, X).

Remark 8 We assume that the set of states of the world Ω is finite because the sequential equilibrium does not apply to infinite games. The reason is that if Ω is infinite and countable, then the set of events an agent can certify may be uncountable (because the set of subsets of an infinite and countable set is uncountable), and thus it may be impossible to consider "trembling" strategies assigning a positive probability to each message.

For any $i \in N$, remember that Π_i is the set of mixed communication strategies of player i, i.e., the set of H_i measurable functions $\pi_i : \Omega \to \Delta(\mathcal{X}_i)$ such that $\text{supp}(\pi_i(\omega)) \subseteq X_i(\omega)$ for all $\omega \in \Omega$. Let $\pi(x \mid \omega) = \prod_{i \in N} \pi_i(x_i \mid \omega)$, where $x = (x_i)_{i \in N} \in X(\omega)$ is a vector of certified events. Σ_i is the set of (second (second stage) effective strategies of player i, i.e., the set of functions $\sigma_i : \mathcal{X} \times \Omega \Delta(A_i)$ such that $\sigma_i(x, \omega) = \sigma_i(x, \omega')$ for all $\omega \in \Omega$, $\omega' \in h_i(\omega)$ and $x \in \mathcal{X}$. Hence, $\sigma_i(a_i \mid x, \omega)$ is the probability that player i chooses action $a_i \in A_i$ at $\omega \in \Omega$ when the vector of messages $x \in X(\omega) \subseteq \mathcal{X}$ has been sent in the communication stage. We have $\Sigma = \prod_{i \in N} \Sigma_i$ and $\sigma(a \mid x, \omega) = \prod_{i \in N} \sigma_i(a_i \mid x, \omega)$, where $a \in A$.

A (second stage) *belief* of player i on Ω is given by a probability distribution $\mu_i : \mathcal{X} \times \Omega \to \Delta(\Omega)$, where $\mu_i(\omega' \mid x, \omega)$ is player i's belief about ω' when the vector of messages $x \in X(\omega)$ has been sent at ω. A *system of beliefs* is denoted by $\mu = (\mu_i)_{i \in N}$.[37] An *assessment* is a tuple (σ, π, μ), where $\sigma \in \Sigma$ is a profile of effective strategies, $\pi \in \Pi$ is a profile of (mixed) communication strategies, and μ is a system of beliefs. A partial assessment is given by (π, μ).

Let Π^0 be the set of all strictly positive communication strategy profiles, i.e., $\Pi^0 \equiv \{\pi \in \Pi : \pi(x \mid \omega) > 0, \ \forall\, \omega \in \Omega, \ \forall\, x \in X(\omega)\}$.[38] If $\pi \in \Pi^0$, then μ is associated with π and p via Bayes' rule:

$$\mu_i(\omega' \mid x, \omega) = \begin{cases} 0 & \text{if } \omega' \notin h_i(\omega) \cap \bigcap_{k \in N} x_k \\ \dfrac{\pi(x \mid \omega') p(\omega')}{\sum_{\omega'' \in h_i(\omega)} p(\omega'') \pi(x \mid \omega'')} & \text{otherwise,} \end{cases} \tag{9}$$

for all $\omega \in \Omega$ and $x \in X(\omega)$.

[37] Since every information set belonging to the first stage game (before messages are received) is reached with positive probability, we need only consider belief consistency in the second stage game.

[38] Notice that $\pi \in \Pi^0$ implies $\text{supp}(\pi_i(\omega)) = X_i(\omega)$ for all $\omega \in \Omega$ and $i \in N$.

Let Ψ^0 be the set of (partial) assessments (π, μ) where $\pi \in \Pi^0$ is a strictly positive communication strategy profile and μ is defined from p and π by Bayes' rule. An assessment (σ, π, μ) is *consistent* if $(\pi, \mu) = \lim_{t \to \infty}(\pi^t, \mu^t)$ for some sequence $\{(\pi^t, \mu^t)\} \subseteq \Psi^0$.

Given a system of beliefs μ and a profile of effective strategies σ, let

$$U_i(\sigma, x, \mu_i, \omega) \equiv \sum_{\omega' \in \Omega} \mu_i(\omega' \mid x, \omega) \sum_{a \in A} \sigma(a \mid x, \omega')\, u_i(a, \omega'),$$

be player i's expected utility at the beginning of the second stage game at $\omega \in \Omega$ when $x \in X(\omega)$ has been revealed. Let

$$EU_i(\sigma, \pi, \mu_i \mid h_i(\omega)) \equiv \sum_{\omega' \in \Omega} p(\omega' \mid h_i(\omega)) \sum_{x \in X(\omega')} \pi(x \mid \omega')\, U_i(\sigma, x, \mu_i, \omega'),$$

be player i's expected utility when he receives his initial information $h_i(\omega)$ at the beginning of the first stage game.

An assessment (σ, π, μ) is *sequentially rational* if for all $i \in N$, $\omega \in \Omega$, and $x \in X(\omega)$ we have,

$$U_i(\sigma, x, \mu_i, \omega) \geq U_i(a_i, \sigma_{-i}, x, \mu_i, \omega), \quad \forall\, a_i \in A_i,$$

and for all $i \in N$ and $\omega \in \Omega$,

$$EU_i(\sigma, \pi, \mu_i \mid h_i(\omega)) \geq EU_i(\sigma, x_i, \pi_{-i}, \mu_i \mid h_i(\omega)), \quad \forall\, x_i \in X_i(\omega).$$

Finally, a sequential equilibrium of (G, X) is an assessment (σ, π, μ) which is sequentially rational and consistent.

B. Additional Lemmas

In this section we show two intuitive but useful lemmas. These lemmas are needed to prove our theorems. The first lemma, used to prove Theorem 2, shows that if a unilateral deviation $x = (x_1, \ldots, x_n) \in X(\omega)$ from $c \in C$ is observable by some player $i \in N$ at $\omega \in \Omega$, then for all states $\omega' \in h_i(\omega) \cap \bigcap_{k \in N} x_k$, (i) There exists a potential deviant for player i whose expected message $c_j(\omega')$ at ω' differs from its actual message x_j, or (ii) There exist two players j and j' (not necessarily potential deviants) whose expected messages $c_j(\omega')$ and $c_{j'}(\omega')$ at ω' differ from their actual messages x_j and $x_{j'}$. The second lemma is a corollary of the first one, and is used to prove Theorem 1. It states that if a unilateral deviation is j-identifiable by some player at ω, then player j is effectively the deviant at ω.

Lemma 1 *Let $x \in X(c, \omega)$ be a unilateral deviation from $c \in C$ at $\omega \in \Omega$ which is observable by some player $i \in N$ at ω. Then, for all $\omega' \in h_i(\omega) \cap \bigcap_{k \in N} x_k$, one or both of the following properties hold:*

(i) There exists $j \in N_i(c, x, \omega)$ such that $x_j \neq c_j(\omega')$;

(ii) There exist j, $j' \in N$, $j \neq j'$, such that $x_j \neq c_j(\omega')$ and $x_{j'} \neq c_{j'}(\omega')$.

Proof. First note that if $N_i(c, x, \omega) = N$, then the result is immediate since the deviation is observable at ω. Indeed, in that case, $x \neq c(\omega')$ for all $\omega' \in h_i(\omega)$, i.e., there exists $j \in N$ such that $x_j \neq c_j(\omega')$; hence, property (i) is satisfied. Now, assume that $N_i(c, x, \omega) \neq N$. From the definition of the set $N_i(c, x, \omega)$ we have $h_i(\omega) \cap c_{-l}^{-1}(x_{-l}) \cap x_l = \emptyset$ for all $l \notin N_i(c, x, \omega)$. Let $\omega' \in h_i(\omega) \cap \bigcap_{k \in N} x_k$. This implies that for all $l \notin N_i(c, x, \omega)$, there exists $j \neq l$ such that $c_j(\omega') \neq x_j$. It is not difficult to verify that this property necessarily implies either (i), or (ii), or both. $\square$

Lemma 2 *If a unilateral deviation $x \in X(c, \omega)$ from $c \in C$ at $\omega \in \Omega$ is j-identifiable by some player at ω, then $x_j \neq c_j(\omega)$ and $x_k = c_k(\omega)$ for all $k \neq j$.*

Proof. If a unilateral deviation x from c is j-identifiable by player i at ω, then $N_i(c, x, \omega) = \{j\}$. Using the fact that $\omega \in h_i(\omega) \cap \bigcap_{k \in N} x_k$, Lemma 1 gives $x_j \neq c_j(\omega)$ (property (ii) in Lemma 1 is impossible with $\omega' = \omega$ since only unilateral deviations are considered). Hence, we also have $x_k = c_k(\omega)$ for all $k \neq j$ because x is a unilateral deviation. □

C. Proof of Theorem 2

In this section, we prove that a knowledge equilibrium (see Definition 6) forms a sequential equilibrium as described in Appendix A. It is easy to verify that sequential rationality is satisfied in the communication stage and in the second stage game. Indeed, the first two conditions in the definition of a knowledge equilibrium are equivalent to the conditions of sequential rationality given in Appendix A. Therefore, we have to show that beliefs associated with consistent possibility correspondences are consistent in the sense of Kreps and Wilson (1982) for all unilateral deviations during the communication stage.

Let $c \in C$ be a pure communication strategy, $\mathcal{P} = (\mathcal{P}_i)_{i \in N}$ a second stage information structure, and let $\pi(c(\omega) \mid \omega) = 1$ for all $\omega \in \Omega$. Thus, $\pi \in \Pi$ is the mixed communication strategy associated with the pure communication strategy $c \in C$. For all $i \in N$, ω, $\omega' \in \Omega$, and $x \in X(\omega)$, let $\mu_i(\omega' \mid x, \omega) = p(\omega' \mid \mathcal{P}_i(x, \omega))$. We show that consistent information structures can be associated with consistent beliefs in the sense of Kreps and Wilson (1982). More precisely, if $\mathcal{P}$ is consistent with (c, X), then we show that (σ, π, μ) is consistent at every information set reachable with at most one unilateral deviation.

Assume that the second stage information structure $\mathcal{P} = (\mathcal{P}_i)_{i \in N}$ is consistent with (c, X), let $(H_j, \succeq_j)_{j \in N}$ be an associated system of orderings over players' partitions, and let ρ be an associated bijection over N. We must find a sequence of strictly positive profiles of strategies $\{\pi^t\} \subseteq \Pi^0$ such that for all ω, $\omega' \in \Omega$, $i \in N$, and $x \in X(c, \omega)$ we have

$$\begin{array}{rrcl} & \lim_{t \to \infty} \pi^t(c(\omega) \mid \omega) & = & 1, \\ \text{and} & \lim_{t \to \infty} \mu_i^t(\omega' \mid x, \omega) & = & p(\omega' \mid \mathcal{P}_i(x, \omega)), \end{array}$$

where μ^t is defined from p and π^t by Bayes' rule, i.e.,

$$\mu_i(\omega' \mid x, \omega) = \begin{cases} 0, & \text{if } \omega' \notin h_i(\omega) \cap \bigcap_{k \in N} x_k \\ \dfrac{\pi^t(x \mid \omega') p(\omega')}{\sum_{\omega'' \in h_i(\omega)} p(\omega'') \pi^t(x \mid \omega'')}, & \text{otherwise,} \end{cases} \tag{10}$$

for all $\omega \in \Omega$ and $i \in N$. If $\omega' \notin h_i(\omega) \cap \bigcap_{k \in N} x_k$, then $\omega' \notin \mathcal{P}_i(x, \omega)$ by the certifiability constraint (condition RR1), and thus $p(\omega' \mid \mathcal{P}_i(x, \omega)) = 0 = \lim_{t \to \infty} \mu_i^t(\omega' \mid x, \omega)$ by Equation (10). Therefore, we must find a sequence $\{\pi^t\} \subseteq \Pi^0$ such that for all $\omega \in \Omega$, $x \in X(c, \omega)$, $i \in N$, and $\omega' \in h_i(\omega) \cap \bigcap_{k \in N} x_k$ we have $\lim_{t \to \infty} \pi^t(c(\omega) \mid \omega) = 1$ and

$$\lim_{t \to \infty} \frac{\pi^t(x \mid \omega') p(\omega')}{\sum_{\omega'' \in h_i(\omega)} p(\omega'') \pi^t(x \mid \omega'')} = p(\omega' \mid \mathcal{P}_i(x, \omega)),$$

i.e.,

$$\lim_{t \to \infty} \frac{\pi^t(x \mid \omega')}{\sum_{\omega'' \in h_i(\omega)} p(\omega'') \pi^t(x \mid \omega'')} = \begin{cases} 0 & \text{if } \omega' \notin \mathcal{P}_i(x, \omega) \\ 1 / \sum_{\omega'' \in \mathcal{P}_i(x, \omega)} p(\omega'') & \text{if } \omega' \in \mathcal{P}_i(x, \omega). \end{cases}$$

This last equality is satisfied if for all $\omega'' \in \mathcal{P}_i(x,\omega)$ we have

$$\omega' \in \mathcal{P}_i(x,\omega) \Rightarrow \lim_{t\to\infty} \frac{\pi^t(x \mid \omega')}{\pi^t(x \mid \omega'')} = 1 \tag{11}$$

$$\omega' \notin \mathcal{P}_i(x,\omega) \Rightarrow \lim_{t\to\infty} \frac{\pi^t(x \mid \omega')}{\pi^t(x \mid \omega'')} = 0. \tag{12}$$

Note that the fractions given just above are well defined since $\omega'' \in \mathcal{P}_i(x,\omega) \Rightarrow \omega'' \in \bigcap_{k\in N} x_k$ by the certifiability constraint. Hence, $x \in X(\omega'')$, which implies that $\pi^t(x \mid \omega'') \neq 0$ because $\pi^t \in \Pi^0$.

Let $\{\varepsilon_t\} \subseteq \mathbb{R}$ be a sequence such that $\lim_{t\to\infty} \varepsilon_t = 0$. To simplify the notations, we drop the subscript t in ε_t. For all $j \in N$, let $\mathcal{I}_j^1 = \{h \in \mathcal{I}_j : h \succeq_j h' \ \forall\, h' \in \mathcal{I}_j\}$, $\mathcal{I}_j^2 = \{h \in \mathcal{I}_j \backslash \mathcal{I}_j^1 : h \succeq_j h' \ \forall\, h' \in \mathcal{I}_j \backslash \mathcal{I}_j^1\}$, $\mathcal{I}_j^3 = \{h \in \mathcal{I}_j \backslash (\mathcal{I}_j^1 \cup \mathcal{I}_j^2) : h \succeq_j h' \ \forall\, h' \in \mathcal{I}_j \backslash (\mathcal{I}_j^1 \cup \mathcal{I}_j^2)\}$, and so on. For example, if $H_j = \{h_j^1, \ldots, h_j^5\}$ and $h_j^1 \succ_j h_j^2 \sim_j h_j^3 \sim_j h_j^4 \succ_j h_j^5$, then $\mathcal{I}_j = \{h_j^1, h_j^2 \cup h_j^3 \cup h_j^4, h_j^5\}$, $\mathcal{I}_j^1 = \{h_j^1\}$, $\mathcal{I}_j^2 = \{h_j^2 \cup h_j^3 \cup h_j^4\}$, and $\mathcal{I}_j^3 = \{h_j^5\}$.

For all $j \in N$ and $\omega \in \Omega$, let $l_j(\omega)$ be the integer l satisfying $\omega \in \mathcal{I}_j^l$. Note that we necessarily have $l_j(\omega) \leq |\Omega| = m$. We consider the following profile of "trembling" communication strategies: For all $j \in N$ and $\omega \in \Omega$,

$$\pi_j^t(x_j \mid \omega) = \begin{cases} \varepsilon^n \varepsilon^{n+1-\rho(j)} \varepsilon^{\frac{l_j(\omega)}{m+1}} & \text{if } x_j \neq c_j(\omega),\ x_j \in X_j(\omega) \\ 1 - (|X_j(\omega)| - 1)\varepsilon^n \varepsilon^{n+1-\rho(j)} \varepsilon^{\frac{l_j(\omega)}{m+1}} & \text{if } x_j = c_j(\omega) \\ 0 & \text{otherwise.} \end{cases}$$

Hence, $\sum_{x_j \in X_j(\omega)} \pi_j^t(x_j \mid \omega) = 1$ and $\lim_{t\to\infty} \pi_j^t(c_j(\omega) \mid \omega) = 1$ for all $j \in N$ and $\omega \in \Omega$. We will differentiate two cases: (1) First we will assume that x is not an observable deviation for player i at ω; this configuration is the simplest one. (2) Second, we will assume that x is an observable deviation for player i at ω. In both cases, we will always assume that $\omega'' \in \mathcal{P}_i(x,\omega)$ in order to show that conditions (11) and (12) are satisfied. As mentioned before, we also assume $\omega' \in h_i(\omega) \cap \bigcap_{k\in N} x_k$ (otherwise, the result was already proved).

(1) Unobservable Deviation

By definition, if x is not an observable deviation for player i at ω, then $\mathcal{P}_i(x,\omega) = h_i(\omega) \cap c^{-1}(x)$. Hence, $\omega'' \in c^{-1}(x)$, i.e., $\lim_{t\to\infty} \pi^t(x \mid \omega'') = 1$. Similarly, if $\omega' \in \mathcal{P}_i(x,\omega)$, then $\lim_{t\to\infty} \pi^t(x \mid \omega') = 1$. In this case, condition (11) is satisfied. On the contrary, if $\omega' \notin \mathcal{P}_i(x,\omega)$, then $\omega' \notin c^{-1}(x)$,[39] which implies that $\lim_{t\to\infty} \pi^t(x \mid \omega') = 0$. Thus, condition (12) is also satisfied.

(2) Observable Deviation

Assume that x is an observable deviation from c for player i at ω. In this case we have $\mathcal{P}_i(x,\omega) = \text{Maxi}\{h_i(\omega) \cap c_{-\eta}^{-1}(x_{-\eta}) \cap x_\eta \mid H_\eta, \succeq_\eta\}$, where

$$\eta = \overline{N}_i(c,x,\omega \mid \rho) \in \arg\max_{k\in N_i(c,x,\omega)} \rho(k).^{40}$$

If ω', $\omega'' \in \mathcal{P}_i(x,\omega)$, then ω', $\omega'' \in c_k^{-1}(x_k)$ for all $k \neq \eta$, and thus $x_k = c_k(\omega') = c_k(\omega'')$ for all $k \neq \eta$. Therefore, $\lim_{t\to\infty} \pi_k^t(x_k \mid \omega') = \lim_{t\to\infty} \pi_k^t(x_k \mid \omega'') = 1$ for all $k \neq \eta$. Moreover, since the deviation is observable, $x_\eta \neq c_\eta(\omega')$ and $x_\eta \neq c_\eta(\omega'')$.[41] Consequently,

[39] Remember that we assume that $\omega' \in h_i(\omega) \cap \bigcap_{k\in N} x_k$.

[40] Of course, if x is a j-identifiable deviation for player i at ω, then $\eta = j$.

[41] Otherwise, ω' or ω'' belongs to $h_i(\omega) \cap c^{-1}(x)$, a contradiction with the fact that the deviation is observable (see Definition 3).

$\lim_{t\to\infty} \pi^t_\eta(x_\eta \mid \omega') = \lim_{t\to\infty} \pi^t_\eta(x_\eta \mid \omega'') = 0$. We get, $\lim_{t\to\infty} \frac{\pi^t(x|\omega')}{\pi^t(x|\omega'')} = \lim_{t\to\infty} \frac{\pi^t_\eta(x_\eta|\omega')}{\pi^t_\eta(x_\eta|\omega'')} = \lim_{t\to\infty} \frac{\varepsilon^n \varepsilon^{n+1-\rho(\eta)} \varepsilon^{\frac{l_\eta(\omega')}{m+1}}}{\varepsilon^n \varepsilon^{n+1-\rho(\eta)} \varepsilon^{\frac{l_\eta(\omega'')}{m+1}}} = 1$ since ω', $\omega'' \in \mathcal{P}_i(x,\omega) \Rightarrow h_\eta(\omega') \sim_\eta h_\eta(\omega'') \Rightarrow l_\eta(\omega') = l_\eta(\omega'')$.

Now, let $\omega'' \in \mathcal{P}_i(x,\omega)$ but $\omega' \notin \mathcal{P}_i(x,\omega)$. This last condition implies that either

(a) $\omega' \in h_i(\omega) \cap c^{-1}_{-\eta}(x_{-\eta}) \cap x_\eta$ but $h_\eta(\omega') \prec_\eta h_\eta(\omega'')$, or
(b) $\omega' \notin \cap c^{-1}_{-\eta}(x_{-\eta})$ (and, as usual, $\omega' \in h_i(\omega) \cap \bigcap_{k\in N} x_k$).

If (a), then ω', $\omega'' \in c^{-1}_k(x_k)$ for all $k \neq \eta$, which implies that $\lim_{t\to\infty} \pi^t_k(x_k \mid \omega') = \lim_{t\to\infty} \pi^t_k(x_k \mid \omega'') = 1$ for all $k \neq \eta$, i.e.,

$$\lim_{t\to\infty} \frac{\pi^t(x \mid \omega')}{\pi^t(x \mid \omega'')} = \lim_{t\to\infty} \frac{\pi^t_\eta(x_\eta \mid \omega')}{\pi^t_\eta(x_\eta \mid \omega'')} = \lim_{t\to\infty} \frac{\varepsilon^n \varepsilon^{n+1-\rho(\eta)} \varepsilon^{\frac{l_\eta(\omega')}{m+1}}}{\varepsilon^n \varepsilon^{n+1-\rho(\eta)} \varepsilon^{\frac{l_\eta(\omega'')}{m+1}}} = \lim_{t\to\infty} \frac{\varepsilon^{\frac{l_\eta(\omega')}{m+1}}}{\varepsilon^{\frac{l_\eta(\omega'')}{m+1}}} = 0,$$

since $h_\eta(\omega') \prec_\eta h_\eta(\omega'') \Rightarrow l_\eta(\omega') > l_\eta(\omega'') \Rightarrow \frac{l_\eta(\omega')}{m+1} > \frac{l_\eta(\omega'')}{m+1}$.

If (b), then there exists $k \neq \eta$ such that $c_k(\omega') \neq x_k$. Moreover, from Lemma 1, we have to differentiate two cases: (i) There exists $j \in N_i(c,x,\omega)$ such that $c_j(\omega') \neq x_j$ (j might be equal to k if $N_i(c,x,\omega)$ is not a singleton, i.e., if the deviation is not identifiable), or (ii) There exist j, $j' \in N$, $j \neq j'$, such that $x_j \neq c_j(\omega')$ and $x_{j'} \neq c_{j'}(\omega')$.

In case (i) we have to distinguish again two subcases: (b') $k \in N_i(c,x,\omega)$, and (b") $k \notin N_i(c,x,\omega)$. For example, if the deviation is j-identifiable, we necessarily have $j = \eta$, and thus $k \notin N_i(c,x,\omega)$ (subcase (b")), which implies that $j \neq k$. In both subcases, we get $\lim_{t\to\infty} \pi^t_k(x_k \mid \omega') = \lim_{t\to\infty} \pi^t_j(x_j \mid \omega') = 0$. Under condition (b') we obtain

$$\begin{aligned}\lim_{t\to\infty} \frac{\pi^t(x \mid \omega')}{\pi^t(x \mid \omega'')} &< \lim_{t\to\infty} \frac{\pi^t_k(x_k \mid \omega')}{\pi^t_\eta(x_\eta \mid \omega'')} \\ &= \lim_{t\to\infty} \frac{\varepsilon^n \varepsilon^{n+1-\rho(k)} \varepsilon^{\frac{l_k(\omega')}{m+1}}}{\varepsilon^n \varepsilon^{n+1-\rho(\eta)} \varepsilon^{\frac{l_\eta(\omega'')}{m+1}}} = 0,\end{aligned}$$

because $k \in N_i(c,x,\omega)$, $k \neq \eta \Rightarrow \rho(\eta) > \rho(k)$ and because $\frac{l_k(\omega')}{m+1}$, $\frac{l_\eta(\omega')}{m+1} < 1$. Under condition (b"), we necessarily have $j \neq k$, and thus

$$\begin{aligned}\lim_{t\to\infty} \frac{\pi^t(x \mid \omega')}{\pi^t(x \mid \omega'')} &< \lim_{t\to\infty} \frac{\pi^t_k(x_k \mid \omega')\pi^t_j(x_j \mid \omega')}{\pi^t_\eta(x_\eta \mid \omega'')} \\ &= \lim_{t\to\infty} \frac{\varepsilon^n \varepsilon^{n+1-\rho(k)} \varepsilon^{\frac{l_k(\omega')}{m+1}} \varepsilon^n \varepsilon^{n+1-\rho(j)} \varepsilon^{\frac{l_j(\omega')}{m+1}}}{\varepsilon^n \varepsilon^{n+1-\rho(\eta)} \varepsilon^{\frac{l_\eta(\omega'')}{m+1}}} \\ &< \lim_{t\to\infty} \frac{\varepsilon^{2n} \varepsilon^2}{\varepsilon^{2n} \varepsilon^{\frac{l_\eta(\omega'')}{m+1}}} = 0,\end{aligned}$$

since $\frac{l_\eta(\omega'')}{m+1} < 2$.

Finally, assume that (ii) holds. In that case, we can take $j \neq k$, and thus we can apply the same reasoning as in subcase (b").

Since a knowledge equilibrium is associated with consistent beliefs and with communication and effective strategies which are sequentially rational, a knowledge equilibrium forms a sequential equilibrium. This completes the proof.

References

AUMANN, R. J. (1976): "Agreeing to Disagree," *The Annals of Statistics*, 4, 1236–1239.

BEN-PORATH, E. (2001): "Cheap Talk in Games with Incomplete Information," mimeo, The Eiten Berglas School of Economics, Tel-Aviv University.

CRAWFORD, V. P. AND J. SOBEL (1982): "Strategic Information Transmission," *Econometrica*, 50, 1431–1451.

DEKEL, E., B. L. LIPMAN, AND A. RUSTICHINI (1998): "Standard State-Space Models Preclude Unawareness," *Econometrica*, 66, 159–173.

FAGIN, R., J. Y. HALPERN, Y. MOSES, AND M. Y. VARDI (1995): *Reasoning about Knowledge*, Cambridge, Massachusetts: MIT Press.

FARRELL, J. AND M. RABIN (1996): "Cheap Talk," *Journal of Economic Perspectives*, 10, 103–118.

FISHMAN, M. J. AND K. M. HAGERTY (1990): "The Optimal Amount of Discretion to Allow in Disclosure," *Quarterly Journal of Economics*, 105, 427–444.

FORGES, F. (1990): "Universal Mechanisms," *Econometrica*, 58, 1341–1364.

FUDENBERG, D. AND J. TIROLE (1991): "Perfect Bayesian Equilibrium and Sequential Equilibrium," *Journal of Economic Theory*, 53, 236–260.

GEANAKOPLOS, J. (1989): "Game Theory without Partitions, and Applications to Speculation and Consensus," Cowles Foundation Discussion Paper 914, Yale University.

——— (1994): "Common Knowledge," in *Handbook of Game Theory*, ed. by R. J. Aumann and S. Hart, Elsevier Science B. V., vol. 2, chap. 40, 1437–1496.

GEANAKOPLOS, J. AND H. M. POLEMARCHAKIS (1982): "We Can't Disagree Forever," *Journal of Economic Theory*, 28, 192–200.

GLAZER, J. AND A. RUBINSTEIN (2001): "Debates and Decisions: On a Rationale of Argumentation Rules," *Games and Economic Behavior*, 36, 158–173.

GÄRDENFORS, P. (1988): *Knowledge in Flux: Modeling the Dynamics of Epistemic States*, Cambridge, Massachusetts: MIT Press.

GROSSMAN, S. J. (1981): "The Informational Role of Warranties and Private Disclosure about Product Quality," *Journal of Law and Economics*, 24, 461–483.

HARSANYI, J. C. (1967–1968): "Games with Incomplete Information Played by Bayesian Players. Parts I, II, III," *Management Science*, 14, 159–182, 320–334, 486–502.

KOESSLER, F. (2002): "Strategic Knowledge Sharing in Bayesian Games: Applications," Working Paper 2002–02, BETA, Université Louis Pasteur, Strasbourg.

KRASUCKI, P. (1996): "Protocols Forcing Consensus," *Journal of Economic Theory*, 70, 266–272.

KREPS, D. M. AND R. WILSON (1982): "Sequential Equilibria," *Econometrica*, 50, 863–894.

LIPMAN, B. L. AND D. SEPPI (1995): "Robust Inference in Communication Games with Partial Provability," *Journal of Economic Theory*, 66, 370–405.

MILGROM, P. (1981): "Good News and Bad News: Representation Theorems and Applications," *Bell Journal of Economics*, 12, 380–391.

MILGROM, P. AND J. ROBERTS (1986): "Relying on the Information of Interested Parties," *Rand Journal of Economics*, 17, 18–32.

MODICA, S. AND A. RUSTICHINI (1999): "Unawareness and Partitional Information Structures," *Games and Economic Behavior*, 27, 265–298.

MORRIS, S. AND H. S. SHIN (2000): "Global Games: Theory and Applications," invited lecture at the Eight World Congress of the Econometric Society (Seattle, 2000).

OKUNO-FUJIWARA, A., M. POSTLEWAITE, AND K. SUZUMURA (1990): "Strategic Information Revelation," *Review of Economic Studies*, 57, 25–47.

PARIKH, R. AND P. KRASUCKI (1990): "Communication, Consensus, and Knowledge," *Journal of Economic Theory*, 52, 178–189.

SEIDMANN, D. J. AND E. WINTER (1997): "Strategic Information Transmission with Verifiable Messages," *Econometrica*, 65, 163–169.

SELTEN, R. (1975): "Reexamination of the Perfectness Concept for Equilibrium Points in Extensive Games," *International Journal of Game Theory*, 4, 25–55.

SHIN, H. S. (1994a): "The Burden of Proof in a Game of Persuasion," *Journal of Economic Theory*, 64, 253–264.

——— (1994b): "News Management and the Value of Firms," *Rand Journal of Economics*, 25, 58–71.